Hacettepe Journal of Mathematics and Statistics
Volume 37 (2) (2008), 97 – 106

COMMON FIXED POINT THEOREMS IN FUZZY METRIC SPACES UNDER IMPLICIT RELATIONS

K. P. R. Rao*, G. Ravi Babu* and B. Fisher†

Received 12 : 06 : 2008 : Accepted 21 : 10 : 2008

Abstract

In this paper we introduce the notion of a pair (f, g) being weakly f-compatible and obtain a common fixed point theorem for self maps in fuzzy metric spaces which modifies and generalizes some known results. We also give a common fixed point theorem for self maps in sequentially compact fuzzy metric spaces.

Keywords: Weakly f-compatible pair (f, g), Sequentially compact, Common fixed points.

2000 AMS Classification: 47 H 10, 54 H 25.

1. Introduction and Preliminaries

The concept of a fuzzy set was introduced by Zadeh [18]. In the last two decades there has been a tremendous development and growth in fuzzy mathematics. George and Veeramani [7] modified the concept of fuzzy metric space which was introduced by Kramosil and Michalek [11]. Grabiec [8] extended the well known fixed point theorems of Banach [1] and Edelstein [4] to fuzzy metric spaces in the sense of [11]. Later many authors, for example, [2, 3, 5, 7, 8, 10, 11, 13, 16, 17] proved fixed and common fixed point theorems in fuzzy metric spaces. In this paper we formulate the definition of the pair (f, g) being weakly f-compatible or weakly g-compatible, and obtain a common fixed point theorem for such pairs of maps under an implicit relation, which generalizes [17, Theorem 3.1], [10, Corollary 1], [2, Theorems 3.1 and 3.5] and [13, Corollary 2]. We also prove a common fixed point theorem for pairs of weakly compatible maps in a sequentially compact fuzzy metric space using an implicit relation.

First of all we give some known definitions and lemmas.

*Department of Applied Mathematics, Acharya Nagarjuna University - Dr. M. R. Appa Row Campus, NUZVID- 521201, Krishna Dt., A. P., India.
E-mail: (K. P. Rao) kprrao2004@yahoo.com (G. R. Babu) ravibabugollapudi@yahoo.co.in
†B.Fisher, Department of Mathematics, University of Leicester, Leicester, LE1 7RH, U.K.
E-mail: fbr@leicester.ac.uk

1.1. Definition. [15] A binary operation $*: [0,1]^2 \to [0,1]$ is called a *continuous t-norm*, if $([0,1], *)$ is an abelian topological monoid with a unit 1 such that $a*b \le c*d$, whenever $a \le c,\ b \le d\ \forall\, a,b,c,d \in [0,1]$.

Two examples of t-norms are $a*b = ab$ and $a*b = \min\{a,b\}$.

1.2. Definition. [7] The 3-tuple $(X, M, *)$ is called a *fuzzy metric space* if X is an arbitrary set, $*$ a continuous t - norm and M a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions:

(1) $M(x,y,t) > 0$,
(2) $M(x,y,t) = 1$ if and only if $x = y$,
(3) $M(x,y,t) = M(y,x,t)$,
(4) $M(x,y,t) * M(y,z,s) \le M(x,z,t+s)$,
(5) $M(x,y,.) : (0,\infty) \to [0,1]$ is continuous, for all $x,y,z \in X$ and $t,s > 0$.

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x,r,t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x,r,t) = \{y \in X : M(x,y,t) > 1 - r\}.$$

Now let $(X, M, *)$ be a fuzzy metric space and τ the set of all $A \subset X$ with $x \in A$ if and only if there exist $t > 0$ and $0 < r < 1$ such that $B(x,r,t) \subset A$. Then τ is a topology on X induced by the fuzzy metric M.

1.3. Definition. [8] A sequence $\{x_n\}$ in a fuzzy metric $(X, M, *)$ is said to be *convergent* to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$. The sequence $\{x_n\}$ is said to be *Cauchy* if $\lim_{n,m\to\infty} M(x_n, x_m, t) = 1$. The space $(X, M, *)$ is said to be *complete* if every Cauchy sequence in X is convergent in X.

1.4. Lemma. [8] *Let $(X, M, *)$ be a fuzzy metric space. Then $M(x,y,t)$ is non-decreasing for all $x,y \in X$.* □

1.5. Lemma. [12] *Let $(X, M, *)$ be a fuzzy metric space. Then M is a continuous function on $X^2 \times (0,\infty)$.* □

Throughout this paper, we now assume that $\lim_{t\to\infty} M(x,y,t) = 1$ and that $\mathbb{N}$ is the set of all natural numbers.

