

FIXED POINTS OF CONTRACTIVE MAPPINGS ON COMPLETE CONE METRIC SPACES

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Abstract

In this paper we present some fixed point results for contractive mappings in complete cone metric spaces. Under special conditions, our results are generalizations of the results of Huang Long-Guang and Zhang Xian (*Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **332**, 1468–1476, 2007), and Sh. Rezapour and R. Hamlbarani (*Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings"*, J. Math. Anal. Appl. **345**, 719–724, 2008).

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1. Introduction

It is well known that the classic contraction mapping principle of Banach is a fundamental result in fixed point theory. Several authors have obtained various extensions and generalizations of Banach's theorem by considering contractive mappings on many different metric spaces. For example, [2, 4, 5, 6, 7, 8, 9, 11], and others. Recently, Guang and Xian [3] introduced the notion of cone metric spaces and proved some fixed point theorems in cone metric spaces for mappings satisfying various contractive conditions. In [10], Rezapour and Hamlbarani generalized some results of [3] by omitting the assumption of normality in the results.

The main purpose of this paper is to present some fixed point results for contractive mappings in complete cone metric spaces.

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2. Preliminaries

Throughout this paper, we denote by \mathbb{N} the set of positive integers and by \mathbb{R} the set of real numbers.

2.1. Definition. Let E be a real Banach space and P a subset of E . Then P is called a *cone* if and only if:

- (i) P is closed, nonempty and satisfies $P \neq \{0\}$,
- (ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P$ implies $ax + by \in P$,
- (iii) $x \in P$ and $-x \in P$ implies $x = 0$.

Given a cone $P \subseteq E$, we define a partial ordering \leq with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ if $x \leq y$ and $x \neq y$, and $x \ll y$ if $y - x \in \text{int } P$, where $\text{int } P$ is the interior of P .

The cone P is called *normal* if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq K\|y\|.$$

The least positive number satisfying the above is then called the *normal constant* of P .

2.2. Lemma. ([12]) *Let E be a real Banach space with a cone P . Then:*

- (i) *If $x \leq y$ and $0 \leq a \leq b$, then $ax \leq by$,*
- (ii) *If $x \leq y$ and $u \leq v$, then $x + u \leq y + v$,*
- (iii) *If $x_n \leq y_n$ for each $n \in \mathbb{N}$, and $\lim_{n \rightarrow \infty} x_n = x$, $\lim_{n \rightarrow \infty} y_n = y$, then $x \leq y$.*

2.3. Lemma. *If P is a cone, $x \in P$, $\alpha \in \mathbb{R}$, $0 \leq \alpha < 1$, and $x \leq \alpha x$, then $x = 0$.*

Proof. If $x \leq \alpha x$, then $\alpha x - x = (\alpha - 1)x \in P$. Since $x \in P$ and $0 \leq \alpha < 1$, we have, from Definition 2.1. (ii), $(1 - \alpha)x \in P$. It follows from Definition 2.1. (iii) that $x = 0$. \square

In the following definition, we suppose that E is a real Banach space, P a cone in E with $\text{int } P \neq \emptyset$ and that \leq is the partial ordering with respect to P .

2.4. Definition. Let X be a non-empty set. Suppose the mapping $d : X \times X \rightarrow E$ satisfies:

- (d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,
- (d2) $d(x, y) = d(y, x)$ for all $x, y \in X$,
- (d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a *cone metric* on X , and (X, d) is called a *cone metric space*. This definition is more general than that of a metric space.

2.5. Example. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subset \mathbb{R}^2$, $X = \mathbb{R}^2$ and suppose that $d : X \times X \rightarrow E$ is defined by

$$d(x, y) = d((x_1, x_2), (y_1, y_2)) = (|x_1 - y_1| + |x_2 - y_2|, \alpha \max\{|x_1 - y_1|, |x_2 - y_2|\}),$$

where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

3. Definitions and lemmas

In this section we shall give some definitions and lemmas.

