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NOTIONS OF OPENNESS AND CLOSEDNESS FOR MAPS BETWEEN L-FUZZY CLOSURE SPACES

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Abstract

In this paper the authors introduce and characterize r-open, r-semiopen sets (resp. r-closed, r-semiclosed sets) and open, semiopen and semi-continuous maps (resp. closed, semiclosed maps) in L-fuzzy closure spaces.

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1. Introduction

Chang introduced fuzzy topological spaces in [1]. In a Chang fuzzy topological space, each fuzzy set is either open or not. Later Chang's idea was developed by Goguen [8] who replaced the lattice $[0, 1]$ by a more general lattice L.

An essentially more general notion of fuzzy topology, in which each fuzzy set has a certain degree of openness, was introduced by Šostak [13], and independently by Ramadan [12], Chattopadhyay, Hazra and Sammanta [3, 2].

Mashhour [7] introduced fuzzy closure spaces in the sense of Chang. On the other hand, L-closure operators corresponding to L-topological spaces (originally called L-fuzzy topological spaces by Chang [1] and Goguen [8]) in the case of a general lattice L were first considered by Ghanim and Hasan in [6]. Klein [11] used fuzzy closure operators to describe L-topological spaces, Šostak [15] applied L-fuzzy closure operators to describe L-fuzzy topologies in the sense of [14], and Chattopadhyay and Sammanta [4] in the

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case of $L = [0, 1]$. Kim [9, 10], defined subspaces and products of fuzzy closure spaces and L-fuzzy closure spaces, respectively.

In this paper we introduce open and closed maps (resp. semiopen, semiclosed and semicontinuous maps), and give some characterization theorems.

2. Preliminaries

Throughout this paper let X be a non-empty set and $(L, \leq, \vee, \wedge, ')$ a complete, completely distributive lattice with an order reversing involution $'$. The smallest and largest elements in L will be denoted by 0 and 1, respectively. Let $L_0 = L \setminus \{0\}$.

If $a \leq b$ or $b \leq a$ for each $a, b \in L$ then L is called a *chain*. A lattice L is called *order dense* if for each $a, b \in L$ such that $a < b$, there exists $c \in L$ such that $a < c < b$.

Note that $(L^X, \leq, \vee, \wedge, ')$ is a complete, completely distributive lattice with an order reversing involution $'$ if L is, the operations are defined point-wise and $\underline{0}$, $\underline{1}$ denotes the smallest and largest elements of L^X. The elements of L^X are called *L-fuzzy sets*. All undefined notations are standard notations of L-fuzzy set theory.

2.1. Definition. [3, 2, 12] Let $\mathcal{T} : L^X \to L$ be a mapping. Then $\mathcal{T}$ is said to be an *L-fuzzy topology* on X if it satisfies the following conditions:

(1) $\mathcal{T}(\underline{0}) = \mathcal{T}(\underline{1}) = 1$.
(2) $\mathcal{T}(\mu \wedge \nu) \geq \mathcal{T}(\mu) \wedge \mathcal{T}(\nu)$.
(3) $\mathcal{T}(\bigvee_{i \in \Gamma} \mu_i) \geq \bigwedge_{i \in \Gamma} \mathcal{T}(\mu_i)$.

The pair $(X, \mathcal{T})$ is called an *L-fuzzy topological space.*

If $\mathcal{T}_1, \mathcal{T}_2$ are L-fuzzy topologies on X, we say $\mathcal{T}_1$ is *finer* than $\mathcal{T}_2$ ($\mathcal{T}_2$ is *coarser* than $\mathcal{T}_1$) if $\mathcal{T}_2(\lambda) \leq \mathcal{T}_1(\lambda)$ for each $\lambda \in L^X$.

2.2. Definition. [3, 2, 12] Let $\mathcal{F} : L^X \to L$ be a mapping. Then $\mathcal{F}$ is said to be an *L-fuzzy cotopology* on X if it satisfies the following conditions:

(1) $\mathcal{F}(\underline{0}) = \mathcal{F}(\underline{1}) = 1$.
(2) $\mathcal{F}(\lambda_1 \vee \lambda_2) \geq \mathcal{F}(\lambda_1) \wedge \mathcal{F}(\lambda_2)$.
(3) $\mathcal{F}(\bigwedge_{i \in \Gamma} \lambda_i) \geq \bigwedge_{i \in \Gamma} \mathcal{F}(\lambda_i)$.

The pair $(X, \mathcal{F})$ is called an *L-fuzzy cotopological space.*

2.3. Proposition. [3, 2, 12] *Let $\mathcal{T}$ be an L-fuzzy topology on X and $\mathcal{T}' : L^X \to L$ the mapping defined by*

$$\mathcal{T}'(\lambda) = \mathcal{T}(\lambda'),$$

Then $(X, \mathcal{T}')$ is an L-fuzzy cotopological space.

2.4. Definition. [3, 2, 12] Let $(X, \mathcal{T}_1), (Y, \mathcal{T}_2)$ be L-fuzzy topological spaces. Then the map $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is called *LF-continuous* iff

$$\mathcal{T}_2(\nu) \leq \mathcal{T}_1(f^{-1}(\nu)) \text{ for every } \nu \in L^Y.$$

2.5. Lemma. [5] *If $f : X \to Y$ we have the following properties for the direct and inverse images of L-fuzzy sets. Here $\mu, \mu_i \in L^X$ and $\nu, \nu_i \in L^Y$.*

(1) $\nu \geq f(f^{-1}(\nu))$*, with equality if f is surjective.*
(2) $\mu \leq f^{-1}(f(\mu))$*, with equality if f is injective.*
(3) $f^{-1}(\nu') = f^{-1}(\nu)'$.
(4) $f^{-1}(\bigvee_{i \in \Gamma} \nu_i) = \bigvee_{i \in \Gamma} f^{-1}(\nu_i)$.
(5) $f^{-1}(\bigwedge_{i \in \Gamma} \nu_i) = \bigwedge_{i \in \Gamma} f^{-1}(\nu_i)$.
(6) $f(\bigvee_{i \in \Gamma} \mu_i) = \bigvee_{i \in \Gamma} f(\mu_i)$.
(7) $f(\bigwedge_{i \in \Gamma} \mu_i) \leq \bigwedge_{i \in \Gamma} f(\mu_i)$*, with equality if f is injective.*

3. L-fuzzy closure spaces

3.1. Definition. [4] An operator $C\colon L^X \times L_0 \to L^X$ is called an *L-fuzzy closure operator* if it satisfies the following conditions:

(1) $C(\underline{0}, r) = \underline{0}$.
(2) $\lambda \leq C(\lambda, r)$ *for each* $\lambda \in L^X$.
(3) $C(\lambda \vee \mu, r) = C(\lambda, r) \vee C(\mu, r)$ for every $r \in L_0$.
(4) $C(\lambda, r) \leq C(\mu, r)$ if $\lambda \leq \mu$.
(5) $C(\lambda, r) \leq C(\lambda, r^*)$ if $r \leq r^*$.