1.6. Lemma. [13] *Let $\{y_n\}$ be a sequence in $(X, M, *)$. If there exists a positive number $k < 1$ such that*

$$M(y_{n+2}, y_{n+1}, kt) \ge M(y_{n+1}, y_n, t),\ \ t > 0,\ \ n \in \mathbb{N},$$

then $\{y_n\}$ is a Cauchy sequence in X. □

1.7. Lemma. [13] *If there exists $k \in (0,1)$ such that $M(x,y,kt) \ge M(x,y,t)$ for all $x,y \in X$ and $t > 0$, then $x = y$.* □

1.8. Definition. [13] Let f and g be self maps on a fuzzy metric space $(X, M, *)$. The pair (f,g) is said to be *compatible* if $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$, for some $z \in X$.

1.9. Definition. [9] Let f and g be self mappings on a fuzzy metric space $(X, M, *)$. Then the mappings are said to be *weakly compatible* if they commute at their coincidence point, that is, $fx = gx$ implies that $fgx = gfx$.

Now we give:

1.10. Definition. [14] The pair (f,g) is said to be *weakly f-compatible* if either $\lim_{n\to\infty} gfx_n = fz$ or $\lim_{n\to\infty} ggx_n = fz$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ and $\lim_{n\to\infty} fgx_n = \lim_{n\to\infty} ffx_n = fz$, for some $z \in X$.

Similarly, we can define weak g-compatibility of the pair (f, g).

Clearly, both Definition 1.8 and 1.10 imply that the pair (f, g) is coincidentally commuting or a weakly compatible pair.

We observe that Definition 1.8 implies Definition 1.10. We also note that a weakly f-compatible pair (f, g) need not be compatible in view of the following example.

1.11. Example. Let $X = [0, 1]$, $a * b = \min\{a, b\}$ and

$$M(x, y, t) = \frac{t}{t + |x - y|}.$$

Define

$$fx = 1 - x, \quad gx = \begin{cases} x & \text{if } 0 \le x \le 1/2, \\ 1 & \text{if } 1/2 < x \le 1. \end{cases}$$

Let $\{x_n\}$ be a sequence in X such that $x_n < 1/2 \ \forall n$ and $\lim_{n\to\infty} x_n = 1/2$. Then

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} 1 - x_n = 1/2, \quad \lim_{n\to\infty} gx_n = \lim_{n\to\infty} x_n = 1/2,$$
$$\lim_{n\to\infty} fgx_n = \lim_{n\to\infty} 1 - x_n = 1/2 = f(1/2),$$
$$\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} x_n = 1/2 = f(1/2)$$

and

$$\lim_{n\to\infty} gfx_n = 1, \quad \lim_{n\to\infty} ggx_n = \lim_{n\to\infty} x_n = 1/2 = f(1/2).$$

Since

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 1/2, \quad \lim_{n\to\infty} fgx_n = f(1/2), \quad \lim_{n\to\infty} ffx_n = f(1/2)$$

implies

$$\lim_{n\to\infty} ggx_n = f(1/2),$$

it follows that (f, g) is weakly f-compatible.

Since

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t) = \lim_{n\to\infty} \frac{t}{t + x_n} = \frac{t}{t + 1/2} \neq 1,$$

the pair (f, g) is not compatible.

1.12. Definition. [14] The pair (f, g) is said to be f-*continuous* if

$$\lim_{n\to\infty} ffx_n = \lim_{n\to\infty} fgx_n = fz,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z,$$

for some $z \in X$.

Recently Seong Hoon Cho [2] fallaciously proved the following theorem:

1.13. Theorem. [2, Theorem 3.1] *Let $(X, M, *)$ be a complete fuzzy metric space with $t * t \ge t$, $\forall t \in [0, 1]$, and let f, g, S and T be self maps on X such that*

(1) $f(X) \subset T(X), \quad g(X) \subset S(X)$,
(2) *S and T are continuous,*
(3) *The pairs (f, S) and (g, T) are compatible,*
(4) *There exists $k \in (0, 1)$ such that for every $x, y \in X$ and $t > 0$,*

$$M(fx, gy, kt) \ge M(Sx, Ty, t) * M(fx, Sx, t) * M(gy, Ty, t) * M(fx, Ty, t),$$

(5) $\lim_{t\to\infty} M(x,y,t) = 1,\ \forall\, x,y \in X$.

