Hacettepe Journal of Mathematics and Statistics
Volume 36 (1) (2007), 43 – 47

ON A THEOREM OF POSNER FOR 3-PRIME NEAR-RINGS WITH (σ, τ)-DERIVATION

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Received 05 : 03 : 2007 : Accepted 16 : 05 : 2007

Abstract

The analog of Posner's theorem on the composition of two derivations in prime rings is proved for 3–prime near-rings with d_1 a (σ, τ)–derivation and d_2 a derivation.

Keywords: 3–prime near-rings, (σ, τ)–derivation, Commutativity.

2000 AMS Classification: 16 Y 30, 19 W 25, 16 U 80.

1. Introduction

Let N be a zero-symmetric left near-ring, σ, τ two near-ring automorphisms of N. An additive mapping $d : N \to N$ is called a (σ, τ)*–derivation* if $d(xy) = \tau(x)d(y) + d(x)\sigma(y)$ holds for all $x, y \in N$.

For $x, y \in N$, the symbol $[x, y]$ will denote $xy - yx$, while the symbol (x, y) will denote the additive commutator $x + y - x - y$. Given $x, y \in N$, we write $[x, y]_{\sigma,\tau} = x\sigma(y) - \tau(y)x$; in particular, $[x, y]_{1,1} = [x, y]$ in the usual sense. As for terminology used here without mention, we refer to G.Pilz [5].

During the last couple of decades, a lot of work has been done on commutativity of prime rings with derivations. The study of derivations of near-rings was initiated by H. E. Bell and G. Mason in [3]. A well-known theorem due to Posner [6] states that if the composition of two derivations of a prime ring of characteristic not equal to two is again a derivation, then at least one of them must be zero. An analogue of this result in 3–prime near-rings was obtained by Beidar *et. al.* (for reference see [1]). Bell and Argaç generalized this result for a nonzero semigroup ideal of N in [2]. It is our aim in this note to extend the result due to Beidar *et. al.* to (σ, τ)–derivations.

Throughout this paper N will denote a zero-symmetric and 3–prime left near-ring with multiplicative center Z. Moreover, d_1 will be a (σ, τ)–derivation and d_2 a derivation of N.

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2. Results

2.1. Lemma. [4, Lemma 2] *Let d be a (σ,τ)–derivation on a near-ring N. Then*

$$(\tau(x)d(y) + d(x)\sigma(y))\sigma(a) = \tau(x)d(y)\sigma(a) + d(x)\sigma(y)\sigma(a)$$

for all $x, y \in N$. □

2.2. Lemma. [4, Lemma 3] *Let d be a nonzero (σ,τ)–derivation on a near-ring N and $a \in N$.*

i) *If $d(N)\sigma(a) = 0$ then $a = 0$.*
ii) *If $ad(N) = 0$ then $a = 0$.* □

2.3. Lemma. *Let N be a 3–prime near-ring, d_1 a nonzero (σ,τ)–derivation and d_2 a derivation of N such that $d_1(x)\sigma(d_2(y)) = -\tau(d_2(x))d_1(y)$ for all $x, y \in N$. Then $(N,+)$ is abelian.*

Proof. Let u, v be in N. Writing $u + v$ instead of y, we have:

$$\begin{aligned}0 &= d_1(x)\sigma(d_2(u+v)) + \tau(d_2(x))d_1(u+v)\\ &= d_1(x)\sigma(d_2(u)) + d_1(x)\sigma(d_2(v)) + \tau(d_2(x))d_1(u) + \tau(d_2(x))d_1(v).\end{aligned}$$

Using the hypothesis, we get

$$0 = d_1(x)\sigma(d_2(u)) + d_1(x)\sigma(d_2(v)) - d_1(x)\sigma(d_2(u)) - d_1(x)\sigma(d_2(v)),$$

and so,

$$d_1(x)\sigma(d_2(u,v)) = 0, \quad \text{for all } x, u, v \in N. \tag{2.1}$$

By Lemma 2.2 (i), we obtain that $d_2(u, v) = 0$ for all $u, v \in N$.

For any $w \in N$, we have $d_2(wu, wv) = 0$. Using (2.1), we obtain that $d_2(w(u, v)) = 0$. This yields

$$d_2(w)(u, v) = 0, \quad \text{for all } w, u, v \in N.$$

By [3, Lemma 3 (iii)], we get $(u, v) = 0$, for all $u, v \in N$. □

2.4. Theorem. *Let N be a 3–prime near-ring with $2N \neq \{0\}$, d_1 a (σ,τ)–derivation and d_2 a derivation such that*

$$d_1\tau = \tau d_1,\ d_1\sigma = \sigma d_1,\ d_2\tau = \tau d_2 \text{ and } d_2\sigma = \sigma d_2.$$

If $d_1(x)\sigma(d_2(y)) = -\tau(d_2(x))d_1(y)$ for all $x, y \in N$, then either $d_1 = 0$ or $d_2 = 0$.

