Hacettepe Journal of Mathematics and Statistics
Volume 37 (2) (2008), 89 – 95

# ON THE NORMS OF TOEPLITZ MATRICES INVOLVING FIBONACCI AND LUCAS NUMBERS

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Received 06 : 06 : 2008 : Accepted 15 : 08 : 2008

**Abstract**

Let us define $A = [a_{ij}]$ and $B = [b_{ij}]$ as $n \times n$ Toeplitz matrices such that $a_{ij} \equiv F_{i-j}$ and $b_{ij} \equiv L_{i-j}$ where $F$ and $L$ denote the usual Fibonacci and Lucas numbers, respectively. We have found upper and lower bounds for the spectral norms of these matrices.

**Keywords:** Fibonacci numbers, Lucas numbers, Toeplitz matrix, Spectral norm, Euclidean norm.

*2000 AMS Classification:* 11 B 39, 15 A 60, 15 A 36.

## 1. Introduction

Let $\{t_n\}_{n=-\infty}^{\infty}$ be a doubly infinite sequence. A Toeplitz matrix is an $n \times n$ matrix

$$T_n = [t_{ij}]_{i,\ j=0}^{n-1}$$

where $t_{ij} = t_{i-j}$, i.e., a matrix of the form

$$T_n = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-(n-1)} \\ t_1 & t_0 & t_{-1} & \cdots & t_{-(n-2)} \\ t_2 & t_1 & t_0 & \cdots & t_{-(n-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_0 \end{bmatrix}.$$

The Fibonacci and Lucas sequences $F_n$ and $L_n$ are defined by the recurrence relations

(1) $\quad F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$

and

(2) $\quad L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

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*Material based on part of the PhD thesis of the first author

If we start from $n=0$, then the Fibonacci and Lucas sequences are given by

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|
| $F_n$ | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | $\cdots$ |
| $L_n$ | 2 | 1 | 3 | 4 | 7 | 11 | 18 | 29 | $\cdots$ . |

The rules (1) and (2) can be used to extend the sequences backwards, thus

$$F_{-1}=F_1-F_0,\quad F_{-2}=F_0-F_{-1},$$

$$L_{-1}=L_1-L_0,\quad L_{-2}=L_0-L_{-1}$$

and so on [7]. These produce

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|
| $F_{-n}$ | 0 | 1 | $-1$ | 2 | $-3$ | 5 | $-8$ | 13 | $\cdots$ |
| $L_{-n}$ | 2 | $-1$ | 3 | $-4$ | 7 | $-11$ | 18 | $-29$ | $\cdots$ . |

Generally,

$$F_{-n}=(-1)^{n+1}F_n$$

and

$$L_{-n}=(-1)^n L_n.$$

In [6], Solak has defined $n\times n$ circulant matrices with Fibonacci and Lucas numbers of the forms

$$A=\left[F_{(\bmod(j-i,n))}\right]_{i,j=1}^{n}$$

and

$$B=\left[L_{(\bmod(j-i,n))}\right]_{i,j=1}^{n}.$$

He has given lower and upper bounds for the spectral norms of these matrices.

In [2], Kayabaş has defined Toeplitz and Hankel matrices given by

$$T_n^k=\left(g_{r-s}^k\right)_{r,s=1}^{n} \tag{3}$$

and

$$H_n^k=\left(g_{r+s}^k\right)_{r,s=1}^{n},$$

where $g_i^k$ is the $i$th element of the $k$-Fibonacci sequence [1]. When $k=2$, the usual Fibonacci sequence is obtained. Moreover she has found upper bounds for the Euclidean norms of $T_n^2$ and $H_n^2$ as follows:

$$\left\|T_n^2\right\|_E\le\sqrt{n\left(F_nF_{n+1}-1\right)}$$

and

$$\left\|H_n^2\right\|_E\le\sqrt{n\left(F_{2n}F_{2n+1}-F_{n+1}F_{n+2}\right)}.$$

In this study, we define Toeplitz matrices involving Lucas numbers of the form

$$B=\left[L_{i-j}\right]_{i,j=1}^{n}. \tag{4}$$

We have found the Euclidean norms, and the upper and lower bounds for the spectral norms of the matrices (3) and (4).

## 2. Preliminaries

Let $A = (a_{ij})$ be an $m \times n$ matrix. The $\ell_p$ norm of the matrix $A$ is defined by

$$\|A\|_p = \left( \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^p \right)^{1/p} \quad (1 \leq p < \infty).$$

If $p = \infty$, then

$$\|A\|_\infty = \lim_{p \to \infty} \|A\|_p = \max_{i,j} |a_{ij}| .$$

The well-known Frobenius (Euclidean) norm of the matrix $A$ is

$$\|A\|_E = \left( \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

and also the spectral norm of the matrix $A$ is

$$\|A\|_2 = \sqrt{\max_{1 \leq i \leq n} |\lambda_i|}$$

where the numbers $\lambda_i$ are the eigenvalues of the matrix $A^H A$ and the matrix $A^H$ is the conjugate transpose of the matrix $A$.