Then f, g, S and T have a unique common fixed point in X.

We observe that this theorem is not valid in view of the following example of Fisher [6] in metric spaces, even when $S = T = I$, the identity map.

1.14. Example. Let $X = \{0,1,2,\ldots\}$, $a * b = \min\{a,b\}$ and

$$M(x,y,t) = \frac{t}{t + \mathrm{d}(x,y)},$$

where $\mathrm{d}(n,n) = 0,\ \forall n \in X$ and for $n \neq m$,

$$\mathrm{d}(m,n) = \begin{cases} 1 & \text{if } m+n \text{ is odd,} \\ 2 & \text{if } m+n \text{ is even.} \end{cases}$$

Define $f, g, S, T : X \to X$ by $S = T = I$, the identity map, and

$$f(2n) = f(2n+1) = 2n+2,\ g(2n) = 2n+1,\ g(2n+1) = 2n+3,$$

for $n = 0,1,2,3,\ldots$. Then all the conditions of Theorem 1.13 are satisfied with $k = 1/2$, but neither f nor g has a fixed point in X.

2. Implicit relations

Let Φ_6 denote the set of all continuous functions $\phi : [0,1]^6 \longrightarrow \mathbb{R}$ satisfying the conditions

(ϕ_1): ϕ is decreasing in t_2, t_3, t_4, t_5 and t_6,
(ϕ_2): $\phi(u,v,v,v,v,v) \geq 0$ implies $u \geq v$ for all $u, v \in [0,1]$.

2.1. Example. $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \min\{t_2,t_3,t_4,t_5,t_6\}$.

2.2. Example. $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^2 - \min\{t_i t_j : i,j \in \{2,3,4,5,6\}\}$.

2.3. Example. $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = t_1^3 - \min\{t_i t_j t_k : i,j,k \in \{2,3,4,5,6\}\}$.

3. Main result

3.1. Theorem. *Let f, g, S and T be self maps on a complete fuzzy metric space $(X, M, *)$ with $t * t \geq t\ \forall\, t \in [0,1]$ such that*

(3.1.1) $f(X) \subseteq T(X),\ g(X) \subseteq S(X)$,

(3.1.2) $$\phi\left(\begin{matrix} M(fx,gy,kt),\ M(Sx,Ty,t),\ M(fx,Sx,t), \\ M(gy,Ty,t),\ M(fx,Ty,\alpha t),\ M(gy,Sx,(2-\alpha)t) \end{matrix}\right) \geq 0,$$

for all $x, y \in X$, $\forall\, t > 0$ and $\forall\, \alpha \in (0,2)$, where $k \in (0,1)$ and $\phi \in \Phi_6$.

Further assume that

(3.1.3) *(f,S) is weakly S-compatible, (g,T) is weakly T-compatible and either (f,S) is S-continuous or (g,T) is T-continuous,*

or

(3.1.4) *(f,S) is weakly f-compatible, (g,T) is weakly g-compatible and either (f,S) is f-continuous or (g,T) is g-continuous.*

Then f, g, S and T have a unique common fixed point $z \in X$, and z is the unique common fixed point of f and S and of g and T.

Proof. Let $x_0 \in X$ be an arbitrary point. By (3.1.1), we can choose a sequence $\{x_n\}$ in X such that $y_{2n} = fx_{2n} = Tx_{2n+1},\ y_{2n+1} = gx_{2n+1} = Sx_{2n+2}$ for $n = 0, 1, 2, \ldots$, Let $d_m(t) = M(y_m, y_{m+1}, t),\ \forall\, t > 0$.

Step 1. Putting $x = x_{2n},\ y = x_{2n+1},\ \alpha = 1 - q_1$ in (3.1.2), where $q_1 \in (k, 1)$, we have

$$
\begin{aligned}
0 &\leq \phi\begin{pmatrix} M(y_{2n}, y_{2n+1}, kt),\ M(y_{2n}, y_{2n-1}, t),\ M(y_{2n}, y_{2n-1}, t), \\ M(y_{2n+1}, y_{2n}, t),\ M(y_{2n}, y_{2n}, (1-q_1)t),\ M(y_{2n+1}, y_{2n-1}, (1+q_1)t) \end{pmatrix} \\
&\leq \phi\begin{pmatrix} M(y_{2n}, y_{2n+1}, kt),\ M(y_{2n}, y_{2n-1}, t),\ M(y_{2n}, y_{2n-1}, t), \\ M(y_{2n+1}, y_{2n}, t),\ 1,\ M(y_{2n}, y_{2n-1}, t) * M(y_{2n+1}, y_{2n}, q_1 t) \end{pmatrix}
\end{aligned}
$$

and so

(i) $\quad \phi(d_{2n}(kt),\ d_{2n-1}(t),\ d_{2n-1}(t),\ d_{2n}(t),\ 1,\ d_{2n-1}(t) * d_{2n}(q_1 t)) \geq 0.$