The pair (X, C) is then called an *L-fuzzy closure space*. It is called *topological* if it also satisfies the condition

$$C(C(\lambda, r), r) = C(\lambda, r)\ \forall\, \lambda \in L^X,\ r \in L_0.$$

Let C_1 and C_2 be L-fuzzy closure operators on X. Then C_1 is called *finer* than C_2 (C_2 is *coarser* than C_1) if $C_1(\lambda, r) \leq C_2(\lambda, r)$ for all $\lambda \in L^X,\ r \in L_0$.

3.2. Proposition. [4] *Let $(X, \mathcal{F})$ be an L-fuzzy cotopological space. Define the map $C_{\mathcal{F}}\colon L^X \times L_0 \to L^X$ by*

$$C_{\mathcal{F}}(\lambda, r) = \bigwedge \{\, \mu \in L^X \mid \mu \geq \lambda,\ \mathcal{F}(\mu) \geq r \}.$$

Then $(X, C_{\mathcal{F}})$ is a topological L-fuzzy closure space and if $r = \bigvee \{s \in L \mid C_{\mathcal{F}}(\lambda, s) = \lambda\}$ then $C_{\mathcal{F}}(\lambda, r) = \lambda$.

3.3. Proposition. [4] *Let (X, C) be L-fuzzy closure space. Define a map $\mathcal{T}_C\colon L^X \to L$ by*

$$\mathcal{F}_C(\lambda) = \bigvee \{r \in L_0 \mid C(\lambda, r) = \lambda\}$$

Then:

(1) *$(X, \mathcal{F}_C)$ is an L-fuzzy cotopological space.*
(2) *We have $C = C_{\mathcal{F}_C}$ iff the L-fuzzy closure space (X, C) satisfies the following conditions:*
  a *It is topological.*
  b *If $r = \bigvee \{s \in L \mid C(\lambda, s) = \lambda\}$ then $C(\lambda, r) = \lambda$.*

3.4. Theorem. [4] *Let $(X, \mathcal{F})$ be an L-fuzzy cotopological space. If $(X, C_{\mathcal{F}})$ is the corresponding L-fuzzy closure space, then $\mathcal{F}_{C_{\mathcal{F}}}$ is an L-fuzzy cotopology on X such that $\mathcal{F}_{C_{\mathcal{F}}} = \mathcal{F}$.*

4. r-open and r-closed sets in L-fuzzy closure spaces

4.1. Definition. Let (X, C) be an L-fuzzy closure space. An L-fuzzy set $\lambda \in L^X$ is said to be *r-closed* if $C(\lambda, r) = \lambda$ and *r-open* if λ' is *r-closed*.

4.2. Proposition. *We have the following:*

(1) (a) *A finite union of r-closed sets is r-closed.*
  (b) *An arbitrary intersection of r-closed sets is r-closed.*
(2) (a) *A finite intersection of r-open sets is r-open.*
  (b) *An arbitrary union of r-open sets is r-open.*

Proof. (1) (a) Let $\{\mu_i \mid i \in \Gamma\}$ be a finite set of r-closed sets, then

$$C(\bigvee_{i\in\Gamma} \mu_i, r) = \bigvee_{i\in\Gamma} C(\mu_i, r) = \bigvee_{i\in\Gamma} \mu_i.$$

(1) (b) Let $\{\mu_i \mid i \in \Gamma\}$ be an arbitrary set of r-closed sets. Since $\bigwedge_{i\in\Gamma} \mu_i \le \mu_i$ we have $C(\bigwedge_{i\in\Gamma} \mu_i, r) \le C(\mu_i, r) = \mu_i$ for each $i \in \Gamma$. Hence, $C(\bigwedge_{i\in\Gamma} \mu_i, r) \le \bigwedge_{i\in\Gamma} \mu_i$, which is sufficient to prove that $\bigwedge_{i\in\Gamma} \mu_i$ is r-closed.

(2) This follows from (1) by applying the involution $'$. □

4.3. Definition. Let (X, C) be an L-fuzzy closure space. The map $\mathcal{I}_C \colon L^X \times L_0 \to L^X$ defined by:

$$\mathcal{I}_C(\lambda, r) = (C(\lambda', r))', \ \lambda \in L^X, \ r \in L_0$$

is called the *L-fuzzy interior operator associated with C*. For $\lambda \in L^X$, $\mathcal{I}_C(\lambda, r)$ will be called the *C-interior* of λ.

4.4. Proposition. *Let (X, C) be an L-fuzzy closure space. Then the C-interior operator $\mathcal{I}_C$ has the following properties:*

(1) $\mathcal{I}_C(\underline{1}, r) = \underline{1}$.
(2) $\mathcal{I}_C(\lambda, r) \le \lambda$ *for every* $\lambda \in L^X$.
(3) $\mathcal{I}_C(\lambda \wedge \mu, r) = \mathcal{I}_C(\lambda, r) \wedge \mathcal{I}_C(\mu, r)$ *for every* $\lambda, \mu \in L^X, r \in L_0$.
(4) $\mathcal{I}_C(\lambda, r) \le \mathcal{I}_C(\mu, r)$ *if* $\lambda \le \mu$.
(5) $\mathcal{I}_C(\lambda, s) \le \mathcal{I}_C(\lambda, r)$ *if* $r \le s$.

Proof. Straightforward. □

One may easily verify the following statements:

(a) For $\mu \in L^X$, μ is r-open iff $\mathcal{I}_C(\mu, r) = \mu$.
(b) μ is r-closed iff μ' is r-open.

