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ON CLASSES OF MULTIVALENT FUNCTIONS INVOLVING LINEAR OPERATOR AND MULTIPLIER TRANSFORMATIONS

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Abstract

Using the results of first order differential subordinations and superordinations, we define and discuss new classes of p-valent functions involving the Dziok-Srivastava operator and multiplier transformation.

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1. Preliminaries

Let $\mathcal{H}$ be the class of functions analytic in $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(a, n)$ the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$.

Let $\mathcal{V}_p$ denote the class of all analytic functions of the form

$$(1.1) \qquad f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \ (z \in \Delta)$$

and let $\mathcal{V} := \mathcal{V}_1$. For two functions $f(z)$ given by (1.1) and $g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k$, the Hadamard product (or convolution) of f and g is defined by

$$(1.2) \qquad (f * g)(z) := z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k =: (g * f)(z).$$

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For $\alpha_j \in \mathbb{C}$ $(j = 1, 2, \dots, l)$ and $\beta_j \in \mathbb{C}\setminus\{0, -, 1, -2, \dots\}$ $(j = 1, 2, \dots m)$, the generalized hypergeometric function ${}_lF_m(\alpha_1, \dots \alpha_l;\ \beta_1, \dots, \beta_m; z)$ is defined by the infinite series

$$ {}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{z^n}{n!} $$
$$ (l \le m + 1; m \in \mathbb{N}_0 := \{0, 1, 2, \dots\}), $$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$ (\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1)(\lambda + 2)\cdots(\lambda + n - 1) & (n \in N := \{1, 2, 3, \dots\}). \end{cases} $$

Corresponding to the function

$$ h_p(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) := z^p\ {}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z), \tag{1.3} $$

the Dziok-Srivastava operator [8] (see also [21]) $H_p^{(l,m)}(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m)$ is defined by the Hadamard product

$$ \begin{aligned} &H_p^{(l,m)}(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m) f(z) \\ &\qquad = h_p(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) * f(z) \\ &\qquad = z^p + \sum_{n=p+1}^{\infty} \frac{(\alpha_1)_{n-p} \cdots (\alpha_l) n - p}{(\beta_1)_{n-p} \cdots (\beta_n)_{n-p}} \frac{a_n z^n}{(n - p)!}\ . \end{aligned} \tag{1.4} $$

Special cases of the Dziok-Srivastava linear operator include the Hohlov linear operator [9], the Carlson-Shaffer linear operator [5], the Ruscheweyh derivative operator [19], the generalized Bernardi-Libera-Livingston operator (cf. [2], [11],[12]) and the Srivastava-Owa fractional derivative operators (cf. [17], [18]).

Corresponding to the function $h_p(\alpha_1, \dots, \alpha_\ell; \beta_1, \dots, \beta_m; z)$, defined by (1.3), we introduce a function $F_\mu(\alpha_1, \dots, \alpha_\ell, \beta_1, \dots, \beta_m; z)$ given by

$$ h_p(\alpha_1, \dots, \alpha_\ell; \beta_1, \dots, \beta_m; z) * F_\mu(\alpha_1, \dots, \alpha_\ell, \beta_1, \dots, \beta_m; z) = \frac{z^p}{(1 - z)^{\mu + p - 1}}, $$
$$ (z \in \mathbb{U},\ \mu > 0). $$

Analogous to $H_p(\alpha_1, \dots, \alpha_\ell,\ \beta_1, \dots, \beta_m)$, the linear operator $J_\mu(\alpha_1, \dots, \alpha_\ell, \beta_1, \dots, \beta_m)$ on $\mathcal{H}$ is defined as follows:

$$ J_\mu(\alpha_1, \dots, \alpha_\ell, \beta_1, \dots, \beta_m) f(z) = F_\mu(\alpha_1, \dots, \alpha_\ell, \beta_1, \dots, \beta_m; z) * f(z) $$
$$ (\alpha_1; \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-; i = 1, \dots, l; j = 1, \dots, m, \mu > 0; z \in \mathbb{U}; f \in \mathcal{V}_p). $$

For convenience, we write

$$ J_\mu^{\ell,m}(\alpha_1) := J_\mu(\alpha_1, \dots, \alpha_\ell; \beta_1, \dots, \beta_m). \tag{1.5} $$

Special cases of this operator are when $p = 1$ [10], the generalized integral operator in [1] when $p = 1$ and $\mu = 2$, and Noor's integral operator [15].

For two analytic functions f and F, we say that F is superordinate to f if f is subordinate to F. Recently Miller and Mocanu [14] considered certain second order differential superordinates. Using the results of Miller and Mocanu [14], Bulboa has considered certain classes of first order differential superordinations [4] and superordination-preserving integral operators [3].

In the present investigation, we introduce new classes of p-valent functions defined by the Dziok-Srivastava linear operator and the multiplier transformation, and study their properties by using certain first order differential subordinations and superordinations.

