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ON THE CONSTRUCTION OF ORTHOGONAL BALANCED INCOMPLETE BLOCK DESIGNS

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Abstract

Orthogonal Incomplete Block Designs are a generalization of Balanced Incomplete Block Designs, and are obtained by piling s Balanced Incomplete Block Designs, $s > 1$, on top of one another. Orthogonal Balanced Incomplete Block Designs can be generated from a set of Initial Block Design. A convenient way to represent the properties of the initial blocks is to use difference squares. The purpose of this study is to point out some connections between Orthogonal Balanced Incomplete Block Designs and Difference Squares. It also gives the essential definitions and properties of Orthogonal Incomplete Block Designs.

Keywords: Balanced incomplete block design, Orthogonal balanced incomplete block design, Difference squares.

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1. Introduction

In standard notation (Mathon and Rosa [3], Wallis [9]), a balanced incomplete block design (BIBD) with parameters (v, b, r, k, λ) is an arrangement of v treatments (sometimes called 'varieties' or 'points') in b blocks, each of size k, where $k < v$, such that

(i) Each treatment appears exactly $r = bk/v$ times overall,
(ii) Each treatment occurs no more than once per block, and
(iii) Each unordered pair of its treatments appears in exactly $\lambda = r(k-1)/(v-1)$ blocks, the parameter λ often being referred to as the 'concurrence parameter' of the BIBD.

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A BIBD is 'unreduced' if its blocks are the distinct k-subsets of the treatments, each such subset occurring just once. Thus b and k in an unreduced BIBD can be calculated as the following combinations (Mathon and Rosa [3])

$$b = \binom{v}{k}, \; r = \binom{v-1}{k-1}$$

The designs under consideration here are a generalization of balanced incomplete block designs (BIBDs) in which the single set of v treatments is replaced by s sets of v, each of which forms a BIBD(v, b, r, k, λ). We superimpose these s BIBDs on the same block set, applied in order, so that our 1 by bk array of v treatments can be regarded as being replaced by an s by bk ordered array, and the single treatment at each plot can be regarded as being replaced by an array of s treatments, one drawn from each set, in the given order. Moreover, each treatment from a given set appears in exactly one column of the array with every treatment, bar one, of every other set, and appears in a block exactly k times with every treatment, bar one, of every other set, and $k-1$ times with every other treatment of its own set. (Giving the same numbering to each of the sets, it may be assumed that treatment x of one set appears with every treatment, except x, of every other set.) Note that the replication is $r = (v-1)$ and that $\lambda = (k-1)$, (Wallis [9], Morgan *et al* [4]).

In short, we may suppose that a set of s BIBDs (with identical parameters) have been superimposed one upon the other, in order, so that a single treatment at a given plot is replaced by a vector of s treatments.

Further, if the incidence matrix of the ith set of treatment blocks is N_{i0} and that of the jth is N_{ij} then $N_{ij} - N_{i0}N'_{j0} = 0$ (Rees [8]).

These designs are a subclass of the orthogonal balanced incomplete block designs.

2. Properties of OBIBDs

Within a given treatment set, each treatment appears within a block equally often with every other treatment.

Across treatment sets, suppose that treatment A of the ith set is superimposed on treatment B of the jth in m blocks. Then over all the blocks, treatment A of the ith must occur in the same block as treatment B of the jth on mk occasions (including the m superimpositions).

Three main types of OBIBD have been investigated to date.

1. Those in which each treatment of a given set occurs an equal number of times with every treatment of every other set, except itself.
2. The second type is similar, save that now each treatment in a given set is allowed to occur with its analogue in every other set.
3. The third kind has each treatment of one set occurring with a subset of the other set, different treatments having different subsets, but any pair of subsets having a fixed number of treatments in common.

In all cases, they are balanced. The original definition of designs of this type is due to Preece [6], the current definition and terminology (of OBIBD) is essentially due to Morgan & Uddin [5].

All three can be summarized by saying that $N_{ij}N'_{ij} = aI + bJ$ for some integers a and b, where I is the identity matrix of order v, and J is the matrix of order v comprising all 1's. In the first case, N is the incidence matrix of the unreduced design for v treatments in blocks of $(v-1)$; in the second case, the same design is augmented by occurrences of the missing treatment, and in the third case N is the incidence matrix of a non-trivial symmetric BIBD [1].