If $d_{2n}(t) < d_{2n-1}(t)$, then

$$d_{2n}(q_1 t) * d_{2n-1}(t) \geq d_{2n}(q_1 t) * d_{2n}(q_1 t) \geq d_{2n}(q_1 t)$$

and from (ϕ_1), we have

$$\phi(d_{2n}(kt),\ d_{2n}(q_1 t),\ d_{2n}(q_1 t),\ d_{2n}(q_1 t),\ d_{2n}(q_1 t),\ d_{2n}(q_1 t)) \geq 0.$$

Then again from (ϕ_2), we have

$$d_{2n}(kt) > d_{2n}(q_1 t),$$

a contradiction. Hence $d_{2n}(t) \geq d_{2n-1}(t)$ for every $n \in \mathbb{N}$ and $\forall t > 0$.

Now from (i) and (ϕ_1) we have

$$\phi(d_{2n}(kt),\ d_{2n-1}(q_1 t),\ d_{2n-1}(q_1 t),\ d_{2n-1}(q_1 t),\ d_{2n-1}(q_1 t),\ d_{2n-1}(q_1 t) \geq 0$$

and from (ϕ_2), we have

(ii) $\quad d_{2n}(kt) > d_{2n-1}(q_1 t).$

Step 2. Similarly, putting $x = x_{2n},\ y = x_{2n-1},\ \alpha = 1 - q_2$ in (3.1.2), where $q_2 \in (k, 1)$, we can show that

(iii) $\quad d_{2n-1}(kt) \geq d_{2n-2}(q_2 t),$

Now let $q = \min\{q_1, q_2\}$ so that $q \in (k, 1)$. Then from (ii) and (iii) we have

$$d_n(kt) \geq d_{n-1}(qt)$$

for every $n \in \mathbb{N}$, and so

$$
\begin{aligned}
M(y_n, y_{n+1}, t) &\geq M(y_{n-1}, y_n, (q/k)t) \\
&\geq M(y_{n-2}, y_{n-1}, (q/k)^2 t) \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\geq M(y_0, y_1, (q/k)^n t).
\end{aligned}
$$

Hence, by Lemma 1.6, $\{y_n\}$ is a Cauchy sequence and from the completeness of X, $\{y_n\}$ converges to some point z in X.

Now suppose that the conditions in (3.1.3) are true.

Step 3. Suppose that (f, S) is S-continuous. Then $Sfx_{2n} \to Sz$ and $SSx_{2n} \to Sz$ as $n \to \infty$. Since (f, S) is weakly S-compatible we have either $fSx_{2n} \to Sz$ or $ffx_{2n} \to Sz$ as $n \to \infty$.

Case 1. Suppose that $fSx_{2n} \to Sz$ as $n \to \infty$. Then putting $x = Sx_{2n},\ y = x_{2n+1},\ \alpha = 1$ in (3.1.2), we get

$$\phi\begin{pmatrix} M(fSx_{2n}, gx_{2n+1}, kt),\ M(SSx_{2n}, Tx_{2n+1}, t),\ M(fSx_{2n}, SSx_{2n}, t), \\ M(gx_{2n+1}, Tx_{2n+1}, t),\ M(fSx_{2n}, Tx_{2n+1}, t),\ M(gx_{2n+1}, SSx_{2n}, t) \end{pmatrix} \geq 0.$$

Letting $n \to \infty$, we have

$$0 \le \phi(M(Sz,z,kt),\ M(Sz,z,t),\ 1,\ 1,\ M(Sz,z,t),\ M(z,Sz,t))$$
$$\le \phi\begin{pmatrix} M(Sz,z,kt),\ M(Sz,z,t),\ M(Sz,z,t), \\ M(Sz,z,t),\ M(Sz,z,t),\ M(z,Sz,t) \end{pmatrix}.$$

From (ϕ_2), we have $M(Sz,z,kt) \ge M(Sz,z,t)$, which implies by Lemma 1.7 that $Sz = z$.

