RESULTS ON THE COMMUTATIVE NEUTRIX CONVOLUTION PRODUCT OF DISTRIBUTIONS

Biljana Jolevska-Tuneska[*] and Arpad Takači[†]

Abstract

Let $\mathrm{Li}(x)$ denote the dilogarithm integral. The goal of this paper is to evaluate several commutative neutrix convolution products involving the dilogarithm integral and its associated functions $\mathrm{Li}_+(x)$ and $\mathrm{Li}_-(x)$.

1. Introduction

The dilogarithm integral $\mathrm{Li}(x)$ is defined by

$$(1) \qquad \mathrm{Li}(x) = -\int_0^x \frac{\ln|1-t|}{t}\, dt$$

(see [1]), and the associated functions $\mathrm{Li}_+(x)$ and $\mathrm{Li}_-(x)$ are defined by

$$\mathrm{Li}_+(x) = H(x)\,\mathrm{Li}(x),\ \ \mathrm{Li}_-(x) = H(-x)\,\mathrm{Li}(x) = \mathrm{Li}(x) - \mathrm{Li}_+(x),$$

where $H(x)$ denotes Heaviside's function.

Next, the distribution $\ln|1-x|x^{-1}$ is defined by

$$\ln|1-x|x^{-1} = -\mathrm{Li}'(x),$$

and its associated distributions $\ln|1-x|x_+^{-1}$ and $\ln|1-x|x_-^{-1}$ are defined by

$$\ln|1-x|x_+^{-1} = H(x)\ln|1-x|x^{-1} = -\mathrm{Li}_+'(x),$$
$$\ln|1-x|x_-^{-1} = H(-x)\ln|1-x|x^{-1} = -\mathrm{Li}_-'(x).$$

[*]Faculty of Electrical Engineering and Informational Technologies, Karpos II bb, Skopje, Republic of Macedonia.   E-mail: `biljanaj@feit.ukim.edu.mk`
[†]Faculty of Natural sciences, Tgr Dositeja Obradovića 4, 21000 Novi Sad, Serbia. E-mail: `takaci@im.ns.ac.yu`

We define the function $I_r(x)$ by

$$I_r(x) = \int_0^x u^r \ln|1 - u|\, du$$

for $r = 0, 1, 2, \ldots$, and it was shown in [5] that

$$I_r(x) = \frac{1}{r+1}(x^{r+1} - 1)\ln|1 - x| - \frac{1}{r+1}\sum_{i=0}^r \frac{x^{i+1}}{i+1}.$$

The classical definition of the convolution product of two functions f and g is as follows:

1.1. Definition. Let f and g be functions. Then the *convolution* $f * g$ is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x - t)\, dt$$

for all points x for which the integral exist.

It follows easily from the definition that if $f * g$ exists then $g * f$ exists and $f * g = g * f$. Further, if $(f * g)'$ and $f * g'$ (or $f' * g$) exist, then

(2) $(f * g)' = f * g'$ (or $f' * g$).

Definition 1.1 can be extended to define the convolution $f * g$ of two distributions f and g in $\mathcal{D}'$, the space of infinitely differentiable functions with compact support, see Gel'fand and Shilov [6].

1.2. Definition. Let f and g be distributions in $\mathcal{D}'$. Then the *convolution* $f * g$ is defined by the equation

$$\langle (f * g)(x), \varphi \rangle = \langle f(y), \langle g(x), \varphi(x + y)\rangle \rangle$$

for arbitrary φ in $\mathcal{D}$, provided f and g satisfy either of the conditions

   (a) Either f or g has bounded support,
   (b) The supports of f and g are bounded on the same side.

Note that if f and g are locally summable functions satisfying either of the above conditions, and the classical convolution $f * g$ exists, then it is in agreement with Definition 1.1.

The following convolutions were proved in [5] for $r = 0, 1, 2, \ldots$.