4.5. Definition. A map $\mathcal{I} \colon L^X \times L_0 \to L^X$ is said to be an *interior operator* if it satisfies the conditions (1)–(5).

4.6. Proposition. *Let $\mathcal{I}$ be an interior operator and define $C_{\mathcal{I}} \colon L^X \times L_0 \to L^X$ by*

$$C_{\mathcal{I}}(\mu, r) = (\mathcal{I}(\mu', r))'$$

for every $\mu \in L^X$. Then $C_{\mathcal{I}}$ is an L-fuzzy closure operator and $\mathcal{I}_{C_{\mathcal{I}}} = \mathcal{I}$.

Proof. We first verify conditions (1)–(5).

(1). $C_{\mathcal{I}}(\underline{0}, r) = (\mathcal{I}(\underline{0}', r))' = (\mathcal{I}(\underline{1}, r))' = (\underline{1}') = \underline{0}$.

(2). $C_{\mathcal{I}}(\mu, r) = (\mathcal{I}(\mu', r))'$ since $\mathcal{I}(\mu', r) \le \mu'$, then $\mu \le (\mathcal{I}(\mu', r))'$, $\mu \le C_{\mathcal{I}}(\mu, r)$.

(3).
$$\begin{aligned} C_{\mathcal{I}}(\lambda \vee \mu, r) &= \mathcal{I}((\lambda \vee \mu)', r))' = \mathcal{I}((\lambda' \wedge \mu')', r)' \\ &= (\mathcal{I}(\lambda', r) \wedge \mathcal{I}(\mu', r))' = \mathcal{I}(\lambda', r)' \vee \mathcal{I}(\mu', r)' \\ &= C_{\mathcal{I}}(\lambda, r) \vee C_{\mathcal{I}}(\mu, r). \end{aligned}$$

(4). If $\lambda \le \mu$ then $\mu' \le \lambda'$, so $\mathcal{I}(\mu', r) \le \mathcal{I}(\lambda', r)$. Taking the complement and using the definition of $C_{\mathcal{I}}$ this leads to

$$C_{\mathcal{I}}(\lambda, r) \le C_{\mathcal{I}}(\mu, r).$$

(5). If $r \le r^*$ then $\mathcal{I}(\lambda', r^*) \le \mathcal{I}(\lambda', r)$. By taking the complement this leads to $(\mathcal{I}(\lambda', r))' \le (\mathcal{I}(\lambda', r^*))'$, hence $C_{\mathcal{I}}(\lambda, r) \le C_{\mathcal{I}}(\lambda, r^*)$.

To prove that $\mathcal{I}_{C_{\mathcal{I}}} = \mathcal{I}$, we note that:

$$\mathcal{I}_{C_{\mathcal{I}}}(\mu, r) = (C_{\mathcal{I}}(\mu', r))' = (\mathcal{I}(\mu, r)')' = \mathcal{I}(\mu, r)$$

for each $\mu \in I^X$, $r \in I_0$. □

4.7. Definition. Let $(X, C_1), (Y, C_2)$ be L-fuzzy closure spaces. A function $f\colon (X, C_1) \to (Y, C_2)$ is called an *open map* (resp. a *closed map*) if $f(\lambda)$ is an r-open set (resp. an r-closed set) for each r-open (resp. r-closed) set $\lambda \in L^X$.

4.8. Definition. [10] Let $(X, C_1), (Y, C_2)$ be L-fuzzy closure spaces. Then $f\colon (X, C_1) \to (Y, C_2)$ is called a *continuous map* if

$$f(C_1(\lambda, r)) \le C_2(f(\lambda), r), \forall\, \lambda \in L^X,\ r \in L_0.$$

4.9. Definition. Let $(X, C_1), (Y, C_2)$ be L-fuzzy closure spaces. A function $f\colon (X, C_1) \to (Y, C_2)$ is called a *homeomorphism* iff f is bijective and $f,\ f^{-1}$ are continuous maps.

4.10. Theorem. *Let (X, C_1), (Y, C_2) be topological L-fuzzy closure spaces. Then the following statements are equivalent for the map $f\colon (X, C_1) \to (Y, C_2)$.*

(1) *f is an open map.*
(2) *$f(\mathcal{I}_{C_1}(\lambda, r)) \le \mathcal{I}_{C_2}(f(\lambda), r)$ for each $\lambda \in L^X,\ r \in L_0$.*
(3) *$\mathcal{I}_{C_1}(f^{-1}(\mu), r) \le f^{-1}(\mathcal{I}_{C_2}(\mu, r))$ for each $\mu \in L^Y,\ r \in L_0$.*
(4) *For any $\mu \in L^Y$ and any r-closed $\lambda \in L^X$ with $f^{-1}(\mu) \le \lambda$, there exists an r-closed set $\rho \in L^Y$ with $\mu \le \rho$ such that $f^{-1}(\rho) \le \lambda$.*

Proof. (1) $\Longrightarrow$ (2). Since (X, C_1) is topological it is easy to see that $\mathcal{I}_{C_1}(\mathcal{I}_{C_1}(\lambda, r), r) = \mathcal{I}_{C_1}(\lambda, r)$, whence $\mathcal{I}_{C_1}(\lambda, r)$ is r-open. Since f is an open map, $f(\mathcal{I}_{C_1}(\lambda, r))$ is r-open in (Y, C_2) and so

$$f(\mathcal{I}_{C_1}(\lambda, r)) = \mathcal{I}_{C_2}(f(\mathcal{I}_{C_1}(\lambda, r)), r).$$

On the other hand, $\mathcal{I}_{C_1}(\lambda, r) \le \lambda$ so $f(\mathcal{I}_{C_1}(\lambda, r)) \le f(\lambda)$, and hence

$$\mathcal{I}_{C_2}(f(\mathcal{I}_{C_1}(\lambda, r)), r) \le \mathcal{I}_{C_2}(f(\lambda), r).$$

From the above inequalities we obtain $f(\mathcal{I}_{C_1}(\lambda, r)) \le \mathcal{I}_{C_2}(f(\lambda), r)$ for each $\lambda \in L^X,\ r \in L_0$.