Proof. Suppose that $d_1 \neq 0$ and $d_2 \neq 0$. We know that $(N,+)$ is abelian from Lemma 2.3. Now, for all $u, v \in N$ we take $x = uv$ in the given equation, obtaining

$$\begin{aligned}0 &= d_1(uv)\sigma(d_2(y)) + \tau(d_2(uv))d_1(y)\\ &= (\tau(u)d_1(v) + d_1(u)\sigma(v))\sigma(d_2(y)) + \tau(ud_2(v) + d_2(u)v)d_1(y).\end{aligned}$$

By the partial distributive law, for all $u, v, y \in N$ we get:

$$0 = \tau(u)d_1(v)\sigma(d_2(y)) + d_1(u)\sigma(v)\sigma(d_2(y)) + \tau(u)\tau(d_2(v))d_1(y) + \tau(d_2(u))\tau(v)d_1(y)$$

and so,

$$0 = \tau(u)(d_1(v)\sigma(d_2(y) + \tau(d_2(v))d_1(y)) + d_1(u)\sigma(v)\sigma(d_2(y)) + d_2(\tau(u))\tau(v)d_1(y).$$

Using the hypothesis, we have

$$d_1(u)\sigma(v)\sigma(d_2(y)) + d_2(\tau(u))\tau(v)d_1(y) = 0, \quad \text{for all } y, u, v \in N. \tag{2.2}$$

Replacing y by $yt, t \in N$ in (2.2) and using (2.2), we have

$$\begin{aligned}0 &= d_1(u)\sigma(v)\sigma(d_2(yt)) + d_2(\tau(u))\tau(v)d_1(yt)\\ &= d_1(u)\sigma(v)\sigma(y)\sigma(d_2(t)) + d_1(u)\sigma(v)\sigma(d_2(y))\sigma(t)\\ &\qquad + d_2(\tau(u))\tau(v)\tau(y)d_1(t) + d_2(\tau(u))\tau(v)d_1(y)\sigma(t)\\ &= d_1(u)\sigma(vy)\sigma(d_2(t)) + d_2(\tau(u))\tau(vy)d_1(t)\\ &\qquad + d_1(u)\sigma(v)\sigma(d_2(y))\sigma(t) + d_2(\tau(u))\tau(v)d_1(y)\sigma(t),\end{aligned}$$

and using (2.2) gives

(2.3) $\quad d_1(u)\sigma(v)\sigma(d_2(y))\sigma(t) + d_2(\tau(u))\tau(v)d_1(y)\sigma(t) = 0, \quad \text{for all } u, v, y, t \in N.$

Substituting $d_1(t)$ for t in (2.3), we get, for all $u, v, y, t \in N$.

(2.4) $\quad d_1(u)\sigma(v)\sigma(d_2(y))\sigma(d_1(t)) + d_2(\tau(u))\tau(v)d_1(y)\sigma(d_1(t)) = 0.$

Now if we take $\tau(y)$ instead of y in (2.4), we have, for all $u, v, y, t \in N$,

(2.5) $\quad d_1(u)\sigma(v)\sigma(d_2(\tau(y)))\sigma(d_1(t)) + d_2(\tau(u))\tau(v)d_1(\tau(y))\sigma(d_1(t)) = 0.$

Substituting $vd_1(y)$ and $\sigma(t)$ instead of v and y, respectively in (2.2), we obtain that, for all $u, v, y, t \in N$,

(2.6) $\quad d_1(u)\sigma(v)\sigma(d_1(y))\sigma(d_2(\sigma(t))) + d_2(\tau(u))\tau(v)\tau(d_1(y))d_1(\sigma(t)) = 0.$

Now, if we subtract (2.5) from (2.6) and use

$$d_1\sigma = \sigma d_1, d_1\tau = \tau d_1,$$

we obtain that

$$d_1(u)\sigma(v)(d_1(\sigma(y))\sigma(d_2(\sigma(t)) - \sigma(d_2(\tau(y))\sigma(d_1(t)) = 0,$$

and so

(2.7) $\quad d_1(N)N(\sigma(d_1(y)d_2(\sigma(t)) - d_2(\tau(y))d_1(t)) = 0, \quad \text{for all } t, y \in N.$

Since N is a 3–prime near-ring and $d_1 \neq 0$ we conclude that,

$$d_1(y)d_2(\sigma(t)) - d_2(\tau(y))d_1(t) = 0.$$

Using $d_2\sigma = \sigma d_2$ and the hypothesis, we get

(2.8) $\quad d_1(y)\sigma(d_2(t) + d_2(t)) = 0, \text{ for all } y, t \in N.$

By Lemma 2.2(i) and $d_1 \neq 0$ we get,

$$d_2(t) + d_2(t) = 0, \text{ for all } t \in N.$$

Hence $0 = d_2(st) + d_2(st) = d_2(s)(t + t)$ for all $s, t \in N$, and so

$$d_2(N)(t + t) = 0, \text{ for all } t \in N.$$

From [3, Lemma 3 (iii)], we have $t + t = 0$ for all $t \in N$, which contradicts the hypothesis that $2N \neq \{0\}$. This completes the proof. □

2.5. Theorem. *Let N be a 3–prime near-ring with $2N \neq \{0\}$, d_1 a (σ, τ)–derivation and d_2 a derivation such that*

$$d_1\tau = \tau d_1,\ d_1\sigma = \sigma d_1,\ d_2\tau = \tau d_2 \text{ and } d_2\sigma = \sigma d_2.$$

If d_1d_2 acts as a (σ, τ)–derivation on N, then either $d_1 = 0$ or $d_2 = 0$.