1.1. Definition. A function $f \in \mathcal{V}(p,n)$ is said to be in the class

$$\mathcal{V}(p,n,\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;\varphi)$$

if it satisfies the following subordination:

$$\frac{H_p^{\ell,m}[\alpha_1+1]f(z)}{z^p} \prec \varphi(z), \ (f \in \mathcal{V}(p,n)), \tag{1.6}$$

and is said to be in

$$\overline{\mathcal{V}}(p,n,\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;\varphi)$$

if f satisfies the following superordination:

$$\varphi(z) \prec \frac{H_p^{l,m}[\alpha_1+1]f(z)}{z^p}, \ (f \in \mathcal{V}(p,n)), \tag{1.7}$$

where $\varphi(z)$ is analytic in Δ, $\varphi(0)=1$ and

$$H_p^{l,m}[\alpha_1]f(z) := H_p^{l,m}(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m)f(z).$$

To make the notation simple, we also write

$$\mathcal{V}(p,n,\alpha_1;\varphi) := \mathcal{V}(p,n,\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;\varphi)$$

and

$$\overline{\mathcal{V}}(p,n,\alpha_1;\varphi) = \overline{\mathcal{V}}(p,n,\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;\varphi).$$

Also we define the class $\mathcal{V}(p,n,\alpha_1;\varphi_1,\varphi_2)$ by the following:

$$\mathcal{V}(p,n,\alpha_1;\varphi_1,\varphi_2) := \overline{\mathcal{V}}(p,n,\alpha_1;\varphi_1) \cap \mathcal{V}(p,n,\alpha_1;\varphi_2).$$

1.2. Definition. A function $f \in \mathcal{V}(p,n)$ is said to be in the class

$$A(p,n,\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;\varphi)$$

if it satisfies the following subordination:

$$\frac{J_\mu^{l,m}[\alpha_1+1]f(z)}{z^p} \prec \varphi(z), \ (f \in \mathcal{V}(p,n)), \tag{1.8}$$

and is said to be in

$$\overline{A}(p,n,\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;\varphi)$$

if f satisfies the following superordination:

$$\varphi(z) \prec \frac{J_\mu^{l,m}[\alpha_1+1]f(z)}{z^p}, \ (f \in \mathcal{V}(p,n)), \tag{1.9}$$

where $\varphi(z)$ is analytic in Δ, $\varphi(0)=1$ and

$$J_\mu^{l,m}[\alpha_1]f(z) = J_\mu(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m)f(z).$$

To make the notation simple, we also write

$$A(p,n,\alpha_1;\varphi) := A(p,n,\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;\varphi)$$

and

$$\overline{A}(p,n,\alpha_1;\varphi) := \overline{A}(p,n,\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;\varphi).$$

Also we define the class $A(p,n,\alpha_1;\varphi_1,\varphi_2)$ by the following:

$$A(p,n,\alpha_1;\varphi_1,\varphi_2) := \overline{A}(p,n,\alpha_1;\varphi_1) \cap A(p,n,\alpha_1;\varphi_2).$$

Motivated by the multiplier transformation on $\mathcal{V}$, we define the operator $I_p(r,\lambda)$ on $\mathcal{V}_p$ by the following infinite series

$$(1.10)\quad I_p(r,\lambda)f(z)=z^p+\sum_{n=p+1}^{\infty}\left(\frac{n+\lambda}{p+\lambda}\right)^r a_n z^n,\ (\lambda\geq 0).$$

The operator $I_p(r,\lambda)$ is closely related to the Salagean derivative operator [20]. The operator $I_\lambda^r := I_1(r,\lambda)$ was studied recently by Cho and Srivastva [6], and by Cho and Kim [7] The operator $I_r := I_1(r,1)$ was studied by Uralagaddi and Somanatha [22]. By using the Hadamard product:

$$(1.11)\quad I_p(r,\lambda)f(z):=\mathcal{F}_\lambda^r(z)*f(z) \text{ where } \mathcal{F}_\lambda^r(z)=z^p+\sum_{n=p+1}^{\infty}\left(\frac{n+\lambda}{p+\lambda}\right)^r z^n,\ (\lambda\geq 0).$$

Corresponding to the function $\mathcal{F}_\lambda^r(z)$ defined by (1.5), we introduce a function $\mathcal{F}_{\lambda,\mu}^r(z)$ given by

$$(1.12)\quad \mathcal{F}_\lambda^r(z)*\mathcal{F}_{\lambda,\mu}^r(z)=\frac{z^p}{(1-z)^{\mu+p-1}},\ (z\in\mathbb{U},\ \mu>0).$$

Using $I_p(r,\lambda)$, we define the multiplier transformations $T_\mu(r,\lambda)$ as follows:

$$(1.13)\quad T_\mu(r,\lambda)f(z)=\mathcal{F}_{\lambda,\mu}^r(z)*f(z),\ (\lambda\geq 0,\ \mu>0,\ z\in\mathbb{U},\ f\in\mathcal{V}_p).$$

For $p=1$, we note that a special case of this operator is the integral operator defined in [16].