(2) $\Longrightarrow$ (3). For all $\mu \in L^Y$, $r \in L_0$, put $\lambda = f^{-1}(\mu)$. From (2) we have

$$f(\mathcal{I}_{C_1}(f^{-1}(\mu), r)) \le \mathcal{I}_{C_2}(f(f^{-1}(\mu)), r) \le \mathcal{I}_{C_2}(\mu, r)$$

by Lemma 2.5 (1). By Lemma 2.5 (2) this gives

$$\mathcal{I}_{C_1}(f^{-1}(\mu), r) \le f^{-1}(\mathcal{I}_{C_2}(\mu, r)).$$

(3) $\Longrightarrow$ (4). Let λ be r-closed such that $f^{-1}(\mu) \le \lambda$, whence $\lambda' \le f^{-1}(\mu')$. Since $\mathcal{I}_{C_1}(\lambda', r) = \lambda'$ then

$$\lambda' = \mathcal{I}_{C_1}(\lambda', r) \le \mathcal{I}_{C_1}(f^{-1}(\mu'), r).$$

From (3),

$$\lambda' \le \mathcal{I}_{C_1}(f^{-1}(\mu'), r) \le f^{-1}(\mathcal{I}_{C_2}(\mu', r)).$$

This implies that

$$\lambda \ge (f^{-1}(\mathcal{I}_{C_2}(\mu', r)))' = f^{-1}((\mathcal{I}_{C_2}(\mu', r))') = f^{-1}(C_2(\mu, r)).$$

Since (Y, C_2) is topological, $\rho = C_2(\mu, r) \in L^Y$ is r-closed and satisfies $\mu \le \rho$ and $f^{-1}(\rho) \le \lambda$.

(4) $\Longrightarrow$ (1). Let ν be an r-open set, put $\mu = f(\nu)'$ and $\lambda = \nu'$ so that λ is r-closed. Then:

$$f^{-1}(\mu) = f^{-1}(f(\nu)') = (f^{-1}(f(\nu)))' \le \nu' = \lambda.$$

From (4), there exists an r-closed set ρ with $\mu \le \rho$ such that $f^{-1}(\rho) \le \lambda = \nu'$. Hence, $\nu \le (f^{-1}(\rho))' = f^{-1}(\rho')$. Thus $f(\nu) \le f(f^{-1}(\rho')) \le \rho'$. On the other hand, since $\mu \le \rho$, $f(\nu) = (\mu)' \ge \rho'$. Hence $f(\nu) = \rho'$. That is, $f(\nu)$ is r-open. □

4.11. Theorem. *Let (X, C_1) and (Y, C_2) be topological L-fuzzy closure spaces. Then the following statements are equivalent for the map $f\colon (X, C_1) \to (Y, C_2)$.*

(1) *f is a closed map.*
(2) $f(C_1(\lambda, r)) \geq C_2(f(\lambda), r),\ \forall \lambda \in L^X,\ r \in L_0$.
(3) $C_1(f^{-1}(\mu), r) \geq f^{-1}(C_2(\mu, r)),\ \forall \mu \in L^Y,\ r \in L_0$
(4) *For any $\mu \in L^Y$ and any r-open $\lambda \in L^X$, with $f^{-1}(\mu) \leq \lambda$, there exists an r-open $\rho \in L^Y$ with $\mu \leq \rho$ such that $f^{-1}(\rho) \leq \lambda$.*

Proof. Similar to the proof of Theorem 4.10. □

4.12. Theorem. *Let (X, C_1), (Y, C_2) be topological L-fuzzy closure spaces. Then the following statements are true for a bijective map $f\colon (X, C_1) \to (Y, C_2)$.*

(1) *f is a closed map iff $f^{-1}(C_2(\mu, r)) \leq C_1(f^{-1}(\mu), r)$ for each $\mu \in L^Y,\ r \in L_0$.*
(2) *f is a closed map iff f is open.*

Proof. (1) $\Longrightarrow$. Let f be a closed map. From Theorem 4.11 (2), for each $\lambda \in L^X,\ r \in L_0$,

$$f(C_1(\lambda, r)) \geq C_2(f(\lambda), r).$$

For all $\mu \in L^Y,\ r \in L_0$, put $\lambda = f^{-1}(\mu)$. Since f is onto, $f(f^{-1}(\mu)) = \mu$. Thus

$$f(C_1(f^{-1}(\mu), r)) \geq C_2(f(f^{-1}(\mu)), r) = C_2(\mu, r)$$

This implies that

$$C_1(f^{-1}(\mu), r) = f^{-1}(f(C_1(f^{-1}(\mu), r))) \geq f^{-1}(C_2(\mu, r)).$$

$\Longleftarrow$. Put $\mu = f(\lambda)$. Since f is injective,

$$f^{-1}(C_2(f(\lambda), r)) \leq C_1(f^{-1}(f(\lambda)), r) = C_1(\lambda, r).$$

Since f is onto, $C_2(f(\lambda), r) \leq f(C_1(\lambda, r))$.

(2). This follows easily from:

$$\begin{aligned} & f^{-1}(C_2(\mu, r)) \leq C_1(f^{-1}(\mu), r) \\ \Longleftrightarrow\ & f^{-1}((\mathfrak{I}_{C_2}(\mu', r))') \leq (\mathfrak{I}_{C_1}(f^{-1}(\mu'), r))' \\ \Longleftrightarrow\ & f^{-1}(\mathfrak{I}_{C_2}(\mu', r)) \geq \mathfrak{I}_{C_1}(f^{-1}(\mu'), r). \end{aligned}$$

□

From the above theorems we obtain the following result.