(3) $\mathrm{Li}_+(x) * x_+^r = \dfrac{1}{r+1}\sum_{i=0}^r \binom{r+1}{i}(-1)^{r-i}I_{r-i}(x)x_+^i + \dfrac{1}{r+1}x^{r+1}\mathrm{Li}_+(x)$

(4) $\ln|1 - x|x_+^{-1} * x_+^r = \sum_{i=0}^{r-1}\binom{r}{i}(-1)^{r-i}I_{r-i-1}(x)x_+^i - x^r\,\mathrm{Li}_+(x).$

The convolution product of distributions may be defined without any restriction on the supports. One of the best known is the definition given by V.S.Vladimirov, although several other definitions of the convolution product are equivalent to that of Vladimirov. However, the convolution product in the sense of any of these definitions does not exist for many pairs of distributions.

In [3] the *commutative neutrix convolution product* is defined, and this exists for a considerably larger class of pairs of distributions. In that definition, unit-sequences of function in $\mathcal{D}$ are used which allows one to approximate a given distribution by a sequence of distributions of bounded support.

To recall the definition of the commutative neutrix convolution product we first let τ be a function in $\mathcal{D}$, see [7], satisfying the conditions:

    (i) $\tau(x) = \tau(-x)$,
   (ii) $0 \leq \tau(x) \leq 1$,
  (iii) $\tau(x) = 1$ for $|x| \leq \frac{1}{2}$,
  (iv) $\tau(x) = 0$ for $|x| \geq 1$.

The function τ_n is then defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \leq n, \\ \tau(n^n - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n, \end{cases}$$

for $n = 1, 2, \ldots$.

We now have the following definition of the commutative neutrix convolution product.

1.3. Definition. Let f and g be distributions in $\mathcal{D}'$, and let $f_n = f\tau_n$ and $g_n = g\tau_n$ for $n = 1, 2, \ldots$. Then the commutative neutrix convolution product $f \boxed{*} g$ is defined as the neutrix limit of the sequence $\{f_n * g_n\}$, provided that the limit h exists in the sense that

$$\underset{n \to \infty}{\mathrm{N-lim}} \langle f_n * g_n, \varphi \rangle = \langle h, \varphi \rangle$$

for all φ in $\mathcal{D}$, where N is the neutrix (see van der Corput [2]), having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range N'' the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \ \ln^r n, \quad (\lambda \neq 0, \ r = 1, 2, \ldots),$$

and all functions which converge to zero in the usual sense as n tends to infinity.

Note that in this definition the convolution product $f_n * g_n$ is in the sense of Definition 1.1, the distributions f_n and g_n having bounded support since the support of τ_n is contained in the interval $[-n - n^{-n}, n + n^{-n}]$. This neutrix convolution product is also commutative.

It is obvious that any results proved with the original definition hold with the new definition. The following theorem (proved in [3]) therefore holds, showing that the commutative neutrix convolution product is a generalization of the convolution product. So the idea of a neutrix lies in neglecting certain numerical sequences diverging to $\pm\infty$, which makes for a wider class of pairs of distributions f and g for which the product exists. It should be noted that, in general, the definition of a commutative neutrix convolution product depends on the choice of the sequence τ_n as well as the set of negligible sequences.

1.4. Theorem. *Let f and g be distributions in $\mathcal{D}'$, satisfying either condition (a) or condition (b) of Gel'fand and Shilov's definition. Then the commutative neutrix convolution product $f \boxed{*} g$ exists and*

$$f \boxed{*} g = f * g.$$

Note, however, that $(f \boxed{*} g)'$ is not necessarily equal to $f' \boxed{*} g$, although we do have the following theorem proved in [4].

1.5. Theorem. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the commutative neutrix convolution product $f \boxed{*} g$ exists. If $\underset{\nu \to \infty}{\mathrm{N-lim}} \langle (f\tau_n') * g_n, \varphi \rangle$ exists and equals $\langle h, \varphi \rangle$ for all φ in $\mathcal{D}$, then $f' \boxed{*} g$ exists and*

$$(f \boxed{*} g)' = f' \boxed{*} g + h.$$

2. Main result

In the following, we need to extend our set of negligible functions to include finite linear sums of the functions $n^s \operatorname{Li}(n^r)$, for $s = 0, 1, 2, \dots$ and $r = 1, 2, \dots$. Before proving some further results we need the following lemma proved in [5]:

2.1. Lemma.

$$\operatorname*{N-lim}_{n\to\infty} I_r(n) = -\frac{1}{(r+1)^2}$$

2.2. Theorem. *The commutative neutrix convolution product* $\operatorname{Li}_+(x) \boxed{*} x^r$ *exists and*

$$(5) \qquad \operatorname{Li}_+(x) \boxed{*} x^r = \frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} \frac{(-1)^{r-i}}{(r-i+1)^2} x^i$$

for $r = 0, 1, 2, \dots$.