Case 2. Suppose $ffx_{2n} \to Sz$ as $n \to \infty$. Putting $x = fx_{2n},\ y = x_{2n+1},\ \alpha = 1$ in (3.1.2), we get

$$\phi\begin{pmatrix} M(ffx_{2n}, gx_{2n+1}, kt),\ M(Sfx_{2n}, Tx_{2n+1}, t),\ M(ffx_{2n}, Sfx_{2n}, t), \\ M(gx_{2n+1}, Tx_{2n+1}, t),\ M(ffx_{2n}, Tx_{2n+1}, t),\ M(gx_{2n+1}, Sfx_{2n}, t) \end{pmatrix} \ge 0.$$

Letting $n \to \infty$, we have

$$0 \le \phi(M(Sz,z,kt),\ M(Sz,z,t),\ 1,\ 1,\ M(Sz,z,t),\ M(z,Sz,t))$$
$$\le \phi\begin{pmatrix} M(Sz,z,kt),\ M(Sz,z,t),\ M(Sz,z,t), \\ M(Sz,z,t),\ M(Sz,z,t),\ M(z,Sz,t) \end{pmatrix}.$$

From (ϕ_2), we have $M(Sz,z,kt) \ge M(Sz,z,t)$, which implies that $Sz = z$.

Step 4. Putting $x = z,\ y = x_{2n+1},\ \alpha = 1$ in (3.1.2) we have

$$\phi\begin{pmatrix} M(fz, gx_{2n+1}, kt),\ M(Sz, Tx_{2n+1}, t),\ M(fz, Sz, t), \\ M(gx_{2n+1}, Tx_{2n+1}, t),\ M(fz, Tx_{2n+1}, t),\ M(gx_{2n+1}, Sz, t) \end{pmatrix} \ge 0.$$

Letting $n \to \infty$, we have

$$\phi(M(fz,z,kt),\ 1,\ M(fz,z,t),\ 1,\ M(fz,z,t),\ 1) \ge 0.$$

From (ϕ_1) and (ϕ_2), we have $M(fz,z,kt) \ge M(fz,z,t)$, which implies that $fz = z$.

Step 5. Since $f(X) \subseteq T(X)$, there exists $w \in X$ such that $z = fz = Tw$. Putting $x = x_{2n},\ y = w,\ \alpha = 1$ in (3.1.2), we have

$$\phi\begin{pmatrix} M(fx_{2n}, gw, kt),\ M(Sx_{2n}, Tw, t),\ M(fx_{2n}, Sx_{2n}, t), \\ M(gw, Tw, t),\ M(fx_{2n}, Tw, t),\ M(gw, Sx_{2n}, t) \end{pmatrix} \ge 0.$$

Letting $n \to \infty$, we have

$$\phi(M(z,gw,kt),\ 1,\ 1,\ M(gw,z,t),\ 1,\ M(gw,z,t)) \ge 0.$$

From (ϕ_1) and (ϕ_2), we have $M(z,gw,kt) \ge M(z,gw,t)$, which implies that $gw = z$. Thus $Tw = gw$.

Since (g,T) is weakly T-compatible it follows that (g,T) is a weakly compatible pair. Hence $Tgw = gTw$, so that $Tz = gz$.

Step 6. Putting $x = x_{2n},\ y = z,\ \alpha = 1$ in (3.1.2) we have

$$\phi\begin{pmatrix} M(fx_{2n}, gz, kt),\ M(Sx_{2n}, Tz, t),\ M(fx_{2n}, Sx_{2n}, t), \\ M(gz, Tz, t),\ M(fx_{2n}, Tz, t),\ M(gz, Sx_{2n}, t) \end{pmatrix} \ge 0.$$

Letting $n \to \infty$,we have

$$\phi(M(z,Tz,kt),\ M(z,Tz,t),\ 1,\ 1,\ M(z,Tz,t),\ M(Tz,z,t)) \ge 0.$$

From (ϕ_1) and (ϕ_2), we have $M(z,Tz,kt) \ge M(Tz,z,t)$, which implies that $Tz = z$. Hence $gz = Tz = z$ and so z is a common fixed point of f, g, S and T.

Step 7. Suppose that z_0 is another common fixed point of f, g, S and T. Putting $x = z,\ y = z_0,\ \alpha = 1$ in (3.1.2), we have

$$\phi(M(z,z_0,kt),\ M(z,z_0,t),\ 1,\ 1,\ M(z,z_0,t),\ M(z_0,z,t)) \ge 0.$$

From (ϕ_1) and (ϕ_2), we have $M(z, z_0, kt) \geq M(z, z_0, t)$, which implies that $z = z_0$. Hence z is the unique common fixed point of f, g, S and T.