Proof. Note that

$$d_2d_1(xy) = \tau(x)d_2d_1(y) + d_2d_1(x)\sigma(y),$$

and

$$d_2d_1(xy) = d_2(\tau(x))d_1(y) + \tau(x)d_2d_1(y) + d_2d_1(x)\sigma(y) + d_1(x)d_2(\sigma(y)).$$

Comparing these two expressions gives

$$\tau(d_2(x))d_1(y) + d_1(x)\sigma(d_2(y)) = 0, \text{ for all } x, y \in N.$$

From Theorem 2.4, we get $d_1 = 0$ or $d_2 = 0$. □

2.6. Theorem. *If N is a 3–prime near-ring with d a nonzero (σ, τ)–derivation such that $d(xy) = d(yx)$ for all $x, y \in N$, then N is commutative near-ring.*

Proof. By hypothesis,

$$\tau(x)d(y) + d(x)\sigma(y) = \tau(y)d(x) + d(y)\sigma(x), \text{ for all } x, y \in N. \tag{2.9}$$

We substitute yx for x, thereby obtaining

$$\tau(y)\tau(x)d(y) + d(yx)\sigma(y) = \tau(y)d(yx) + d(y)\sigma(y)\sigma(x).$$

By the partial distributive law for $x, y \in N$ and using $d(yx) = d(xy)$ we get

$$\begin{aligned}&\tau(y)\tau(x)d(y) + \tau(y)d(x)\sigma(y) + d(y)\sigma(x)\sigma(y)\\ &\qquad = \tau(y)\tau(x)d(y) + \tau(y)d(x)\sigma(y) + d(y)\sigma(y)\sigma(x),\end{aligned}$$

and so,

$$d(y)\sigma(x)\sigma(y) = d(y)\sigma(y)\sigma(x), \text{ for all } x, y \in N. \tag{2.10}$$

Replacing x by $xz, z \in N$ we have:

$$d(y)\sigma(x)\sigma(z)\sigma(y) = d(y)\sigma(y)\sigma(x)\sigma(z) = d(y)\sigma(x)\sigma(y)\sigma(z),$$

and so

$$d(y)N\sigma([z, y]) = 0, \text{ for all } y, z \in N.$$

Since N is a 3–prime near-ring, we obtain that $d(y) = 0$ or $y \in Z$. Since d is a nonzero (σ, τ)–derivation on N, we have $y \in Z$. So, N is a commutative near-ring. □

2.7. Theorem. *Let N be $3-$prime $2-$torsion free near-ring and d_1 a nonzero $(\sigma, \tau)-$ derivation, d_2 a nonzero derivation on N such that $d_1\tau = \tau d_1, d_1\sigma = \sigma d_1, d_2\tau = \tau d_2$ and $d_2\sigma = \sigma d_2$. If $d_1(x)\sigma(d_2(y)) = \tau(d_2(y))d_1(x)$ for $x, y \in N$, then N is a commutative ring.*

Proof. By hypothesis, we get

$$[d_1(x), d_2(y)]_{\sigma,\tau} = 0, \text{ for all } x, y \in N.$$

From [4, Theorem 6], we obtain that $d_1d_2(N) = 0$ or $d_2(N) \subset Z$.

Since d_1 and d_2 are nonzero derivations, we have $d_2(N) \subset Z$. This completes the proof by [3, Theorem 2]. □

References

[1] Beidar, K. I., Fong, Y. and Wang, X. K. *Posner and Herstein Theorems for derivations of 3–prime near-rings*, Communications in Algebra **24** (5), 1581–1589, 1996.

[2] Bell, H. E. and Argaç, N. *Derivations, products of derivations and commutativity in near-rings*, Algebra Colloquium **8** 4, 399–407, 2001.

[3] Bell, H. E. and Mason G. *On Derivations in Near-rings, Near-rings and Near-fields* (North-Holland Mathematical Studies **137**, 1987).

[4] Gölbaşı, Ö. and Aydın, N. *Results on prime near-rings with* (σ, τ)*–derivation*, Mathematical Journal of Okayama University **46**, 1–7, 2004.

[5] Pilz, G. *Near-rings, 2nd Ed.* (North Holland, Amsterdam, 1983).

[6] Posner, E. C. *Derivations in prime rings*, Proc. Amer. Math. Soc. **8**, 1093–1100, 1957.