Proof. We put $[\operatorname{Li}_+(x)]_n = \operatorname{Li}_+(x)\tau_n(x)$ and $[x^r]_n = x^r \tau_n(x)$ for $n = 1, 2, \dots$. Since these functions have compact support, the classical convolution product $[\operatorname{Li}_+(x)]_n * [x^r]_n$ exists by Definition 1.1, and we have:

$$
\begin{aligned}
[\operatorname{Li}_+(x)]_n * [x^r]_n &= \int_{-\infty}^{\infty} \operatorname{Li}_+(t)(x-t)^r \tau_n(x-t)\tau_n(t)\, dt \\
&= \int_0^n \operatorname{Li}(t)(x-t)^r \tau_n(x-t)\, dt \\
& \qquad\qquad + \int_n^{n+n^{-n}} \operatorname{Li}(t)(x-t)^r \tau_n(x-t)\tau_n(t)\, dt \\
&= I_1 + I_2.
\end{aligned}
$$

(6)

If $0 \le x \le n$ we see that

$$
\begin{aligned}
I_1 &= \int_0^n \operatorname{Li}(t)(x-t)^r \tau_n(x-t)\, dt \\
&= -\int_0^n (x-t)^r \int_0^t \frac{\ln|1-u|}{u}\, du\, dt \\
&= -\int_0^n \frac{\ln|1-u|}{u} \int_u^n (x-t)^r\, dt\, du \\
&= \frac{1}{r+1} \sum_{i=0}^{r+1} (-1)^{r-i} x^i \binom{r+1}{i} \int_0^n u^{-1} \ln|1-u| \{u^{r-i+1} - n^{r-i+1}\}\, du \\
&= \frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} (-1)^{r-i} x^i I_{r-i}(n) + \frac{1}{r+1} x^{r+1} \operatorname{Li}(n) + \\
& \qquad\qquad\qquad + \frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} (-1)^{r-i} x^i \operatorname{Li}(n) n^{r-i+1}.
\end{aligned}
$$

Thus using Lemma 2.1 we have,

$$(7) \qquad \operatorname*{N-lim}_{n\to\infty} I_1 = \frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} \frac{(-1)^{r-i}}{(r-i+1)^2} x^i.$$

Next, if $-n \le x \le 0$ we have

(8)
$$
\begin{aligned}
I_1 &= \int_0^n \mathrm{Li}(t)(x-t)^r \tau_n(x-t)\,dt \\
&= \int_0^{x+n} \mathrm{Li}(t)(x-t)^r\,dt + \int_{x+n}^{x+n+n^{-n}} \mathrm{Li}(t)(x-t)^r \tau_n(x-t)\,dt,
\end{aligned}
$$

where

(9)
$$
\begin{aligned}
\int_0^{x+n} \mathrm{Li}(t)(x-t)^r\,dt &= -\int_0^{x+n} (x-t)^r \int_0^t u^{-1}\ln|1-u|\,du\,dt \\
&= -\int_0^{x+n} u^{-1}\ln|1-u| \int_u^{x+n} (x-t)^r\,dt\,du \\
&= \frac{1}{r+1}\int_0^{x+n} u^{-1}\ln|1-u|\left[(-n)^{r+1}-(x-u)^{r+1}\right]\,du \\
&= \frac{1}{r+1}\left((-n)^{r+1}-x^{r+1}\right)\mathrm{Li}(x+n) \\
&\qquad + \frac{1}{r+1}\sum_{i=0}^r \binom{r+1}{i}(-1)^{r-i}x^i I_{r-i}(x+n).
\end{aligned}
$$