1.3. Definition. A function $f\in\mathcal{V}(p,n)$ is said to be the class $\mathcal{V}(p,n,r,\lambda;\varphi)$ if it satisfies the following subordination:

$$(1.14)\quad \frac{I_p(r+1,\lambda)f(z)}{z^p}\prec\varphi(z),\ (f\in\mathcal{V}(p,n)),$$

and is said to be in $\overline{\mathcal{V}}(p,n,r,\lambda;\varphi)$ if f satisfies the following superordination:

$$(1.15)\quad \varphi(z)\prec\frac{I_p(r+1,\lambda)f(z)}{z^p},\ (f\in\mathcal{V}(p,n)),$$

where $\varphi(z)$ is analytic in Δ and $\varphi(0)=1$. Also we define the class $\mathcal{V}(p,n,r,\lambda;\varphi_1,\varphi_2)$ by the following:

$$\mathcal{V}(p,n,r,\lambda,\varphi_1,\varphi_2):=\overline{\mathcal{V}}(p,n,r,\lambda;\varphi_1)\cap\mathcal{V}(p,n,r,\lambda;\varphi_2).$$

1.4. Definition. A function $f\in\mathcal{V}(p,n)$ is said to be in the class $\mathcal{A}(p,n,r,\lambda;\varphi)$ if it satisfies the following subordination:

$$(1.16)\quad \frac{T_\mu(r+1,\lambda)f(z)}{z^p}\prec\varphi(z),\ (f\in\mathcal{V}(p,n)),$$

and is said to be in $\overline{A}(p,n,r,\lambda;\varphi)$ if f satisfies the following superordination:

$$(1.17)\quad \varphi(z)\prec\frac{T_\mu(r+1,\lambda)f(z)}{z^p},\ (f\in\mathcal{V}(p,n)),$$

where $\varphi(z)$ is analytic in Δ and $\varphi(0)=1$. Also we define the class $A(p,n,r,\lambda;\varphi_1,\varphi_2)$ by the following:

$$A(p,n,r,\lambda;\varphi_1,\varphi_2):=\overline{A}(p,n,r,\lambda;\varphi_1)\cap A(p,n,r,\lambda;\varphi_2).$$

In our present investigation of the above defined classes, we need the following:

1.5. Definition. [14, Definition 2, p. 817]. Denote by $\mathcal{Q}$, the set of all functions $f(z)$ that are analytic and injective on $\overline{\Delta} - E(f)$, where

$$E(f) = \{\xi \in \partial\Delta : \lim_{z \Longrightarrow \xi} f(z) = \infty\},$$

and are such that $f'(\xi) \neq 0$ for $\xi \in \partial\Delta - E(f)$.

1.6. Lemma. (cf. Miller and Mocanu [13, Theorem 3.4h, p. 132]). *Let $\psi(z)$ be univalent in the unit disc Δ and θ, φ analytic in a domain $D \supset \psi(\Delta)$ with $\varphi(w) \neq 0$, when $w \in \psi(\Delta)$. Set*

$$Q(z) := z\psi'(z)\varphi(\psi(z)),\ h(z) := \theta(\psi(z)) + Q(z).$$

Suppose that

1) *$Q(z)$ is starlike in Δ, and*
2) *$Re\dfrac{zh'(z)}{Q(z)} > 0$ for $z \in \Delta$.*

If $q(z)$ is analytic in Δ, with $q(0) = \psi(0), q(\Delta) \subset D$ and

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(\psi(z)) + z\psi'(z)\varphi(\psi(z)), \tag{1.18}$$

then $q(z) \prec \psi(z)$ and $\psi(z)$ is the best dominant.

1.7. Lemma. [4]. *Let $\psi(z)$ be univalent in the unit disc Δ and θ analytic in a domain D containing $\psi(\Delta)$. Suppose that*

(1) *$Re[\theta'(\psi(z))/\varphi(\psi(z))] > 0$ for $z \in \Delta$.*
(2) *$z\psi'(z)\varphi(\psi(z))$ is starlike in Δ.*

If $q(z) \in \mathcal{H}(\psi(0), 1) \cap \mathcal{Q}$, with $q(\Delta) \subseteq D$, and $\theta(q(z)) + zq'(z)\varphi(q(z))$ is univalent in Δ, then

$$\theta(\psi(z)) + z\psi'(z)\varphi(\psi(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)). \tag{1.19}$$

implies $\psi(z) \prec q(z)$ and $\psi(z)$ is the best subordinant.

2. Results involving the Dziok-Srivastava linear operator

2.1. Theorem. *Let $\psi(z)$ be univalent in $\Delta, \psi(0) = 1$. Assume that ψ is convex in Δ, α_1 be a complex number and $Re\{\alpha_1\} > -1$. Let $\chi(z)$ be defined by*

$$\chi(z) = \psi(z) + \frac{z\psi'(z)}{(\alpha_1 + 1)},\ (\alpha_1 \neq -1). \tag{2.1}$$

If $f \in \mathcal{V}(p, n, \alpha_1 + 1; \chi)$, then $f \in \mathcal{V}(p, n, \alpha_1; \psi)$. If $f \in \overline{\mathcal{V}}(p, n, \alpha_1 + 1; \chi)$,

$$0 \neq \frac{H_p^{l,m}[\alpha_1 + 1]f(z)}{z^p} \in \mathcal{H}(1, 1) \cap \mathcal{Q},\ \frac{H_p^{l,m}[\alpha_1 + 2]f(z)}{z^p} \text{ is univalent in } \Delta, \tag{2.2}$$

then $f \in \overline{\mathcal{V}}(p, n, \alpha_1; \psi)$.