Step 8. Suppose that z_1 is another common fixed point of f and S. Putting $x = z_1,\ y = z,\ \alpha = 1$ in (3.1.2), we have

$$\phi(M(z_1, z, kt),\ M(z_1, z, t),\ 1,\ 1,\ M(z_1, z, t),\ M(z, z_1, t)) \geq 0.$$

From (ϕ_1) and (ϕ_2), we have $M(z_1, z, kt) \geq M(z, z_1, t)$, which implies that $z_1 = z$. Hence z is the unique common fixed point of f and S.

Similarly we can show that z is the unique common fixed point of g and T.

Similarly we can prove the theorem if (g, T) is T-continuous.

Also we can prove the theorem if the conditions in (3.1.4) are true. □

3.2. Example. Let $X = [0, 1]$, $a * b = \min\{a, b\}$ and

$$M(x, y, t) = \frac{t}{t + |x - y|}.$$

Define $fx = gx = 1$ and

$$Sx = Tx = \begin{cases} \frac{1+x}{2} & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1, \end{cases}$$

for all $x \in X$. Then all the conditions of Theorem 3.1 are satisfied with

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min\{t_2, t_3, t_4, t_5, t_6\}.$$

Clearly 1 is the unique common fixed point of f, g, S and T.

Now we give another implicit relation which is useful for the next theorem.

4. An implicit relation.

Let Ψ_6 be the set of all functions $\psi : [0, 1]^6 \longrightarrow \mathbb{R}$ such that

(ψ_1): $\psi(v, u, u, v, w, 1) > 0$ or $\psi(v, u, v, u, 1, w) > 0$ implies $u < v$ for all $u, v \in [0, 1)$ and $w \leq 1$,

(ψ_2): $\psi(v, 1, 1, v, v, 1) \leq 0$, $\psi(v, v, 1, 1, v, v) \leq 0$ and $\psi(v, 1, v, 1, 1, v) \leq 0$ for all $v \in [0, 1)$.

4.1. Example. $\psi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min\{t_2, t_3, t_4\} - b(t_5 + t_6)$, where $b \geq 0$.

4.2. Example. $\psi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - \min\{t_2^2, t_3 t_4\} - b t_5 t_6$, where $b \geq 0$.

4.3. Example. $\psi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^3 - t_2 t_3 t_4 - b(t_5^2 t_6 + t_5 t_6^2)$, where $b \geq 0$.

4.4. Definition. $(X, M, *)$ is said to be a *sequentially compact fuzzy metric space* if every sequence in X has a convergent sub-sequence.

4.5. Theorem. *Let f, g, S and T be self-mappings of a sequentially compact fuzzy metric space $(X, M, *)$ such that*

(1) $S(X) \subseteq g(X)$ *and* $T(X) \subseteq f(X)$,

(2) $$\psi\begin{pmatrix} M(Sx, Ty, t),\ M(fx, gy, t),\ M(fx, Sx, t), \\ M(gy, Ty, t),\ M(fx, Ty, t),\ M(Sx, gy, t) \end{pmatrix} > 0$$
for every $x, y \in X$ with one of $fx \neq gy$, $fx \neq Sx$ and $gy \neq Ty$ and for all $t > 0$, where $\psi \in \Psi_6$,

(3) *The pairs (f, S) and (g, T) are weakly compatible,*

(4) *Either f and S are continuous or g and T are continuous.*

Then f, g, S and T have a unique common fixed point p in X . Further p is the unique common fixed point of f and S and of g and T.

Proof. Suppose that f and S are continuous and for any $t > 0$, let

$$m = \sup\{M(fx, Sx, t) : x \in X\}.$$

Since f and S are continuous on a sequentially compact fuzzy metric space, there exists $u \in X$ such that $m = M(fu, Su, t)$.

Since $S(X) \subseteq g(X)$, there exists $v \in X$ such that

(5) $Su = gv$.

Since $T(X) \subseteq f(X)$, there exists $w \in X$ such that

(6) $Tv = fw$.

Suppose neither f and S nor g and T have a coincidence point in X. Then

$$m = M(fu, Su, t) < 1,\ M(gv, Tv, t) < 1 \text{ and } M(fw, Sw, t) < 1.$$

We have

$$\begin{aligned} 0 &< \psi\begin{pmatrix} M(Su, Tv, t),\ M(fu, gv, t),\ M(fu, Su, t), \\ M(gv, Tv, t),\ M(fu, Tv, t),\ M(Su, gv, t) \end{pmatrix} \\ &= \psi(M(Tv, gv, t),\ m,\ m,\ M(gv, Tv, t),\ M(fu, Tv, t),\ 1), \end{aligned}$$

and by (ψ_1), we have

(7) $m < M(gv, Tv, t)$.