4.13. Theorem. *Let $f\colon (X, C_1) \to (Y, C_2)$ be a bijective map between the topological L-fuzzy closure spaces (X, C_1) and (Y, C_2). Then the following statements are equivalent:*

(1) *f is a homeomorphism.*
(2) *f is a continuous map and an open map.*
(3) *f is a continuous map and a closed map.*
(4) *$f(\mathfrak{I}_{C_1}(\lambda, r)) = \mathfrak{I}_{C_2}(f(\lambda), r)$, for each $\lambda \in L^X,\ r \in L_0$.*
(5) *$f(C_1(\lambda, r)) = C_2(f(\lambda), r)$, for each $\lambda \in L^X,\ r \in L_0$.*
(6) *$\mathfrak{I}_{C_1}(f^{-1}(\mu), r) = f^{-1}(\mathfrak{I}_{C_2}(\mu, r))$, for each $\mu \in L^Y,\ r \in L_0$.*
(7) *$C_1(f^{-1}(\mu), r) = f^{-1}(C_2(\mu, r))$, for each $\mu \in L^Y,\ r \in L_0$.*

4.14. Theorem. *Let $(X, \mathfrak{T}_1), (Y, \mathfrak{T}_2)$ be L-fuzzy topological spaces, and denote the corresponding L-fuzzy closure spaces by (X, C_1), (Y, C_2) respectively. Then a function $f\colon (X, \mathfrak{T}_1) \to (Y, \mathfrak{T}_2)$ is LF-continuous iff $f\colon (X, C_1) \to (Y, C_2)$ is a continuous map.*

Proof. Let f be LF-continuous. Then for all $\lambda \in L^X,\ r \in L_0$

$$C_2(f(\lambda), r) = C_{\mathcal{T}'_2}(f(\lambda),\ r) = \bigwedge\{\mu \mid \mu \in L^Y,\ \mu \geq f(\lambda),\ \mathcal{T}_2(\mu') \geq r\}.$$

(See Propositions 2.3 and 3.2). But from $\mu \geq f(\lambda)$ we will have $f^{-1}(\mu) \geq \lambda$, while from the definition of LF-continuity, $r \leq \mathcal{T}_2(\mu') \leq \mathcal{T}_1(f^{-1}(\mu')) = \mathcal{T}_1(f^{-1}(\mu)')$. Thus we can write:

$$\begin{aligned} C_2(f(\lambda), r) &= \bigwedge \{\mu \in L^Y \mid \mu \geq f(\lambda),\ r \leq \mathcal{T}_2(\mu')\} \\ &\geq \bigwedge \{ff^{-1}(\mu) \mid f^{-1}(\mu) \geq \lambda,\ \mathcal{T}_1(f^{-1}(\mu)') \geq r\} \\ &\geq f\big(\bigwedge \{f^{-1}(\mu) \mid f^{-1}(\mu) \geq \lambda,\ \mathcal{T}_1(f^{-1}(\mu)') \geq r\}\big) \\ &\geq f(C_1(\lambda, r)) \end{aligned}$$

Then $f(C_1(\lambda, r)) \leq C_2(f(\lambda), r)$, i.e f is a continuous map.

Conversely, let f be a continuous map. It will be sufficient to prove that $\mathcal{T}'_2(\mu) \leq \mathcal{T}_1(f^{-1}(\mu))\ \forall \mu \in L^Y$ (see Proposition 2.3). Take $\mu \in L^Y$. By Theorem 3.4 we have $\mathcal{T}'_2 = \mathcal{T}'_{C_2}$, so by Proposition 3.3 we must prove that $\bigvee\{r \in L_0 \mid C_2(\mu, r) = \mu\} \leq \mathcal{T}'_2(f^{-1}(\mu))$. This is true if $C_2(\mu, r) = \mu \implies r \leq \mathcal{T}'_2(f^{-1}(\mu)$, so suppose there exists some $r_0 \in L_0$ satisfying $C_2(\mu, r_0) = \mu$ and $r_0 \nleq \mathcal{T}'_2(f^{-1}(\mu))$.

Since f is a continuous map and using Lemma 2.5 we have $f(C_1(f^{-1}(\mu), r_0)) \leq C_2(\mu, r_0) = \mu$. This leads to:

$$C_1(f^{-1}(\mu, r_0)) \leq f^{-1}(f(C_1(f^{-1}(\mu), r_0))) \leq f^{-1}(\nu),$$

that is,

$$C_1(f^{-1}(\mu), r_0) \leq f^{-1}(\mu).$$

Hence $C_1(f^{-1}(\mu), r_0) = f^{-1}(\mu)$ since $f^{-1}(\mu) \leq C_1(f^{-1}(\mu), r_0)$. Again by Theorem 3.4 and Proposition 3.3 we have $\mathcal{T}'_1(f^{-1}(\mu)) = \mathcal{T}'_{C_1}(f^{-1}(\mu)) = \bigvee\{r \in L_0 \mid C_1(f^{-1}(\mu), r) = f^{-1}(\mu)\} \geq r_0$, which is contradiction. □

4.15. Theorem. *Let $(X, C_1), (Y, C_2)$ be L-fuzzy closure spaces. If $f\colon (X, C_1) \to (Y, C_2)$ is a continuous map then $f\colon (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is LF-continuous, but the converse is false in general. Here, $\mathcal{T}_1, \mathcal{T}_2$ are defined by the equalities $\mathcal{T}_1(\lambda) = \mathcal{T}'_{C_1}(\lambda')$ and $\mathcal{T}_{C_2}(\lambda) = \mathcal{T}'_{C_2}(\lambda')$.*

Proof. The proof of continuity $\implies$ LF-continuity in Theorem 4.14 relies on the equalities $\mathcal{T}'_k = \mathcal{T}'_{C_k}$, $k = 1, 2$. Here these equalities hold by definition, so essentially the same proof holds here too.

To show that the converse is false in general, consider the following example.

4.16. Example. Let $X = \{x, y, z\}$. We denote by χ_A the characteristic function of a subset A of X. Let $L = [0, 1] = I$, so that $I_0 = (0, 1]$.