Proof. Define the function $q(z)$ by

$$q(z) = \frac{H_p^{l,m}[\alpha_1 + 1]f(z)}{z^p}. \tag{2.3}$$

Then, clearly, $q(z)$ is analytic in Δ, we find from (2.3) that

$$\frac{zq'(z)}{q(z)} = \frac{z(H_p^{l,m}[\alpha_1 + 1]f(z))'}{H_p^{l,m}[\alpha_1 + 1]f(z)} - p. \tag{2.4}$$

By making use of the identity

$$z(H_p^{l,m}[\alpha_1]f(z))' = \alpha_1 H_p^{l,m}[\alpha_1 + 1]f(z) - (\alpha_1 - p)H_p^{l,m}[\alpha_1]f(z), \tag{2.5}$$

we have from (2.4) that

$$\frac{H_p^{l,m}[\alpha_1+2]f(z)}{z^p} = \frac{1}{\alpha_1+1}[(\alpha_1+1)q(z)+zq'(z)]. \tag{2.6}$$

Since $f \in \mathcal{V}(p,n,\alpha_1+1;\chi)$, we have from (2.6) that

$$(\alpha_1+1)q(z)+zq'(z) \prec (\alpha_1+1)\psi(z)+z\psi'(z)$$

and this can be written as (1.18), by defining

$$\theta(w) = (\alpha_1+1)w \quad \text{and} \quad \varphi(w) = 1.$$

Note that $\varphi(w) \neq 0$ and $\theta(w)$, $\varphi(w)$ are analytic in $\mathbb{C}$. Set

$$Q(z) = z\psi'(z), \tag{2.7}$$

$$h(z) = \theta(\psi(z)) + Q(z) = (\alpha_1+1)\psi(z)+z\psi'(z). \tag{2.8}$$

By the hypothesis of Theorem 2.1, Q is starlike and

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{(\alpha_1+2)+\frac{z\psi''(z)}{\psi(z)}\right\} > 0.$$

By an application of Lemma 1.6, we obtain that $q(z) \prec \psi(z)$ or $\frac{H_p^{l,m}[\alpha_1+1]f(z)}{z^p} \prec \psi(z)$, which shows that $f \in \mathcal{V}(p,n,\alpha_1;\psi)$.

The other half of Theorem 2.1 follows by a similar application of Lemma 1.7. □

Using Theorem 2.1, we obtain the following "sandwich result":

2.2. Corollary. *Let $\psi_i(z)$ be univalent in Δ and $\psi_i(0)=1$, $(i=1,2)$. Further assume that $\psi_i(z)$ is convex univalent in Δ, α_1 is a complex number and $Re\{\alpha_1\} > -1$, $(i=1,2)$. If $f \in \mathcal{V}(p,n,\alpha_1+1;\chi_1,\chi_2)$ satisfies (2.2), then $f \in \mathcal{V}(p,n,\alpha_1;\psi_1,\psi_2)$, where*

$$\chi_i(z) = \psi_i(z) + \frac{z\psi_i'(z)}{(\alpha_1+1)}, \ (i=1,2,;\alpha_1 \neq -1).$$ □

2.3. Theorem. *Let ψ be univalent in $\Delta, \psi(0)=1$, and λ a complex number. Assume that ψ is convex in Δ and $Re\{\lambda\} > -p$. Define the functions F and χ by*

$$\begin{aligned} F(z) &= \frac{\lambda+p}{z^\lambda}\int_0^z t^{\lambda-1}f(t)dt \\ \chi(z) &= \left[\frac{z\psi'(z)}{(\lambda+p)} + \psi(z)\right], \ (\lambda \neq -p). \end{aligned} \tag{2.9}$$

If $f \in \mathcal{V}(p,n,\alpha_1;\chi)$, then $F \in \mathcal{V}(p,n,\alpha_1;\psi)$. If $f \in \overline{\mathcal{V}}(p,n,\alpha_1;\chi)$,

$$0 \neq \frac{H_p^{l,m}[\alpha_1+1]F(z)}{z^p} \in \mathcal{H}(1,1)\cap Q \ \text{and} \ \frac{H_p^{l,m}[\alpha_1+1]f(z)}{z^p} \ \text{is univalent in } \Delta, \tag{2.10}$$

then $F \in \overline{\mathcal{V}}(p,n,\alpha_1;\psi)$.

Proof. From the definition of $F(z)$, we obtain that

$$(\lambda+p)H_p^{l,m}[\alpha_1+1]f(z) = \lambda H_p^{l,m}[\alpha_1+1]F(z) + z(H_p^{l,m}[\alpha_1+1]F(z))'. \tag{2.11}$$

Define the function $q(z)$ by

$$q(z) = \frac{H_p^{l,m}[\alpha_1+1]F(z)}{z^p}. \tag{2.12}$$

Then, clearly, $q(z)$ is analytic in Δ. Using (2.11) and (2.12), we have

$$\frac{H_p^{l,m}[\alpha_1+1]f(z)}{z^p} = \frac{\lambda q(z) + z\left(\frac{H_p^{l,m}[\alpha+1]F'(z)}{z^p}\right)}{\lambda+p}. \tag{2.13}$$

Upon logarithmic differentiation of (2.13), we get

$$\frac{H_p^{l,m}[\alpha_1+1]f(z)}{z^p} = \frac{zq'(z)}{\lambda+p} + q(z). \tag{2.14}$$

Since $f \in \mathcal{V}(p,n,\alpha_1,\chi)$, we have from (2.14),

$$q(z) + \frac{zq'(z)}{\lambda+p} \prec \psi(z) + \frac{z\psi'(z)}{\lambda+p},$$

and this can be written as (1.18) by defining $\theta(w) = w$ and $\varphi(w) = \frac{1}{\lambda+p}$.