The following inequality holds [8]:

$$\frac{1}{\sqrt{n}} \|A\|_E \leq \|A\|_2 \leq \|A\|_E . \tag{5}$$

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ matrices. Then, the Hadamard product of $A$ and $B$ is the entry-wise product given by [5]

$$A \circ B = (a_{ij} b_{ij}).$$

Define the maximum column length norm $c_1(\cdot)$ and maximum row length norm $r_1(\cdot)$ on $m \times n$ matrices $A = (a_{ij})$ by

$$c_1(A) \equiv \max_j \sqrt{\sum_i |a_{ij}|^2} = \max_j \left\| [a_{ij}]_{i=1}^m \right\|_E$$

and

$$r_1(A) \equiv \max_i \sqrt{\sum_j |a_{ij}|^2} = \max_i \left\| [a_{ij}]_{j=1}^n \right\|_E ,$$

respectively. Let $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ be $m \times n$ matrices. If $C = A \circ B$ then [4]

$$\|C\|_2 \leq r_1(A) c_1(B). \tag{6}$$

The following sum formulae for the Fibonacci and Lucas numbers are well known [3, 7]:

$$\sum_{k=1}^{n-1} F_k^2 = F_n F_{n-1}, \tag{7}$$

$$\sum_{k=1}^{n-1} L_k^2 = L_n L_{n-1} - 2, \tag{8}$$

$$\sum_{k=1}^{n} F_k F_{k-1} = \begin{cases} F_n^2 & n \text{ even}, \\ F_n^2 - 1 & n \text{ odd}, \end{cases} \tag{9}$$

and

$$\sum_{k=1}^{n} L_k L_{k-1} = \begin{cases} L_n^2 - 4 & n \text{ even}, \\ L_n^2 + 1 & n \text{ odd}. \end{cases} \tag{10}$$

## 3. Main results

**3.1. Theorem.** *Let the $n \times n$ matrix $A = [a_{ij}]$ satisfy $a_{ij} \equiv F_{i-j}$ (as in (3) for $k = 2$). Then,*

$$\|A\|_2 \geq \begin{cases} \sqrt{\frac{2}{n} F_n^2} & n \text{ even}, \\ \sqrt{\frac{2}{n}\left(F_n^2 - 1\right)} & n \text{ odd}, \end{cases}$$

*and*

$$\|A\|_2 \leq \sqrt{\left(1 + F_n F_{n-1}\right)\left(F_n F_{n-1}\right)},$$

*where $\|\cdot\|_2$ is the spectral norm, and $F_n$ denotes the $n$th Fibonacci number.*

*Proof.* The matrix $A$ is of the form

$$A = \begin{bmatrix} F_0 & F_{-1} & F_{-2} & \cdots & F_{2-n} & F_{1-n} \\ F_1 & F_0 & F_{-1} & \cdots & F_{3-n} & F_{2-n} \\ F_2 & F_1 & F_0 & \cdots & F_{4-n} & F_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ F_{n-2} & F_{n-3} & F_{n-4} & \cdots & F_0 & F_{-1} \\ F_{n-1} & F_{n-2} & F_{n-3} & \cdots & F_1 & F_0 \end{bmatrix}.$$

We deduce from (7) that

$$\begin{aligned} \|A\|_E^2 &= nF_0^2 + 2\sum_{i=1}^{n-1}\sum_{k=1}^{i} F_k^2 \\ &= 2\sum_{i=1}^{n-1} F_i F_{i+1} \\ &= 2\sum_{k=1}^{n} F_k F_{k-1}. \end{aligned}$$

We conclude from (9) that

$$\|A\|_E = \begin{cases} \sqrt{2F_n^2} & n \text{ even}, \\ \sqrt{2\left(F_n^2 - 1\right)} & n \text{ odd}. \end{cases}$$

Using inequality (5) we obtain

$$\|A\|_2 \geq \begin{cases} \sqrt{\frac{2}{n} F_n^2} & n \text{ even}, \\ \sqrt{\frac{2}{n}\left(F_n^2 - 1\right)} & n \text{ odd}. \end{cases}$$