We define $C_1, C_2 : I^X \times I_0 \longrightarrow I^X$ as follows:

$$C_1(\lambda, r) = \begin{cases} \underline{0} & \text{if } \lambda = \underline{0},\ r \in I_0 \\ \chi_{\{z\}} & \text{if } \lambda = z_s,\ s \in I_0,\ 0 < r \leq \frac{1}{2} \\ \underline{1} & \text{otherwise,} \end{cases}$$

and

$$C_2(\lambda, r) = \begin{cases} \underline{0} & \text{if } \lambda = \underline{0},\ r \in I_0 \\ \chi_{\{x,y\}} & \text{if } \lambda = x_s,\ s \in I_0,\ 0 < r \leq \frac{1}{3} \\ \chi_{\{z\}} & \text{if } \lambda = z_s,\ s \in I_0,\ r \leq \frac{1}{2} \\ \underline{1} & \text{otherwise,} \end{cases}$$

where x_s, z_s denote fuzzy points. Then the identity map $\mathrm{id}_X\colon (X, C_1) \to (X, C_2)$ is not a continuous map because for any $s \in I_0$,

$$\underline{1} = C_1(x_s, \tfrac{1}{4}) \nleq C_2(x_s, \tfrac{1}{4}) = \chi_{\{x,y\}}.$$

On the other hand, from the definition of $\mathcal{T}_{C_1}, \mathcal{T}_{C_2}\colon I^X \to I$:

$$\mathcal{T}_{C_1}(\lambda) = \mathcal{T}_{C_2}(\lambda) \begin{cases} 1 & \text{if } \lambda = \underline{0} \text{ or } \underline{1} \\ \frac{1}{2} & \text{if } \lambda = \chi_{\{x,y\}}, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $\mathrm{id}_X\colon (X, \mathcal{T}_{C_1}) \to (X, \mathcal{T}_{C_2})$ is LF-continuous.

This shows that for the above fuzzy I-closure spaces, id_X is an LF-continuous mapping which is not continuous. □

4.17. Theorem. *Let $(X, C_1), (Y, C_2)$ be L-fuzzy closure spaces and $f\colon (X, C_1) \to (Y, C_2)$ a map. Then the following statements are equivalent:*

(1) *f is a continuous map.*
(2) *$C_1(f^{-1}(\nu), r) \leq f^{-1}(C_2(\nu, r)),\ \forall \nu \in L^Y,\ r \in L_0$.*
(3) *$f^{-1}(\mathcal{I}_{C_2}(\nu, r)) \leq \mathcal{I}_{C_1}(f^{-1}(\nu), r),\ \forall \nu \in L^Y,\ r \in L_0$.*

Proof. (1) $\Longrightarrow$ (2). Let $\nu \in L^Y$ and set $\mu = f^{-1}(\nu)$ in $f(C_1(\mu, r)) \leq C_2(f(\mu), r)$. Since $C_2(ff^{-1}(\nu), r) \leq C_2(\nu, r)$, we get $f(C_1(f^{-1}(\nu), r)) \leq C_2(\nu, r)$. Thus $C_1(f^{-1}(\nu), r) \leq f^{-1}(C_2(\nu, r)), \forall \nu \in L^X$.

(2) $\Longrightarrow$ (1). Take $\mu \in L^X$. Using (2) this leads to $C_1(f^{-1}(f(\mu)), r) \leq f^{-1}(C_2(f(\mu), r))$. Hence, $C_1(\mu, r) \leq f^{-1}(C_2(f(\mu), r))$, and so $f(C_1(\mu, r)) \leq C_2(f(\mu), r)$. So, f is continuous.

(2) $\Longrightarrow$ (3). Since $C_1(f^{-1}(\nu'), r) \leq f^{-1}(C_2(\nu', r))$, then applying the involution $'$ to both sides gives $(f^{-1}(C_2(\nu', r)))' \leq (C_1(f^{-1}(\nu'), r))'$. However, $(f^{-1}(C_2(\nu', r)))' = f^{-1}(C_2(\nu', r)')$, so we have

$$f^{-1}(\mathcal{I}_{C_2}(\nu, r)) \leq \mathcal{I}_{C_1}(f^{-1}(\nu), r).$$

(3) $\Longrightarrow$ (2). Trivial from the definition of $\mathcal{I}_C$. □

If $C\colon L^X \times L_0 \to L^X$ is a L-fuzzy closure operator on X, then for each $r \in L_0$, $C_r\colon L^X \to L^X$ defined by $C_r(\lambda) = C(\lambda, r)$ is a Chang L-fuzzy closure operator on X [4].

5. r-semiopen and r-semiclosed sets in L-fuzzy closure spaces

5.1. Definition. Let (X, C) be L-fuzzy closure space. For $\lambda \in L^X$ and $r \in L_0$:

(1) λ is called an *r-semiopen set* if there exists an r-open set $\nu \in L^X$ such that $\nu \leq \lambda \leq C(\nu, r)$.
(2) λ is called an *r-semiclosed set* if there exists an r-closed set $\nu \in L^X$ such that $\mathcal{I}_C(\nu, r) \leq \lambda \leq \nu$.

5.2. Remark. (1) If λ is r-semiopen then $\lambda \leq C(\mathcal{I}_C(\lambda, r), r)$. Conversely, if this inequality is satisfied and (X, C) is topological then λ is r-semiopen.

(2) If λ is r-semiclosed then $\mathcal{I}_C(C(\lambda, r), r) \leq \lambda$. Conversely, if this inequality is satisfied and (X, C) is topological then λ is r-semiclosed.

5.3. Definition. Let (X, C) be an L-fuzzy closure space. The *r-semiclosure* $SC(\mu, r)$, $r \in L_0$, $\mu \in L^X$, is defined by

$$SC(\mu, r) = \bigwedge \{\rho \in L^X \mid \mu \leq \rho,\ \rho \text{ is } r\text{-semiclosed}\}$$

and the *r-semi-interior* $S\mathcal{I}_C$ is defined by

$$S\mathcal{I}_C(\mu, r) = \bigvee \{\rho \in L^X \mid \mu \geq \rho,\ \rho \text{ is } r\text{-semiopen}\}.$$

From the above definitions we clearly have $S\mathcal{I}_C(\mu, r) \leq \mu \leq SC(\mu, r)$, while if (X, C) is topological,

$$\mathcal{I}_C(\mu, r) \leq S\mathcal{I}_C(\mu, r) \leq \mu \leq SC(\mu, r) \leq C(\mu, r).$$

5.4. Remark. Since an arbitrary union of r-open sets is r-open by Proposition 4.2, it is easy to show that an arbitrary union of r-semiopen sets is r-semiopen. Hence, in particular, the r-semi-interior of $\mu \in L^X$ is r-semiopen.

In just the same way, an arbitrary intersection of r-semiclosed sets is r-semiclosed, and the r-semiclosure of $\mu \in L^X$ is r-semiclosed.