Note that $\varphi(w) \neq 0$ and $\theta(w), \varphi(w)$ are analytic in $\mathbb{C}$. Set

$$Q(z) := \frac{z\psi'(z)}{\lambda+p}$$

$$h(z) := \theta(\psi(z)) + Q(z) = \psi(z) + \frac{z\psi'(z)}{\lambda+p}.$$

By the hypothesis of Theorem 2.3, $Q(z)$ is starlike and

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{(\lambda+p) + \left(1 + \frac{z\psi''(z)}{\psi'(z)}\right)\right\} > 0.$$

By an application of Lemma 1.6, we obtain that

$$q(z) \prec \psi(z),$$

or

$$\frac{H_p^{l,m}[\alpha_1+1]F(z)}{z^p} \prec \psi(z),$$

which shows that $F \in \mathcal{V}(p,n,\alpha_1;\psi)$.

The second half of Theorem 2.3 follows by a similar application of Lemma 1.7. □

Using Theorem 2.3, we have the following result:

2.4. Corollary. *Let ψ_i be univalent in Δ, $\psi_i(0) = 1$, $(i = 1,2)$ and λ a complex number. Assume that $\psi_i(z)$ is convex in Δ and $Re\{\lambda\} > -p$, $(i = 1,2)$. If $f \in \mathcal{V}(p,n,\alpha_1;\chi_1,\chi_2)$ satisfies (2.10), then the function F defined by (2.9) belongs to $\mathcal{V}(p,n,\alpha_1;\psi_1,\psi_2)$, where $\chi_i(z) = \left[\frac{z\psi_i'(z)}{(\lambda+p)} + \psi_i(z)\right]$, $(i = 1,2, \lambda \neq -p)$.*

Now we will give some particular cases of Theorem 2.1 obtained for different choices of $\psi(z)$.

2.5. Example. Let $p = 1, l = m+1$ and $\alpha_2 = \beta_1, \ldots, \alpha_l = \beta_m$, we get $H_1[1]f(z) = f(z)$ and $H_1[2]f(z) = zf'(z)$. Let $\chi(z) = 1 + \lambda z$, $(0 \leq \lambda \leq 1)$. Then:

$$f'(z) \prec 1 + \lambda z \implies \frac{f(z)}{z} \prec 1 + \frac{\lambda}{2}z$$

and

$$1 + \lambda z \prec f'(z) \implies 1 + \frac{\lambda}{2}z \prec \frac{f(z)}{z}.$$

2.6. Theorem. *Let $\psi(z)$ be univalent in Δ and $\psi(0) = 1$. Assume that ψ is convex in Δ, α_1 is a complex number and $Re\{\alpha_1\} > -1$. Let $\chi(z)$ be defined by*

$$\chi(z) := \psi(z) + \frac{z\psi'(z)}{(\alpha_1+1)}, \ (\alpha_1 \neq -1). \tag{2.15}$$

If $f \in A(p,n,\alpha_1;\chi)$, then $f \in A(p,n,\alpha+1;\psi)$. If $f \in \overline{A}(p,n,\alpha_1;\chi)$,

$$0 \neq \frac{J_\mu^{l,m}[\alpha_1+2]f(z)}{z^p} \in \mathcal{H}(1,1)\cap \mathcal{Q},\ \frac{J_\mu^{l,m}[\alpha_1+1]f(z)}{z^p} \text{ is univalent in } \Delta, \tag{2.16}$$

then $f \in \overline{A}(p,n,\alpha_1+1;\psi)$.

Proof. Define the function $q(z)$ by

$$q(z) := \frac{J_\mu^{l,m}[\alpha_1+2]f(z)}{z^p}. \tag{2.17}$$

Then, clearly $q(z)$ is analytic in Δ. We find from (2.9) that

$$\frac{zq'(z)}{q(z)} = \frac{z(J_\mu^{l,m}[\alpha_1+2]f(z))'}{J_\mu^{l,m}[\alpha+2]f(z)} - p. \tag{2.18}$$

By making use of the identity

$$z(J_\mu^{l,m}(\alpha_1+1)f(z))' = \alpha_1(J_\mu^{l,m}(\alpha_1)f(z) - (\alpha_1-p)J_\mu^{l,m}(\alpha_1+1)f(z). \tag{2.19}$$

We complete the proof using the same steps as in proof of Theorem 2.1. □

Using Theorem 2.6, we obtain the following "Sandwich result".

2.7. Corollary. *Let $\psi_i(z)$ be univalent in Δ and $\psi_i(0)=1,\ (i=1,2)$. Further assume that ψ_i is convex in Δ and $Re\{\alpha_1\} > -1,\ (i=1,2)$. If $f \in A(p,n,\alpha_1;\chi_1,\chi_2)$ satisfies (2.16), then $f \in A(p,n,\alpha_1+1;\psi_1,\psi_2)$, where $\chi_i(z) = \psi_i(z) + z\dfrac{\psi_i'(z)}{(\alpha_1+1)},\ (i=1,2,\alpha_1 \neq -1)$.* □

The proof of the next theorem is the same as the proof of Theorem 2.3.