3.1. Definition. [3] Let (X, d) be a cone metric space. A sequence $\{x_n\}$ in X is said to be:

- (a) A *convergent sequence* if for every $c \in E$ with $0 \ll c$, there is $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, x) \ll c$ for some fixed x in X . We denote this by $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$, $n \rightarrow \infty$.

- (b) A *Cauchy sequence* if for every $c \in E$ with $0 \ll c$, there is $N \in \mathbb{N}$ such that for all $n, m \geq N$, $d(x_n, x_m) \ll c$.

A cone metric space (X, d) is said to be *complete* if every Cauchy sequence is convergent in X .

The following lemma was recently proved without assuming normality in [1].

3.2. Lemma. *Let (X, d) be a cone metric space. If $\{x_n\}$ is a convergent sequence in X , then the limit of $\{x_n\}$ is unique.* \square

The proof of the following lemma is straightforward, and is omitted.

3.3. Lemma. *Let (X, d) be a cone metric space, $\{x_n\}$ a sequence in X . If $\{x_n\}$ converges to x and $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$, then $\{x_{n_k}\}$ converges to x .* \square

3.4. Lemma. *Let (X, d) be a cone metric space and $\{x_n\}$ a sequence in X . If there exists a sequence $\{a_n\}$ in \mathbb{R} with $a_n > 0$ for all $n \in \mathbb{N}$ and $\sum a_n < \infty$, which satisfies $d(x_{n+1}, x_n) \leq a_n M$ for all $n \in \mathbb{N}$ and for some $M \in E$ with $M \geq 0$, then $\{x_n\}$ is a Cauchy sequence in (X, d) .*

Proof. Let $n > m$. Then

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\leq (a_{n-1} + a_{n-2} + \cdots + a_m)M \\ (1) \quad &= M \sum_{k=m}^{n-1} a_k. \end{aligned}$$

Take $c \in E$ with $0 \ll c$. Choose $\varepsilon > 0$ such that $c + N_\varepsilon(0) \subseteq P$, where

$$N_\varepsilon(0) = \{y \in E : \|y\| < \varepsilon\}.$$

Since $\sum_{n=1}^{\infty} a_n < \infty$, we can choose a sufficiently large natural number N such that

$$\left\| \sum_{k=m}^{n-1} a_k \right\| \|M\| = \left\| M \sum_{k=m}^{n-1} a_k \right\| < \varepsilon,$$

for all $n > m \geq N$. Therefore, we have $M \sum_{k=m}^{n-1} a_k \in N_\varepsilon(0)$ and $-M \sum_{k=m}^{n-1} a_k \in N_\varepsilon(0)$, for all $n > m \geq N$. Hence $c - M \sum_{k=m}^{n-1} a_k \in c + N_\varepsilon(0)$ and so $c - M \sum_{k=m}^{n-1} a_k \in \text{int } P$, for all $n > m \geq N$. Thus, $M \sum_{k=m}^{n-1} a_k \ll c$, for all $n > m \geq N$. Then from inequality (1), we have $d(x_n, x_m) \ll c$, for all $n > m \geq N$. Therefore, $\{x_n\}$ is a Cauchy sequence in (X, d) . \square

4. Fixed points on complete cone metric spaces

The following theorems were proved in [3].

4.1. Theorem. *Let (X, d) be a complete cone metric space with normal cone P and normal constant K . Suppose the mapping $T : X \rightarrow X$ satisfies the contractive condition*

$$(2) \quad d(Tx, Ty) \leq k(d(Tx, x) + d(Ty, y)),$$

for all $x, y \in X$, where $k \in [0, \frac{1}{2})$ is a constant. Then, T has a unique fixed point in X . For each $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point. \square

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Note that there are cones which are not normal, see [10]. In [10], Rezapour and Hamlbarani proved Theorem 4.1 and Theorem 4.2 without the normality condition.