5.5. Definition. Let $f\colon (X, C_1) \to (Y, C_2)$ be a map from an L-fuzzy closure space (X, C_1) to another L-fuzzy closure space (Y, C_2), and $r \in L_0$. Then f is called:

(1) A *semicontinuous map* if $f^{-1}(\nu)$ is an r-semiopen set for each r-open set $\nu \in L^Y$, or equivalently, if $f^{-1}(\nu)$ is an r-semiclosed set for each r-closed set $\nu \in L^Y$.
(2) A *semiopen map* if $f(\mu)$ is an r-semiopen set for each r-open set $\mu \in L^X$.
(3) A *semiclosed map* if $f(\mu)$ is an r-semiclosed set for each r-closed set $\mu \in L^X$.

5.6. Theorem. *Let (X, C_1), (Y, C_2) be topological L-fuzzy closure spaces. Then the following are equivalent for a map $f\colon (X, C_1) \to (Y, C_2)$.*

(1) *f is a semicontinuous map.*
(2) *$\mathcal{I}_{C_1}(C_1(f^{-1}(\nu), r), r) \leq f^{-1}(C_2(\nu, r))$ for each $\nu \in L^Y$, $r \in L_0$.*
(3) *$f(\mathcal{I}_{C_1}(C(\mu, r), r)) \leq C_2(f(\mu), r)$ for each $\mu \in L^X$, $r \in L_0$.*

Proof. : (1) $\Longrightarrow$ (2). Let f be a semicontinuous map, $\nu \in L^Y$. Then $C_2(\nu, r)$ is r-closed since (X, C_2) is topological, so since f is a semicontinuous map, $f^{-1}(C_2(\nu, r))$ is r-semiclosed. Thus

$$f^{-1}(C_2(\nu, r)) \geq \mathcal{I}_{C_1}(C_1(f^{-1}(C_2(\nu, r)), r), r) \geq \mathcal{I}_{C_1}(C_1(f^{-1}(\nu), r), r).$$

(2) $\Longrightarrow$ (3). Let $\mu \in L^X$. Then $f(\mu) \in L^Y$. By (2),

$$f^{-1}(C_2(f(\mu), r)) \geq \mathcal{I}_{C_1}(C_1(f^{-1}f(\mu), r)) \geq \mathcal{I}_{C_1}(C_1(\mu, r), r).$$

Hence

$$C_2(f(\mu), r) \geq ff^{-1}(C_2(f(\mu), r)) \geq f(\mathcal{I}_{C_1}(C_1(\mu, r), r)).$$

(3) $\Longrightarrow$ (1). Let ν be an r-closed set. Since $f^{-1}(\nu) \in L^X$ we have by (3),

$$f(\mathcal{I}_{C_1}(C_1(f^{-1}(\nu), r), r) \leq C_2(ff^{-1}(\nu), r) \leq C_2(\nu, r) = \nu.$$

So

$$\mathcal{I}_{C_1}(C_1(f^{-1}(\nu), r), r) \leq f^{-1}f(\mathcal{I}_{C_1}(C_1(f^{-1}(\nu), r), r)) \leq f^{-1}(\nu).$$

Since (X, C_1) is topological, $f^{-1}(\nu)$ is an r-semiclosed set by Remark 5.2 (2), and hence f is a semicontinuous map. □

5.7. Remark. Clearly, every continuous (resp. open, closed) map is a semicontinuous (resp. semiopen, semiclosed) map.

5.8. Theorem. *Let (X, C_1), (Y, C_2) be topological L-fuzzy closure spaces. Then the following statements are equivalent for the map $f\colon (X, C_1) \to (Y, C_2)$.*

(1) *f is a semicontinuous map.*
(2) *$f(SC_1(\mu, r)) \leq C_2(f(\mu), r)$ for each $\mu \in L^X$, $r \in L_0$.*
(3) *$SC_1(f^{-1}(\nu), r) \leq f^{-1}(C_2(\nu, r))$ for each $\nu \in L^Y$, $r \in L_0$.*
(4) *$f^{-1}(\mathcal{I}_{C_2}(\nu, r)) \leq S\mathcal{I}_{C_1}(f^{-1}(\nu), r)$ for each $\nu \in L^Y$.*

Proof. Left to the reader. □

5.9. Theorem. *For L-fuzzy closure spaces (X, C_1), (Y, C_2) with (Y, C_2) topological, let $f\colon (X, C_1) \to (Y, C_2)$ be a bijection. Then f is a semicontinuous map iff $\mathcal{I}_{C_2}(f(\mu), r) \leq f(S\mathcal{I}_{C_1}(\mu, r))$ for each $\mu \in L^X$ and $r \in L_0$.*

Proof. Let f be a semicontinuous map and $\mu \in L^X$. By hypothesis $\mathcal{I}_{C_2}(f(\mu), r)$ is r-open, so $f^{-1}(\mathcal{I}_{C_1}(f(\mu), r))$ is r-semiopen. Since f is one to one, we have

$$f^{-1}(\mathcal{I}_{C_2}(f(\mu), r)) \leq S\mathcal{I}_{C_1}(f^{-1}f(\mu), r) = S\mathcal{I}_{C_1}(\mu, r).$$

Since f is onto,

$$\mathcal{I}_{C_2}(f(\mu), r) = ff^{-1}(\mathcal{I}_{C_2}(f(\mu), r)) \leq f(S\mathcal{I}_{C_1}(\mu, r)).$$

Conversely, let ν be an r-open set. Then $\mathcal{I}_{C_2}(\nu, r) = \nu$. Since f is onto,

$$f(S\mathcal{I}_{C_1}(f^{-1}(\nu), r)) \geq \mathcal{I}_{C_2}(ff^{-1}(\nu), r) = \mathcal{I}_{C_2}(\nu, r) = \nu.$$

Since f is one to one, we have

$$f^{-1}(\nu) \leq f^{-1}f(S\mathcal{I}_{C_1}(f^{-1}(\nu), r)) = S\mathcal{I}_{C_1}(f^{-1}(\nu), r) \leq f^{-1}(\nu).$$

Thus $f^{-1}(\nu) = S\mathcal{I}_{C_1}(f^{-1}(\nu), r)$, and hence $f^{-1}(\nu)$ is r-semiopen. Therefore f is a semicontinuous map. □