2.8. Theorem. *Let ψ be univalent in Δ, $\psi(0)=1$ and λ a complex number. Assume that ψ is convex in Δ and $Re\{\lambda\} > -p$. Define the functions F and χ by*

$$\begin{aligned} F(z) &:= \frac{\lambda+p}{z^\lambda}\int_0^z t^{\lambda-1}f(t)dt \\ \chi(z) &:= \left[\psi(z) + \frac{z\psi'(z)}{\lambda+p}\right],\ (\lambda \neq -p). \end{aligned} \tag{2.20}$$

If $f \in A(p,n,\alpha_1;\chi)$, then $F \in A(p,n,\alpha_1;\psi)$. If $f \in \overline{A}(p,n,\alpha_1,\chi)$,

$$0 \neq \frac{J_\mu^{l,m}[\alpha_1+1]F(z)}{z^p} \in \mathcal{H}(1,1)\cap \mathcal{Q} \text{ and } \frac{J_\mu^{l,m}[\alpha_1+1]f(z)}{z^p} \text{ is univalent in } \Delta, \tag{2.21}$$

then $F \in \overline{A}(p,n,\alpha_1;\psi)$. □

Using Theorem 2.8, we have the following result:

2.9. Corollary. *Let ψ_i be univalent in $\Delta, \psi_i(0)=1,\ (i=1,2)$ and λ a complex number. Assume that ψ_i is convex in Δ and $Re\{\lambda\} > -p$. If $f \in A(p,n,\alpha_1;\chi_1,\chi_2)$ satisfies (2.21), then F defined by (2.20) belongs to $A(p,n,\alpha_1;\psi_1,\psi_2)$, where*

$$\chi_i(z) = \psi_i(z) + \frac{z\psi_i'(z)}{\lambda+p},\ (i=1,2,\lambda \neq -p).$$

3. Results Involving Multiplier Transformation

3.1. Theorem. *Let $\psi(z)$ be univalent in Δ, $\psi(0) = 1$, ψ convex in Δ, λ a complex number and $Re\{\lambda\} > -p$. Let $\chi(z)$ be defined by*

$$\chi(z) = \psi(z) + \frac{z\psi'(z)}{(p+\lambda)}, \ (\lambda \neq -p).$$

If $f \in \mathcal{V}(p, n, r+1, \lambda; \chi)$, then $f \in \mathcal{V}(p, n, r, \lambda; \psi)$. If $f \in \overline{\mathcal{V}}(p, n, r+1, \lambda; \chi)$,

$$0 \neq \frac{I_p(r+1,\lambda)f(z)}{z^p} \in \mathcal{H}(1,1) \cap Q, \ \frac{I_p(r+2,\lambda)f(z)}{z^p} \text{ is univalent in } \Delta, \tag{3.1}$$

then $f \in \overline{\mathcal{V}}(p, r, \lambda; \psi)$.

Proof. Define the function $q(z)$ by

$$q(z) = \frac{I_p(r+1,\lambda)f(z)}{z^p}. \tag{3.2}$$

Then, clearly, $q(z)$ is analytic in Δ. Also by a simple computation, we find from (3.2) that

$$\frac{zq'(z)}{q(z)} = \frac{z(I_p(r+1,\lambda)f(z))'}{I_p(r+1,\lambda)f(z)} - p. \tag{3.3}$$

By making use of the identity

$$z(I_p(r,\lambda)f(z))' = (p+\lambda)I_p(r+1,\lambda)f(z) - \lambda I_p(r,\lambda)f(z) \tag{3.4}$$

we have from (3.3) that

$$\frac{I_p(r+2,\lambda)f(z)}{z^p} = q(z) + \frac{zq'(z)}{(p+\lambda)}. \tag{3.5}$$

Since $f \in \mathcal{V}(p, n, r+1, \lambda; \chi)$, and in view of (3.5), we have

$$q(z) + \frac{zq'(z)}{(p+\lambda)} \prec \psi(z) + \frac{z\psi'(z)}{(p+\lambda)}.$$

The first result follows by an application of Lemma 1.6.

Similarly, the second result follows from Lemma 1.7. □

Using Theorem 3.1, we obtain the following "sandwich result":

3.2. Corollary. *Let $\psi_i(z)$ be univalent in Δ, $\psi_i(0) = 1$, $\psi_i(z)$ convex in Δ, λ a complex number and $Re\{\lambda\} > -p$ for $i = 1, 2$. Define*

$$\chi_i(x) = \left[\psi_i(z) + \frac{z\psi_i'(z)}{p+\lambda}\right], \ (i = 1, 2, \lambda \neq -p).$$

If $f \in \mathcal{V}(p, n, r+1, \lambda; \chi_1, \chi_2)$ satisfies (3.1), then $f \in \mathcal{V}(p, n, r, \lambda; \psi_1, \psi_2)$. □

3.3. Theorem. *Let ψ be univalent in Δ, $\psi(0) = 1$, δ a complex number, ψ convex in Δ and $Re\{\delta\} > -p$. Define the functions $F(z)$ and $\chi(z)$ by*

$$\begin{aligned} F(z) &= \frac{\delta+p}{z^p}\int_0^z t^{\delta-1}f(t)dt \\ \chi(z) &= \psi(z) + \frac{z\psi'(z)}{p+\delta}, \qquad (\delta \neq -p). \end{aligned} \tag{3.6}$$

If $f \in \mathcal{V}(p, n, r, \lambda; \chi)$, then $F \in \mathcal{V}(p, n, r, \lambda; \psi)$. If $f \in \overline{\mathcal{V}}(p, n, r, \lambda; \chi)$,

$$0 \neq \frac{I_p(r+1,\lambda)F(z)}{z^p} \in \mathcal{H}(1,1) \cap \mathcal{Q}, \ \frac{I_p(r+1,\lambda)f(z)}{z^p} \text{ is univalent in } \Delta, \tag{3.7}$$

then $F \in \overline{\mathcal{V}}(p, n, r, \lambda; \psi)$.