Now from (2), we have

$$\begin{aligned} 0 &< \psi\begin{pmatrix} M(Sw, Tv, t),\ M(fw, gv, t),\ M(fw, Sw, t), \\ M(gv, Tv, t),\ M(fw, Tv, t),\ M(Sw, gv, t) \end{pmatrix} \\ &= \psi\begin{pmatrix} M(fw, Sw, t),\ M(gv, Tv, t),\ M(fw, Sw, t), \\ M(gv, Tv, t),\ 1,\ M(Sw, gv, t) \end{pmatrix}. \end{aligned}$$

By (ψ_1), we have

(8) $M(gv, Tv, t) < M(fw, Sw, t)$.

Now from the definition of m and the inequalities (7) and (8) we have

$$m \geq M(fw, Sw, t) > M(gv, Tv, t) > m,$$

a contradiction. Hence there exists $\alpha \in X$ such that $f\alpha = S\alpha$ or $g\alpha = T\alpha$.

Case (a): Suppose that $f\alpha = S\alpha$. Since $S(X) \subseteq g(X)$, there exists $\alpha \in X$ such that $S\alpha = g\beta$. Suppose that $M(g\beta, T\beta, t) < 1$. Then from (2) we have

$$\begin{aligned} 0 &< \psi\begin{pmatrix} M(S\alpha, T\beta, t),\ M(f\alpha, g\beta, t),\ M(f\alpha, S\alpha, t), \\ M(g\beta, T\beta, t),\ M(f\alpha, T\beta, t),\ M(S\alpha, g\beta, t) \end{pmatrix} \\ &= \psi(M(g\beta, T\beta, t),\ 1,\ 1,\ M(g\beta, T\beta, t),\ M(g\beta, T\beta, t),\ 1). \end{aligned}$$

By (ψ_2), we have $M(g\beta, T\beta, t) = 1$, so that $g\beta = T\beta$. Thus

(9) $f\alpha = S\alpha = g\beta = T\beta = p$, say.

Since the pair (f, S) is weakly compatible we have

(10) $fp = fS\alpha = Sf\alpha = Sp$.

Suppose that $M(Sp, p, t) < 1$. From (2), we have

$$0 < \psi\begin{pmatrix} M(Sp, T\beta, t),\ M(fp, g\beta, t),\ M(fp, Sp, t), \\ M(g\beta, T\beta, t),\ M(fp, T\beta, t),\ M(Sp, g\beta, t) \end{pmatrix}$$
$$= \psi(M(Sp, p, t),\ M(Sp, p, t),\ 1,\ 1,\ M(Sp, p, t),\ M(Sp, p, t)).$$

Hence from (ψ_2), we have $Sp = p$. Thus

(11) $fp = Sp = p$.

Since the pair (g, T) is weakly compatible we have

$$gp = gT\beta = Tg\beta = Tp.$$

Using (2) with $x = \alpha,\ y = p$ and (ψ_2) we can show that $Tp = p$. Thus,

(12) $gp = Tp = p$.

Hence p is a common fixed point of f, g, S and T.

Case (b): Suppose that $g\alpha = T\alpha$. Since $T(X) \subseteq f(X)$, there exists $\beta \in X$ such that $T\alpha = f\beta$.

Suppose that $M(f\beta, S\beta, t) < 1$. From (2), we have

$$0 < \psi\begin{pmatrix} M(S\beta, T\alpha, t),\ M(f\beta, g\alpha, t),\ M(f\beta, S\beta, t), \\ M(g\alpha, T\alpha, t),\ M(f\beta, T\alpha, t),\ M(S\beta, g\alpha, t) \end{pmatrix}$$
$$= \psi(M(S\beta, f\beta, t),\ 1,\ M(f\beta, S\beta, t),\ 1,\ 1,\ M(S\beta, f\beta, t)).$$

Hence from (ψ_2), we have $f\beta = S\beta$. Thus $S\beta = f\beta = T\alpha = g\alpha = p$, say. Now as in case(a), we can show that p is a common fixed point of f, g, S and T.