Now suppose that E is a Banach space and P a cone in E . For $a, b, c \in E$ we will denote the proposition $(a \leq b) \vee (a \leq c)$ by $a \leq b \vee c$. Moreover, for $r \in \mathbb{R}$, $a \leq r(b \vee c)$ will denote $(a \leq rb) \vee (a \leq rc)$.

This notation is convenient because we do not have to repeat the left-hand element, but it must be stressed that $b \vee c$ has no meaning outside these expressions. In particular the only case where there exists an *element* d of E for which $a \leq b \vee c$ is equivalent to $a \leq d$ for all a is when b and c are comparable and $d = b \vee c = \max\{b, c\}$.

We now note that if $d(Tx, x)$ and $d(Ty, y)$ are comparable, then

$$d(Tx, x) + d(Ty, y) \leq 2[d(Tx, x) \vee d(Ty, y)].$$

Similarly, if $d(Tx, y)$ and $d(Ty, x)$ are comparable, then

$$d(Tx, y) + d(Ty, x) \leq 2[d(Tx, y) \vee d(Ty, x)].$$

In this case, if inequalities (2) and (3) hold, then we obtain, respectively, the inequalities

$$d(Tx, Ty) \leq k[d(Tx, x) \vee d(Ty, y)]$$

and

$$d(Tx, Ty) \leq k[d(Tx, y) \vee d(Ty, x)]$$

where $k \in [0, 1)$. These inequalities can hold for some $k \in [0, 1)$ even if (2) or (3), respectively, do not hold for some $k \in [0, \frac{1}{2})$ (see Examples 5.1, 5.2 below). Moreover, replacing \vee by \bigvee in the above inequalities gives us two more general conditions that are well defined even when $d(Tx, x)$, $d(Ty, y)$ (resp. $d(Tx, y)$, $d(Ty, x)$) are not comparable. It is these conditions that form the basis of the following theorems.

4.3. Theorem. *Let T be a mapping on the complete cone metric space X into itself that satisfies the inequality*

$$(4) \quad d(Tx, Ty) \leq k \left[d(Tx, x) \bigvee d(Ty, y) \right],$$

for all $x, y \in X$, where $0 \leq k < 1$. Then T has a unique fixed point in X . For each $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Proof. Let x be an arbitrary point X . If $T^{n+1}x = T^n x$ for some n , then T has a fixed point. Assume $T^{n+1}x \neq T^n x$ for each n . Using inequality (4), we have

$$d(T^{n+1}x, T^n x) \leq k \left[d(T^{n+1}x, T^n x) \bigvee d(T^n x, T^{n-1}x) \right].$$

If $d(T^{n+1}x, T^n x) \leq kd(T^{n+1}x, T^n x)$, then from Lemma 2.3 we have $d(T^{n+1}x, T^n x) = 0$, since $k < 1$, which contradicts our hypothesis. Therefore we have

$$d(T^{n+1}x, T^n x) \leq kd(T^n x, T^{n-1}x).$$

In general

$$d(T^{n+1}x, T^n x) \leq k^n d(Tx, x)$$

for all $n \in \mathbb{N}$. Since $\sum_{n=1}^{\infty} k^n < \infty$, it follows from Lemma 3.4 that $\{T^n x\}$ is a Cauchy sequence in the complete cone metric space (X, d) and so has a limit z in X .

Now let $c \in E$ with $0 \ll c$. Choose a natural number N such that $d(T^n x, z) \ll c(1-k)$ and $d(T^n x, T^{n-1}x) \ll c$ for all $n \geq N$. From condition (d3) we have

$$(5) \quad d(Tz, z) \leq d(Tz, T^n x) + d(T^n x, z)$$

for all $n \geq N$, and from condition (4),

$$d(Tz, T^n x) \leq k \left[d(Tz, z) \vee d(T^n x, T^{n-1} x) \right].$$

Now we have to consider, for each $n \geq N$, the following two cases:

Case 1. If $d(Tz, T^n x) \leq kd(Tz, z)$, then from (5), we have

$$d(Tz, z) \leq \frac{1}{1-k} d(T^n x, z) \ll c.$$

Case 2. If $d(Tz, T^n x) \leq kd(T^n x, T^{n-1} x)$, then from (5), we have

$$d(Tz, z) \leq d(T^n x, z) + kd(T^n x, T^{n-1} x) \ll c(1-k) + ck = c.$$

It now follows from Cases 1 and 2 that $d(Tz, z) \ll c$. For fixed $0 \ll c$, we have $0 \ll \frac{c}{m}$ for all $m \in \mathbb{N}$. Thus $d(Tz, z) \ll \frac{c}{m}$, for all $m \in \mathbb{N}$. So, $\frac{c}{m} - d(Tz, z) \in P$, for all $m \in \mathbb{N}$. Since $\frac{c}{m} \rightarrow 0$, as $m \rightarrow \infty$, and P is closed, it follows that $-d(Tz, z) \in P$. Also we have $d(Tz, z) \in P$. Hence $d(Tz, z) = 0$ and so $Tz = z$.

Now suppose that T has a second fixed point z' . Then from inequality (4) we have

$$d(z, z') = d(Tz, Tz') \leq k \left[d(Tz, z) \vee d(Tz', z') \right],$$

which is equivalent to $d(z, z') \leq k[d(Tz, z) \vee d(Tz', z')] = 0$ since $d(Tz, z) = 0 = d(Tz', z')$ are comparable. It follows that $z = z'$. So z is the unique fixed point of T . \square

4.4. Theorem. *Let T be a mapping on the complete cone metric space X into itself that satisfies the inequality*

$$(6) \quad d(Tx, Ty) \leq k \left[d(Tx, y) \vee d(Ty, x) \right],$$

for all $x, y \in X$, where $0 \leq k < 1$. Then T has a unique fixed point in X . For each $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Proof. Let x be an arbitrary point X . Then using inequality (6) for $d(T^{n+1}x, T^n x)$ and noting that $d(T^n x, T^n x) = 0$ is comparable with $d(T^{n-1}x, T^{n+1}x)$ we have

$$d(T^{n+1}x, T^n x) \leq k[d(T^{n+1}x, T^{n-1}x) \vee d(T^n x, T^n x)] = kd(T^{n+1}x, T^{n-1}x).$$

Again using (6) we have

$$d(T^{n+1}x, T^n x) \leq k^2 \left[d(T^{n+1}x, T^{n-2}x) \vee d(T^{n-1}x, T^n x) \right],$$

and then

$$d(T^{n+1}x, T^n x) \leq k^3 \left[d(T^{n+1}x, T^{n-3}x) \vee d(T^n x, T^{n-2}x) \right].$$

Continuing this process we have

$$(7) \quad d(T^{n+1}x, T^n x) \leq k^n \vee \left\{ d(T^{n+1-s}x, T^s x) : s = 0, 1, 2, \dots, m \right\}$$

where

$$m = \begin{cases} \frac{1}{2}n & n \text{ even,} \\ \frac{1}{2}(n-1) & n \text{ odd,} \end{cases}$$

and $a \leq \vee \{a_i : i = 1, 2, \dots, n\}$ if and only if $a \leq a_i$ for some i .

Similarly, this process applies to the terms $d(T^{n+1-s}x, T^s x)$ for $s > 0$, so from inequality (7) we have

$$(8) \quad d(T^{n+1}x, T^n x) \leq k^n \vee \left\{ k^s d(T^{n+1-s}x, T^s x) : s = 0, 1, 2, \dots, n-1 \right\}.$$

Using (8) and the inequality

$$(9) \quad d(T^{n+1}x, x) \leq d(T^{n+1}x, T^n x) + d(T^n x, x)$$

we will now show by induction that

$$(10) \quad d(T^{n+1}x, x) \leq \prod_{i=1}^n \frac{1}{1-k^i} d(Tx, x)$$

for $n = 1, 2, \dots$

It is trivial from (8) and (9) that inequality (10) holds for $n = 1$.