Proof. Similar to that of Theorem 2.3. □

Using Theorem 2.3, we have the following result:

3.4. Corollary. *Let ψ_i be univalent in Δ, $\psi_i(0) = 1$ and δ a complex number. Assume that $\psi_i(z)$ is convex in Δ and $Re\{\delta\} > -p$. Define the functions χ_i by*

$$\chi_i(z) = \psi_i(z) + \frac{z\psi_i'(z)}{p+\delta}, \ (i = 1, 2, \ \delta \neq -p).$$

If $f \in \mathcal{V}(p, n, r, \lambda; \chi_1, \chi_2)$ satisfies (3.7), then F defined by (3.6) belongs to the class $\mathcal{V}(p, n, r, \lambda; \psi_1, \psi_2)$. □

3.5. Theorem. *Let $f(z) \in \mathcal{V}(p, n)$. Then $f \in \mathcal{V}(p, n, r, \lambda; \varphi)$ if and only if*

$$F(z) = \frac{p+\lambda}{z^\lambda} \int_0^z t^{\lambda-1} f(t)dt \in \mathcal{V}(p, n, r + 1, \lambda; \varphi). \tag{3.8}$$

Also $f \in \overline{\mathcal{V}}(p, n, r, \lambda; \varphi)$ if and only if $F \in \overline{\mathcal{V}}(p, n, r + 1, \lambda; \varphi)$.

Proof. From (3.8), we have

$$(p + \lambda)f(z) = \lambda F(z) + zF'(z). \tag{3.9}$$

By convoluting (3.9) with $\phi_p(n, \lambda; z) = z^p + \sum_{n=p+1}^{\infty} \left(\frac{n+\lambda}{p+\lambda}\right)^r z^n$ and using the fact that $z(f * g)'(z) = f(z) * zg'(z)$, we obtain

$$(p + \lambda)I_p(r, \lambda)f(z) = \lambda I_p(r, \lambda)F(z) + z(I_p(r, \lambda)F(z))',$$

and by using (3.4) we get

$$I_p(r, \lambda)f(z) = I_p(r + 1, \lambda)F(z) \tag{3.10}$$

and

$$\begin{aligned} (p + \lambda)I_p(r + 1, \lambda)f(z) &= z(I_p(r, \lambda)f(z))' + \lambda I_p(r, \lambda)f(z) \\ &= z(I_p(r + 1, \lambda)F(z))' + \lambda I_p(r + 1, \lambda)F(z) \\ &= (p + \lambda)I_p(r + 2, \lambda)F(z). \end{aligned} \tag{3.11}$$

Therefore, from (3.11), we have

$$\frac{I_p(r + 2, \lambda)F(z)}{z^p} = \frac{I_p(r + 1, \lambda)f(z)}{z^p},$$

and the desired result follows at once. □

Using Theorem 3.5, we have

3.6. Corollary. *Let $f(z) \in \mathcal{V}(p, n)$. Then $f \in \mathcal{V}(p, n, r, \lambda; \varphi_1, \varphi_2)$ if and only if F given by (3.8) is in $\mathcal{V}(p, n, r + 1, \lambda; \varphi_1, \varphi_2)$.* □

Now we will give some particular cases of Theorem 3.1 obtained for different choices of $\psi(z)$.

3.7. Example. Let $p = 1$ and $r = \lambda = 0$. We get $I_1(1, 0)f(z) = zf'(z)$ and $I_1(2, 0)f(z) = z(f'(z) + zf''(z))$. Let $\chi(z) = (1 + z)e^z$. Then

$$f'(z) + zf''(z) \prec (1 + z)e^z \implies f'(z) \prec e^z$$

or

$$(1 + z)e^z \prec f'(z) + zf''(z) \implies e^z \prec f'(z).$$

3.8. Theorem. *Let $\psi(z)$ be univalent in Δ, $\psi(0)=1$, ψ convex, λ a complex number and $Re(\lambda)>-p$. Let $\chi(z)$ be defined by*

$$\chi(z)=\psi(z)+\frac{z\psi'(z)}{(p+\lambda)},\ (\lambda\neq -p).$$

If $f\in A(p,n,r,\lambda;\chi)$, then $f\in A(p,n,r+1,\lambda;\psi)$. If $f\in \overline{A}(p,n,r,\lambda;\chi)$,

$$0\neq \frac{T_\mu(r+2,\lambda)f(z)}{z^p}\in \mathcal{H}(1,1)\cap \mathcal{Q},\ \frac{T_\mu(r+1,\lambda)f(z)}{z^p} \text{ is univalent in } \Delta, \tag{3.12}$$

then $f\in \overline{A}(p,n,r+1,\lambda;\psi)$.