Further, it is easily seen that, for each fixed x and $K = \sup\{|\mathrm{Li}_+(x)|\}$ we have

$$
\left| \int_{x+n}^{x+n+n^{-n}} \mathrm{Li}(t)(x-t)^r \tau_n(x-t)\,dt \right| \le K \int_{x+n}^{x+n+n^{-n}} |x-t|^r\,dt
$$

and so

(10)
$$
\lim_{n\to\infty} \int_{x+n}^{x+n+n^{-n}} \mathrm{Li}(t)(x-t)^r \tau(x-t)\,dt = 0.
$$

Using the fact that

$$
\mathrm{N}\!-\!\lim_{n\to\infty} I_r(x+n) = \mathrm{N}\!-\!\lim_{n\to\infty} I_r(n) = -\frac{1}{(r+1)^2},
$$

and noting that the function $\mathrm{Li}(x+n)$ is negligible we have from (8), (9) and (10) that in this case also equation (7) holds.

Further, it is easily seen that for each fixed x

(11)
$$
\lim_{n\to\infty} I_2 = \lim_{n\to\infty} \int_n^{n+n^{-n}} \mathrm{Li}(t)(x-t)^r \tau_n(x-t)\tau_n(t)\,dt = 0
$$

and now (5) follows immediately from (6), (7) and (11), so proving the theorem. $\square$

2.3. Corollary. *The commutative neutrix convolution product* $\mathrm{Li}_+(x) \boxed{*} x_-^r$ *exists and*

(12)
$$
\begin{aligned}
\mathrm{Li}_+(x) \boxed{*} x_-^r &= \frac{1}{r+1}\sum_{i=0}^r \binom{r+1}{i}(-1)^i\left[\frac{1}{(r-i+1)^2}x^i - I_{r-i}(x)x_+^i\right] \\
&\qquad + \frac{(-1)^{r+1}}{r+1}x^{r+1}\,\mathrm{Li}_+(x)
\end{aligned}
$$

for $r = 0, 1, 2, \ldots$

Proof. Since the commutative neutrix convolution product is distributive with respect to addition, we have

$$
\mathrm{Li}_+(x) \boxed{*} x^r = \mathrm{Li}_+(x) * x_+^r + (-1)^r \mathrm{Li}_+(x) \boxed{*} x_-^r,
$$

and equation (12) follows from equation (5) and (3). $\square$

2.4. Theorem. *The commutative neutrix convolution product* $\ln|1-x|x_+^{-1} \boxed{*} x^r$ *exists and*

$$(13) \qquad \ln|1-x|x_+^{-1} \boxed{*} x^r = \sum_{i=0}^{r-1} \binom{r}{i} \frac{(-1)^{r-i}}{(r-i)^2} x^i$$

for $r = 0, 1, 2, \ldots$

Proof. Using Theorem 2 we have

$$(14) \qquad \ln|1-x|x_+^{-1} \boxed{*} x^r = r \, \mathrm{Li}_+(x) \boxed{*} x^{r-1} - \mathop{\mathrm{N}-\lim}_{n\to\infty}[\mathrm{Li}_+(x)\tau_n'(x)] * [x^r]_n$$

when, on integration by parts we have

$$
\begin{aligned}
(15) \qquad [\mathrm{Li}_+(x)\tau_n'(x)] * (x^r)_n = {}& \int_n^{n+n^{-n}} \mathrm{Li}(t)(x-t)^r \tau_n(x-t)\, d\tau_n(t) \\
= {}& -\mathrm{Li}(n)(x-n)^r \tau_n(x-n) \\
& - \int_n^{n+n^{-n}} \ln|1-t|t^{-1}(x-t)^r \tau_n(t)\tau_n(x-t)\, dt \\
& + r \int_n^{n+n^{-n}} \mathrm{Li}(t)(x-t)^{r-1}\tau_n(t)\tau_n(x-t)\, dt \\
& + \int_n^{n+n^{-n}} \mathrm{Li}(t)(x-t)^r \tau_n(t)\tau_n'(x-t)\, dt.
\end{aligned}
$$

Now $\tau_n(x-n)$ is either 0 or 1 for large enough n, and so

$$(16) \qquad \mathop{\mathrm{N}-\lim}_{n\to\infty} \mathrm{Li}(n)(x-n)^r \tau_n(x-n) = 0.$$