Now assume that

$$d(T^{j+1}x, x) \leq \prod_{i=1}^j \frac{1}{1-k^i} d(Tx, x)$$

holds for $j = 1, 2, \dots, n-1$. Taking $s = 0$ in (8) we may suppose first that

$$d(T^{n+1}x, T^n x) \leq k^n d(T^{n+1}x, x).$$

Then from inequality (9), we get

$$d(T^{n+1}x, x) \leq k^n d(T^{n+1}x, x) + d(T^n x, x)$$

and so

$$\begin{aligned} d(T^{n+1}x, x) &\leq \frac{1}{1-k^n} d(T^n x, x) \\ &\leq \prod_{i=1}^n \frac{1}{1-k^i} d(Tx, x) \end{aligned}$$

by our assumption.

Now taking $s = m$, $1 \leq m \leq n-1$ in (8) we suppose that

$$d(T^{n+1}x, T^n x) \leq k^{n+m} d(T^{n+1-m}x, x).$$

Then from inequality (9) we have that

$$\begin{aligned} d(T^{n+1}x, x) &\leq k^{n+m} d(T^{n+1-m}x, x) + d(T^n x, x) \\ &\leq k^{n+m} \prod_{i=1}^{n-m} \frac{1}{1-k^i} d(Tx, x) + \prod_{i=1}^{n-1} \frac{1}{1-k^i} d(Tx, x), \end{aligned}$$

by our assumption. Thus, we get

$$\begin{aligned} d(T^{n+1}x, x) &\leq \prod_{i=1}^n \frac{1}{1-k^i} (k^{n+m} + 1 - k^n) d(Tx, x) \\ &\leq \prod_{i=1}^n \frac{1}{1-k^i} d(Tx, x), \end{aligned}$$

since $k^{n+m} < k^n$. Thus, in either case we obtain inequality (10). It therefore follows by induction that inequality (10) holds for $n = 1, 2, \dots$

Put

$$\prod_{i=1}^{\infty} \frac{1}{1-k^i} = r.$$

Then we have $r < \infty$ and also

$$\prod_{i=1}^n \frac{1}{1-k^i} < r,$$

since

$$\prod_{i=1}^{\infty} \frac{1}{1-k^i} = e^{\sum_{i=1}^{\infty} \ln\left(\frac{1}{1-k^i}\right)}$$

and

$$\sum_{i=1}^{\infty} \ln\left(\frac{1}{1-k^i}\right) < \infty.$$

Using inequalities (8) and (10) we have

$$d(T^{n+1}x, T^n x) \leq k^n r d(Tx, x)$$

for all $n = 1, 2, \dots$, since $k < 1$. Since $\sum_{n=1}^{\infty} k^n < \infty$, It follows from Lemma 3.4 that $\{T^n x\}$ is a Cauchy sequence in the complete cone metric space (X, d) and so has a limit z in X .

Now let $c \in E$ with $0 \ll c$. Choose a natural number N such that $d(T^n x, z) \ll \frac{c(1-k)}{2}$ for all $n \geq N$. From condition (d3) we have

$$(11) \quad d(Tz, z) \leq d(Tz, T^n x) + d(T^n x, z)$$

for all $n \geq N$, and from condition (6),

$$d(Tz, T^n x) \leq k \left[d(Tz, T^{n-1} x) \vee d(T^n x, z) \right].$$

Now we have to consider, for each $n \geq N$, the following two cases:

Case 1. If

$$d(Tz, T^n x) \leq kd(Tz, T^{n-1} x),$$

then from (11), we have

$$\begin{aligned} d(Tz, z) &\leq d(T^n x, z) + kd(Tz, T^{n-1} x) \\ &\leq d(T^n x, z) + k[d(Tz, z) + d(z, T^{n-1} x)] \end{aligned}$$

and so

$$\begin{aligned} d(Tz, z) &\leq \frac{1}{1-k} d(T^n x, z) + \frac{k}{1-k} d(z, T^{n-1} x) \\ &\leq \frac{1}{1-k} [d(T^n x, z) + d(z, T^{n-1} x)] \\ &\ll \frac{c}{2} + \frac{c}{2} = c. \end{aligned}$$

Case 2. If

$$d(Tz, T^n x) \leq kd(T^n x, z),$$

then from (11), we have

$$d(Tz, z) \leq d(T^n x, z) + kd(T^n x, z) \ll \frac{c}{2} + \frac{c}{2} = c.$$

It now follows from cases 1 and 2 that $d(Tz, z) \ll c$. For fixed $0 \ll c$, we have $0 \ll \frac{c}{m}$ for all $m \in \mathbb{N}$. Thus $d(Tz, z) \ll \frac{c}{m}$, for all $m \in \mathbb{N}$. So $\frac{c}{m} - d(Tz, z) \in P$, for all $m \in \mathbb{N}$. Since $\frac{c}{m} \rightarrow 0$, as $m \rightarrow \infty$, and P is closed, it follows that $-d(Tz, z) \in P$. Also we have $d(Tz, z) \in P$. Hence $d(Tz, z) = 0$ and so $Tz = z$.

Now suppose that T has a second fixed point z' . Then from inequality (6) we have

$$d(z, z') = d(Tz, Tz') \leq k \left[d(Tz, z') \vee d(Tz', z) \right],$$

which is equivalent to $d(z, z') \leq k[d(z, z') \vee d(z', z)]$. However, $d(z, z') = d(z', z)$ are comparable so

$$d(z, z') \leq k[d(z, z') \vee d(z', z)] = kd(z, z').$$

It follows from Lemma 2.3 that $d(z, z') = 0$, since $k < 1$, so that the fixed point is unique. \square

5. Some examples

The following examples show that if the elements of the sets $\{d(Tx, x) \mid x \in X\}$ and $\{d(Tx, y) \mid x, y \in X\}$ are comparable then Theorem 4.3 and Theorem 4.4 are more general than Theorem 4.1 and Theorem 4.2, respectively.

5.1. Example. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subset \mathbb{R}^2$, $X = [0, 1]$ and the mapping $d : X \times X \rightarrow E$ defined by $d(x, y) = (|x - y|, |x - y|)$. Then (X, d) is a complete cone metric space.

Define $T : X \rightarrow X$ by $Tx = x/3$. Then we have

$$\begin{aligned} d(Tx, Ty) &= \left(\left| \frac{x}{3} - \frac{y}{3} \right|, \left| \frac{x}{3} - \frac{y}{3} \right| \right), \\ d(Tx, x) &= \left(\left| \frac{x}{3} - x \right|, \left| \frac{x}{3} - x \right| \right) = \left(\frac{2x}{3}, \frac{2x}{3} \right), \\ d(Ty, y) &= \left(\left| \frac{y}{3} - y \right|, \left| \frac{y}{3} - y \right| \right) = \left(\frac{2y}{3}, \frac{2y}{3} \right). \end{aligned}$$

Since $x, y \in \mathbb{R}$, either $d(Tx, x) - d(Ty, y) \in P$ or $d(Ty, y) - d(Tx, x) \in P$. Hence $d(Tx, x)$ and $d(Ty, y)$ are comparable and we have

$$d(Tx, x) \vee d(Ty, y) = \left(\frac{2x}{3}, \frac{2x}{3} \right) \vee \left(\frac{2y}{3}, \frac{2y}{3} \right)$$

and

$$d(Tx, x) + d(Ty, y) = \left(\frac{2}{3}(x + y), \frac{2}{3}(x + y) \right).$$

Clearly, the inequality (4) is satisfied for $\frac{1}{2} \leq k < 1$. However, the inequality (2) is not satisfied for $0 \leq k < \frac{1}{2}$.