Proof. Define the function $q(z)$ by

$$q(z)=\frac{T_\mu(r+2,\lambda)f(z)}{z^p}. \tag{3.13}$$

Then, clearly, $q(z)$ is analytic in Δ. Also by a simple computation, we find by (3.13) that

$$\frac{zq'(z)}{q(z)}=\frac{z(T_\mu(r+2,\lambda)f(z))'}{T_\mu(r+2,\lambda)f(z)}-p.$$

By making use of the identity

$$z(T_\mu(r+1,\lambda)f)'=(p+\lambda)T_\mu(r,\lambda)f(z)-\lambda T_\mu(r+1,\lambda)f(z), \tag{3.14}$$

we complete the proof using the same steps as in the proof of Theorem 3.1. □

Using Theorem 3.8, we obtain the following "sandwich result":

3.9. Corollary. *Let $\psi_i(z)$ be univalent in $\Delta, \psi_i(0)=1$, $\psi_i(z)$ convex in Δ, λ a complex number and $Re\{\lambda\}>-p$ for $i=1,2$. Define*

$$\chi_i(z)=\psi_i(z)+\frac{z\psi_i'(z)}{(p+\lambda)},\ (\lambda\neq -p).$$

If $f\in A(p,n,r,\lambda;\chi_1,\chi_2)$ satisfies (3.12)*, then $f\in A(p,n,r+1,\lambda;\psi_1,\psi_2)$.* □

The proof of the next theorem is the same as the proof of Theorem 3.3.

3.10. Theorem. *Let ψ be univalent in Δ, $\psi(0)=1$ and λ a complex number. Assume that ψ is convex in Δ and $Re\{\lambda\}>-p$. Define the functions F and χ by*

$$\begin{aligned} F(z)&=\frac{\lambda+p}{z^p}\int_0^z t^{\lambda-1}f(t)dt\\ \chi(z)&=\left[\psi(z)+\frac{z\psi(z)}{\lambda+p}\right],\ (\lambda\neq -p).\end{aligned} \tag{3.15}$$

If $f\in A(p,n,r,\lambda;\chi)$, then $F\in A(p,n,r,\lambda;\psi)$. If $f\in \overline{A}(p,n,r,\lambda;\chi)$,

$$0\neq \frac{T_\mu(r+1,\lambda)F(z)}{z^p}\in \mathcal{H}(1,1)\cap \mathcal{Q},\ \frac{T_\mu(r+1,\lambda)f(z)}{z^p} \text{ is univalent in } \Delta, \tag{3.16}$$

then $F\in \overline{A}(p,n,r,\lambda;\psi)$. □

Using Theorem 3.10, we have the following result:

3.11. Corollary. *Let ψ_i be univalent in Δ, $\psi_i(0)=1$ and λ a complex number. Assume that ψ_i is convex in Δ and $Re\{\lambda\}>-p$. Define the function χ_i by*

$$\chi_i(z)=\left[\psi_i(z)+\frac{z\psi_i(z)}{\lambda+p}\right],\ (i=1,2,\ \lambda\neq -p).$$

If $f\in A(p,n,r,\lambda;\chi_1,\chi_2)$ satisfies (3.16)*, then F defined* (3.15) *belongs to the class $A(p,n,r,\lambda;\psi_1,\psi_2)$.* □

3.12. Theorem. *Let $f(z) \in \mathcal{V}(p,n)$. Then $f \in A(p,n,r+1,\lambda,\varphi)$ if and only if*

$$F(z) = \frac{p+\lambda}{z^{\lambda}} \int_0^z t^{\lambda-1} f(t)\,dt \in A(p,n,r,\lambda,\varphi). \tag{3.17}$$

Also $f \in \overline{A}(p,n,r+1,\lambda;\varphi)$ if and only if $F \in \overline{A}(p,n,r,\lambda;\varphi)$.

Proof. From (3.17), we have

$$(p+\lambda)f(z) = \lambda F(z) + zF'(z). \tag{3.18}$$

By convoluting (3.18) with $\phi_p(n,\lambda;z) = z^p + \sum_{n=p+1}^{\infty}(\mu+p-1)_{n-p}\left(\frac{p+\lambda}{n+\lambda}\right)^r \frac{z^n}{(n-p)!}$, and using the fact that $z(f*g)'(z) = f(z) * zg'(z)$, we obtain

$$(p+\lambda)T_\mu(r+3,\lambda)f(z) = \lambda T_\mu(r+3,\lambda)F(z) + z(T_\mu(r+3,\lambda)F(z))',$$

and by using (3.14), we get

$$T_\mu(r+3,\lambda)f(z) = T_\mu(r+2,\lambda)F(z) \tag{3.19}$$

and

$$\begin{aligned}(p+\lambda)T_\mu(r+2,\lambda)f(z) &= z(T_\mu(r+3,\lambda)f(z))' + \lambda T_\mu(r+3,\lambda)f(z)\\ &= z(T_\mu(r+2,\lambda)F(z))' + \lambda T_\mu(r+2,\lambda)F(z)\\ &= (p+\lambda)T_\mu(r+1,\lambda)F(z).\end{aligned} \tag{3.20}$$

Therefore, from (3.20), we have

$$\frac{T_\mu(r+1,\lambda)F(z)}{z^p} = \frac{T_\mu(r+2,\lambda)f(z)}{z^p},$$

and the desired result follows at once. □

Using Theorem 3.12, we have

3.13. Corollary. *Let $f \in \mathcal{V}(p,n)$. Then $f \in A(p,n,r+1,\lambda;\varphi_1,\varphi_2)$ if and only if F given by (3.17) is in $A(p,n,r,\lambda;\varphi_1,\varphi_2)$.*

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