Next we have

$$\left| \int_n^{n+n^{-n}} \ln|1-t|t^{-1}(x-t)^r \tau_n(t)\tau_n(x-t)\, dt \right| \le \int_n^{n+n^{-n}} \left| \ln|1-t|t^{-1}(x-t)^r \right| dt$$

and it follows that

$$(17) \qquad \lim_{n\to\infty} \int_n^{n+n^{-n}} \ln|1-t|t^{-1}(x-t)^r \tau_n(t)\tau_n(x-t)\, dt = 0.$$

Similarly,

$$(18) \qquad \lim_{n\to\infty} \int_n^{n+n^{-n}} \mathrm{Li}(t)(x-t)^{r-1}\tau_n(t)\tau_n(x-t)\, dt = 0.$$

Noting that $\tau_n'(x-t) = 0$ for large enough n and $x \ne 0$, it follows that

$$(19) \qquad \lim_{n\to\infty} \int_n^{n+n^{-n}} \mathrm{Li}(t)(x-t)^r \tau_n(t)\tau_n'(x-t)\, dt = 0.$$

If $x = 0$, then

$$
\begin{aligned}
& \int_n^{n+n^{-n}} \mathrm{Li}(t)(x-t)^r \tau_n(t)\tau_n'(-t)\, dt \\
= {}& \frac{1}{2} \int_n^{n+n^{-n}} \mathrm{Li}(t)(x-t)^r \, d\tau_n^2(t)\, dt \\
= {}& \frac{1}{2} \mathrm{Li}(n)(x-n)^r + \int_n^{n+n^{-n}} \left[\ln|1-t|t^{-1}(x-t) - r\,\mathrm{Li}(t) \right] (x-t)^{r-1}\tau_n^2(t)\, dt,
\end{aligned}
$$

and it follows that

$$(20) \qquad \underset{n\to\infty}{\mathrm{N}-\lim} \int_n^{n+n^{-n}} \mathrm{Li}(t)(x-t)^r \tau_n(t)\tau_n'(-t)\,dt = 0.$$

It now follows from Equations (15) to (20) that

$$\underset{n\to\infty}{\mathrm{N}-\lim}[\mathrm{Li}_+(x)\tau_n'(x)] * [x^r]_n = 0,$$

and the equation (13) follows directly from (14) and (5), so proving the theorem. $\square$

2.5. Corollary. *The commutative neutrix convolution product* $\ln|1-x|x_+^{-1} \boxed{*}\, x_-^r$ *exists and*

$$(21) \qquad \ln|1-x|x_+^{-1} \boxed{*}\, x_-^r = \sum_{i=0}^{r-1} \binom{r}{i}(-1)^i \left[\frac{1}{(r-i)^2}\,x^i - I_{r-i-1}(x)x_+^i\right] + (-1)^r x^r \,\mathrm{Li}_+(x)$$

for $r = 0,1,2,\ldots$.

Proof. Because we have

$$\ln|1-x|x_+^{-1} \boxed{*}\, x^r = \ln|1-x|x_+^{-1} * x_+^r + (-1)^r \ln|1-x|x_+^{-1} \boxed{*}\, x_-^r\,,$$

equation (21) follows from equation (13) and (4). $\square$

References

[1] Abramowitz, M. and Stegun, I. A. (Eds) *Handbook of Mathematical Functions with formulas, Graphs and Mathematical Tables, 9th printing* (New York: Dover, 1972), 1004-1005.

[2] van der Corput, J. G. *Introduction to the neutrix calculus*, J. Analyse Math. **7**, 291–398, 1959.

[3] Fisher, B. and Li, C. K. *A commutative neutrix convolution product of distributions*, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. **23**, 13–27, 1993.

[4] Fisher, B., Özçağ, E. and Gülen, Ü. *The exponential integral and the commutative neutrix convolution product*, J. Analysis **7**, 7–20, 1999.

[5] Jolevska-Tuneska, B., Fisher, B. and Özçağ, E. *On the dilogarithm integral*, submited for publication.

[6] Gel'fand, I. M. and Shilov, G. E. *Generalized functions, Vol. I* (Academic Press, 1964), Chap. 1.

[7] Jones, D. S. *The convolution of generalized functions*, Quart. J. Math. Oxford **24** (2), 145–163, 1973.