5.2. Example. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subset \mathbb{R}^2$, $X = [0, 1]$ and the mapping $d : X \times X \rightarrow E$ defined by $d(x, y) = (|x - y|, |x - y|)$. Define $T : X \rightarrow X$ by

$$Tx = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{3}{7} & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

Then we have,

$d(Tx, Ty) = (0, 0)$, $d(Tx, y) = (y, y)$ and $d(Ty, x) = (x, x)$ for $0 \leq x, y \leq \frac{1}{2}$ and $d(Tx, Ty) = (0, 0)$, $d(Tx, y) = (\frac{3}{7}, y)$ and $d(Ty, x) = (\frac{3}{7}, x)$ for $\frac{1}{2} < x, y \leq 1$.

Thus, $d(Tx, y)$ and $d(Ty, x)$ are comparable and also inequality (6) is satisfied for $0 \leq x, y \leq \frac{1}{2}$ or $\frac{1}{2} < x, y \leq 1$.

Let $0 \leq x \leq \frac{1}{2}$ and $\frac{1}{2} < y \leq 1$. Then,

$$\begin{aligned} d(Tx, Ty) &= d\left(0, \frac{3}{7}\right) = \left(\frac{3}{7}, \frac{3}{7}\right), \\ d(Tx, y) &= d(0, y) = (y, y), \\ d(Ty, x) &= d\left(\frac{3}{7}, x\right) = \left(\left|\frac{3}{7} - x\right|, \left|\frac{3}{7} - x\right|\right). \end{aligned}$$

If $d(Tx, y) = (y, y) \leq \left(\left| \frac{3}{7} - x \right|, \left| \frac{3}{7} - x \right| \right) = d(Ty, x)$, then $\left| \frac{3}{7} - x \right| - y \geq 0$ and so $\left| \frac{3}{7} - x \right| > y > \frac{1}{2}$. Since $0 \leq x \leq \frac{1}{2}$, we have $\left| \frac{3}{7} - x \right| \leq \frac{3}{7}$, a contradiction. But $d(Tx, y) \geq d(Ty, x)$. Thus, we have

$$d(Tx, Ty) = \left(\frac{3}{7}, \frac{3}{7} \right) \leq k(y, y) = kd(Tx, y) = k[d(Tx, y) \vee d(Ty, x)],$$

for all $0 \leq x \leq \frac{1}{2}$ and $\frac{1}{2} < y \leq 1$ where $\frac{6}{7} \leq k < 1$.

Similarly, it is easy to see that the inequality (6) is satisfied for all $\frac{1}{2} < x \leq 1$ and $0 \leq y \leq \frac{1}{2}$ where $\frac{6}{7} \leq k < 1$.

Thus, $d(Tx, y)$ and $d(Ty, x)$ are comparable and also inequality (6) is satisfied, for all $x, y \in [0, 1]$.

Now let $x = \frac{3}{7}, y = \frac{6}{7}$. Then the inequality (3) is not satisfied for $0 \leq k < \frac{1}{2}$. In fact, if the inequality (3) holds for $x = \frac{3}{7}, y = \frac{6}{7}$ where $0 \leq k < \frac{1}{2}$, then we have

$$d(Tx, Ty) = d\left(0, \frac{3}{7}\right) = \left(\frac{3}{7}, \frac{3}{7}\right),$$

$$d(Tx, y) = d\left(0, \frac{6}{7}\right) = \left(\frac{6}{7}, \frac{6}{7}\right),$$

$$d(Ty, x) = d\left(\frac{3}{7}, \frac{3}{7}\right) = (0, 0)$$

and

$$\left(\frac{3}{7}, \frac{3}{7}\right) \leq k\left(\frac{6}{7}, \frac{6}{7}\right),$$

and so $k \geq \frac{1}{2}$. This is a contradiction because of $0 \leq k < \frac{1}{2}$.

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