5.10. Theorem. *Let (X, C_1), (Y, C_2) be L-fuzzy closure spaces with (X, C_1) topological. Then the following statements are equivalent for a map $f\colon (X, C_1) \to (Y, C_2)$.*

(1) *f is a semiopen map.*
(2) *$f(\mathcal{I}_{C_1}(\mu, r)) \leq S\mathcal{I}_{C_2}(f(\mu), r)$ for each $\mu \in L^X$, $r \in L_0$.*
(3) *$\mathcal{I}_{C_1}(f^{-1}(\nu), r) \leq f^{-1}(S\mathcal{I}_{C_2}(\nu, r))$ for each $\nu \in L^Y$, $r \in L_0$.*

Proof. (1) $\Longrightarrow$ (2). Take $\mu \in L^X$. By hypothesis $\mathcal{I}_{C_1}(\mu, r)$ is an r-open set. Hence, since f is a semiopen map, $f(\mathcal{I}_{C_1}(\mu, r))$ is an r-semiopen set. Thus

$$f(\mathcal{I}_{C_1}(\mu, r)) = S\mathcal{I}_{C_2}(f(\mathcal{I}_{C_1}(\mu, r)), r) \leq S\mathcal{I}_{C_2}(f(\mu), r).$$

(2) $\Longrightarrow$ (3). Let $\nu \in L^Y$. Then $f^{-1}(\nu) \in L^X$. By (2),

$$f(\mathcal{I}_{C_1}(f^{-1}(\nu), r)) \leq S\mathcal{I}_{C_2}(ff^{-1}(\nu), r) \leq S\mathcal{I}_{C_2}(\nu, r).$$

Thus we have

$$\mathcal{I}_{C_1}(f^{-1}(\nu), r) \leq f^{-1}f(\mathcal{I}_{C_1}(f^{-1}(\nu), r)) \leq f^{-1}(S\mathcal{I}_{C_2}(\nu, r)).$$

(3) $\Longrightarrow$ (1). Let μ be an r-open set. Then $\mathcal{I}_{C_1}(\mu, r) = \mu$. Since $f(\mu) \in L^Y$ we have by (3),

$$\mu = \mathcal{I}_{C_1}(\mu, r) \leq \mathcal{I}_{C_1}(f^{-1}f(\mu), r) \leq f^{-1}(S\mathcal{I}_{C_2}(f(\mu), r)).$$

Hence we have

$$f(\mu) \leq ff^{-1}(S\mathfrak{I}_{C_2}(f(\mu), r)) \leq S\mathfrak{I}_{C_2}(f(\mu), r) \leq f(\mu).$$

Thus $f(\mu) = S\mathfrak{I}_{C_2}(f(\mu), r)$, and so $f(\mu)$ is an r-semiopen by Remark 5.4. Therefore, f is a semiopen map. □

5.11. Theorem. *Let (X, C_1), (Y, C_2) be L-fuzzy closure spaces with (X, C_1) topological. Then the following statements are equivalent for a map $f\colon (X, C_1) \to (Y, C_2)$.*

(1) *f is a semiclosed map.*
(2) *$SC_2(f(\mu), r) \leq f(C_1(\mu, r))$ for each $\mu \in L^X$, $r \in L_0$.*

Proof. (1) $\Longrightarrow$ (2). Let $\mu \in L^X$. By hypothesis, $C_1(\mu, r)$ is an r-closed set. Since f is a semiclosed map, $f(C_1(\mu, r))$ is an r-semiclosed set. Thus we have

$$SC_2(f(\mu), r) \leq SC_2(f(C_1(\mu, r)), r) = f(C_1(\mu, r)).$$

(2) $\Longrightarrow$ (1). Let μ be an r-closed set. Then $C_1(\mu, r) = \mu$. By (2),

$$SC_2(f(\mu), r) \leq f(C_1(\mu, r)) = f(\mu) \leq SC_2(f(\mu), r).$$

Thus $f(\mu) = SC_2(f(\mu), r)$, and hence $f(\mu)$ is r-semiclosed by Remark 5.4. Therefore, f is a semiclosed map. □

5.12. Theorem. *For L-fuzzy closure spaces (X, C_1), (Y, C_2) with (X, C_1) topological, let $f\colon (X, C_1) \to (Y, C_2)$ be a bijection. Then f is a semiclosed map iff $f^{-1}(SC_2(\nu, r)) \leq C_1(f^{-1}(\nu), r)$ for each $\nu \in L^Y$ $r \in L_0$.*

Proof. Let f be a semiclosed map and $\nu \in L^Y$. Then $f^{-1}(\nu) \in L^X$. Since f is onto, we have

$$SC_2(\nu, r) = SC_2(ff^{-1}(\nu), r) \leq f(C_1(f^{-1}(\nu), r))$$

by Theorem 5.10. Since f is one to one, we have

$$f^{-1}(SC_2(\nu, r)) \leq f^{-1}f(C_1(f^{-1}(\nu), r)) = C_1(f^{-1}(\nu), r).$$

Conversely, let μ be r-closed, Then $C_1(\mu, r) = \mu$. Since f is onto, we have

$$SC_2(f(\mu), r) = ff^{-1}(SC_2(f(\mu), r)) \leq f(\mu) \leq SC_2(f(\mu), r).$$

Thus $f(\mu) = SC_2(f(\mu), r)$, and hence $f(\mu)$ is r- semiclosed. Therefore f is a semiclosed map. □

5.13. Theorem. *Let (X, C_1), (Y, C_2), (Z, C_3) be L-fuzzy closure spaces. Let $f\colon (X, C_1) \to (Y, C_2)$ and $g\colon (Y, C_2) \to (Z, C_3)$ be open maps. Then the composition $gof\colon X \to Z$ is an open map.*

Proof. Straightforward. □

5.14. Theorem. *Let (X, C_1), (Y, C_2), (Z, C_3) be L-fuzzy closure spaces. Let $f\colon (X, C_1) \to (Y, C_2)$ and $g\colon (Y, C_2) \to (Z, C_3)$ be closed maps. Then the composition $gof\colon X \to Z$ is a closed map.*

Proof. Straightforward. □

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