On the other hand, let the matrices

$$C = (c_{ij}) = \begin{cases} c_{ij} = 1 & j = 1, \\ c_{ij} = F_{i-j} & j \neq 1, \end{cases}$$

and

$$D = (d_{ij}) = \begin{cases} d_{ij} = 1 & j \neq 1, \\ d_{ij} = F_{i-j} & j = 1, \end{cases}$$

satisfy $A = C \circ D$. Then

$$r_1(C) = \max_i \sqrt{\sum_j |c_{ij}|^2} = \sqrt{1 + \sum_{k=1}^{n-1} F_{-k}^2}$$
$$= \sqrt{1 + \sum_{k=1}^{n-1} F_k^2} = \sqrt{1 + F_n F_{n-1}}$$

and

$$c_1(D) = \max_j \sqrt{\sum_i |d_{ij}|^2} = \sqrt{\sum_{k=0}^{n-1} F_k^2}$$
$$= \sqrt{\sum_{k=1}^{n-1} F_k^2} = \sqrt{F_n F_{n-1}}.$$

From (6), we have

$$\|A\|_2 \leq \sqrt{(1 + F_n F_{n-1})(F_n F_{n-1})}.$$

Thus, the proof is completed. □

**3.2. Theorem.** *Let the $n \times n$ matrix $B = [b_{ij}]$ satisfy $b_{ij} \equiv L_{i-j}$. Then*

$$\|B\|_2 \geq \begin{cases} \sqrt{\frac{2}{n}(L_n^2 - 4)} & n \text{ even,} \\ \sqrt{\frac{2}{n}(L_n^2 + 1)} & n \text{ odd,} \end{cases}$$

*and*

$$\|B\|_2 \leq \sqrt{(L_n L_{n-1} - 1)(L_n L_{n-1} + 2)},$$

*where $\|\cdot\|_2$ is the spectral norm, and $L_n$ denotes the $n$ th Lucas number.*

*Proof.* The matrix $B$ is of the form

$$B = \begin{bmatrix} L_0 & L_{-1} & L_{-2} & \cdots & L_{2-n} & L_{1-n} \\ L_1 & L_0 & L_{-1} & \cdots & L_{3-n} & L_{2-n} \\ L_2 & L_1 & L_0 & \cdots & L_{4-n} & L_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ L_{n-2} & L_{n-3} & L_{n-4} & \cdots & L_0 & L_{-1} \\ L_{n-1} & L_{n-2} & L_{n-3} & \cdots & L_1 & L_0 \end{bmatrix}.$$

We deduce from (8) that

$$\|B\|_E^2 = nL_0^2 + 2\sum_{i=1}^{n-1}\sum_{k=1}^{i} L_k^2$$
$$= 4n + 2\sum_{i=1}^{n-1}(L_i L_{i+1} - 2)$$
$$= 2\sum_{k=1}^{n} L_k L_{k-1}.$$

We conclude from (10) that

$$\|B\|_E = \begin{cases} \sqrt{2(L_n^2 - 4)} & n \text{ even,} \\ \sqrt{2(L_n^2 + 1)} & n \text{ odd.} \end{cases}$$

Using inequality (5) we obtain

$$\|B\|_2 \geq \begin{cases} \sqrt{\frac{2}{n}\left(L_n^2-4\right)} & n \text{ even}, \\ \sqrt{\frac{2}{n}\left(L_n^2+1\right)} & n \text{ odd}. \end{cases}$$

On the other hand, let the matrices

$$E=(e_{ij})=\begin{cases} e_{ij}=1 & j=1, \\ e_{ij}=L_{i-j} & j\neq 1, \end{cases}$$

and

$$F=(f_{ij})=\begin{cases} f_{ij}=1 & j\neq 1, \\ f_{ij}=L_{i-j} & j=1, \end{cases}$$

satisfy $B=E\circ F$. Then

$$r_1(E)=\max_i\sqrt{\sum_j\left|e_{ij}\right|^2}=\sqrt{1+\sum_{k=1}^{n-1}L_{-k}^2}$$
$$=\sqrt{1+\sum_{k=1}^{n-1}L_k^2}=\sqrt{L_nL_{n-1}-1}$$

and

$$c_1(F)=\max_j\sqrt{\sum_i\left|f_{ij}\right|^2}=\sqrt{\sum_{k=0}^{n-1}L_k^2}$$
$$=\sqrt{4+\sum_{k=1}^{n-1}L_k^2}=\sqrt{L_nL_{n-1}+2}.$$

From (6), we have

$$\|B\|_2\leq\sqrt{\left(L_nL_{n-1}-1\right)\left(L_nL_{n-1}+2\right)}.$$

Thus, the proof is completed. □

**Acknowledgment** The authors thank the referees for their helpful suggestions concerning the presentation of this paper.

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