Suppose that p_0 is another common fixed point of f, g, S and T. Using (2) with $x = p,\ y = p_0$ and (ψ_2), we can show that $p_0 = p$. Thus p is the unique common fixed point of f, g, S and T.

Now suppose that p_1 is another common fixed point of f and S. Using (2) with $x = p_1,\ y = p$ and (ψ_2) we can show that $p_1 = p$. Thus p is the unique common fixed point of f and S.

Similarly we can show that p is the unique common fixed point of g and T.

Similarly the theorem holds when g and T are continuous. □

4.6. Remark. Theorem 4.5 holds if the inequality (2) is replaced by one of the following inequalities:

(a) $M(Sx, Ty, t) > \min\{M(fx, gy, t),\ M(fx, Sx, t),\ M(gy, Ty, t)\}$,
(b) $M^2(Sx, Ty, t) > \min\{M^2(fx, gy, t),\ M(fx, Sx, t)M(gy, Ty, t)\}$,
(c) $M^3(Sx, Ty, t) > M(fx, gy, t)M(fx, Sx, t)M(gy, Ty, t)$.

4.7. Example. Let $X = [0, 1]$, $a * b = \min\{a, b\}$ and

$$M(x, y, t) = \frac{t}{t + |x - y|}.$$

Define $Sx = Tx = 1,\ fx = \frac{x+1}{2}$ and $gx = \frac{2+x}{3}$ for all $x \in X$. Then all the conditions of Theorem 4.5 are satisfied with

$$\psi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min\{t_2, t_3, t_4\}.$$

Clearly 1 is the unique common fixed point of S, T, f and g.

References

[1] Banach, S. *Theoric les operations Lineairar* (Manograic Mathematic Zne, Warsaw, 1932).

[2] Cho, Seong Hoon *On common fixed points in fuzzy metric spaces,* International Mathematical Forum **1** (10), 471–479, 2006.

[3] Cho, Y. J., Pathak, H, K., Kang, S. M. and Jung, J. S. *Common fixed points of compatible maps of type* (β) *on fuzzy metric spaces,* Fuzzy Sets and Systems, **93**, 99–111, 1998.

[4] Edelstein, M. *On fixed and periodic points under contradiction mapping,* J. London Math. Soc. **37**, 74–79, 1962.

[5] Fang, J.X. *On fixed point theorems in fuzzy metric spaces,* Fuzzy Sets and Systems, **46**, 107–113, 1992.

[6] Fisher, B. *Results on common fixed points on bounded metric spaces,* Math. Sem. Notes **7**, 73–80, 1979.

[7] George, A. and Veeramani, P. *On some results in fuzzy metric spaces,* Fuzzy Sets and Systems **64**, 395–399, 1994.

[8] Grabiec, M. *Fixed points in fuzzy metric spaces,* Fuzzy Sets and Systems **27**, 385–389, 1988.

[9] Jungck, G. and Rhoades, B. E. *Fixed points for set-valued functions without continuity,* Indian J. Pure. Appl. Math., **29** (3), 227–238, 1998.

[10] Khan, M. S., Pathak, H. K. and George, R. *Compatible mapping of type (A-1) and type (A-2) and common fixed points in fuzzy metric spaces,* International Mathematical Forum **2** (11), 515–524, 2007.

[11] Kramosil, O. and Michalek, J. *Fuzzy metric and statistical metric spaces,* Kybernetica **11**, 326–334, 1975.

[12] Lopez, J. Rodrigues and Romaguera, S. *The Hausdorff fuzzy metric on compact sets,* Fuzzy Sets and Systems **147** (2), 273–283, 2004.

[13] Mishra, S. N., Sharma, S. N. and Singh, S. L. *Common fixed points of maps in fuzzy metric spaces,* Internat. J. Math. Sci. **17**, 253–258, 1994.

[14] Naidu, S. V. R. *Fixed point theorems for four self maps on a metric space by altering distance,* J. Indian Math. Soc., **71** (1-4), 153–173, 2004.

[15] Schweizer, B. and Sklar, A. *Statistical metric spaces,* Pacific J. Math. **10**, 314–334, 1960.

[16] Sharma, S. *Common fixed point theorems in fuzzy metric spaces,* Fuzzy Sets and Systems, **127**, 345–352, 2002.

[17] Singh, Bijenra and Chauhan, M. S. *Common fixed points of compatible maps in fuzzy metric spaces,* Fuzzy Sets and Systems *115*, 471–475, 2000.

[18] Zadeh, L. A. *Fuzzy sets,* Inform. and Control, **8**, 338–353, 1965.