Treatment x of set i never occurs in the same plot or block as treatment x of set j, so, when the distinction between the treatment sets is ignored, each block contains ks distinct treatments, and so v≥ks is a necessary condition for the existence of an OBIBD$(v, k, \lambda; s)$.

2.1. Example. The 24 blocks below constitute an OBIBD$(9, 3, 2; 3)$. The blocks are enclosed in parentheses, and the plots separated by semi-colons; within each plot the treatments from the different sets are given in order, and each treatment set is $Z_8 \cup \{\infty\}$, where ∞ is any symbol.

The three blocks in a row form a parallel class for each treatment set.

(∞; 1; 5; 0; 7; 2; 4; 6; 3); (1; 5;∞; 7; 2; 0; 6; 3; 4); (5; ∞; 1; 2; 0; 7; 3; 4; 6);
(∞; 2; 6; 1; 0; 3; 5; 7; 4); (2; 6; ∞; 0; 3; 1; 7; 4; 5); (6; ∞; 2; 3; 1; 0; 4; 5; 7);
(∞; 3; 7; 2; 1; 4; 6; 0; 5); (3; 7; ∞; 1; 4; 2; 0; 5; 6); (7; ∞; 3; 4; 2; 1; 5; 6; 0);
(∞; 4; 0; 3; 2; 5; 7; 1; 6); (4; 0; ∞; 2; 5; 3; 1; 6; 7); (0; ∞; 4; 5; 3; 2; 6; 7; 1);
(∞; 5; 1; 4; 3; 6; 0; 2; 7); (5; 1; ∞; 3; 6; 4; 2; 7; 0); (1; ∞; 5; 6; 4; 3; 7; 0; 2);
(∞; 6; 2; 5; 4; 7; 1; 3; 0); (6; 2; ∞; 4; 7; 5; 3; 0; 1); (2; ∞; 6; 7; 5; 4; 0; 1; 3);
(∞; 7; 3; 6; 5; 0; 2; 4; 1); (7; 3; ∞; 5; 0; 6; 4; 1; 2); (3; ∞; 7; 0; 6; 5; 1; 2; 4);
(∞; 0; 4; 7; 6; 1; 3; 5; 2); (0; 4; ∞; 6; 1; 7; 5; 2; 3); (4; ∞; 0; 1; 7; 6; 2; 3; 5)

OBIBD designs can be generated from a set of initial blocks (an initial block is a block design which has the form $(q3+1, q+1, 1)$). A convenient way to represent the properties of the initial blocks is to use difference squares (Greig and Rees [2]).

3. Difference squares for OBIBDs

Difference squares were introduced by Rees [8]. For each initial block, set up a k by k square in which the entries are the differences between the k treatments of the first set and the k treatments of the second set. The blocks of both sets of treatments must consist of the initial blocks of a BIBD with the appropriate parameters. If we order the k treatments of each treatment set by their plot order, then the combined set of diagonals comprises an equal number, say m, of occurrences of each of these differences, while the total set of differences in the bodies of the tables must comprise km occurrences of each of the differences possible for that difference set. For OBIBDs, $m = 1$, (Greig and Rees [2]).

With multiple sets of treatments, such squares and transversals have to be set up consistently between all pairs of treatment sets.

3.1. Example. The design of Example 2.1 is developed modulo 8 from the initial blocks:

(∞, 1, 5; 0, 7, 2; 4, 6, 3), (1, 5, ∞; 7, 2, 0; 6, 3, 4); (5, ∞, 1; 2, 0, 7; 3, 4, 6).

The difference squares of the first treatments with respect to the second are:

	1	7	6
∞	−∞	−∞	−∞
0	1	7	6
4	5	3	2

	5	2	3
1	4	1	2
7	6	3	4
6	7	4	5

	∞	0	4
5	∞	3	7
2	∞	6	2
3	∞	5	1

The difference squares for the first set of treatments with respect to the third, and of the second with respect to the third, are very similar.

It is immediately seen that each difference occurs 3 (i.e., k) times in the body of the tables, while each occurs once on the main diagonals, corresponding to the pairs of treatments at each plot.

These designs are a subclass of a much wider class of generalizations of BIBDs to several sets of treatments. Designs similar to those under consideration here were first developed for orchard trials, where successive sets of treatments would be applied to a fixed set of blocks (Preece [7]), and were first introduced by Preece [6]; see also Morgan and Uddin [5], Rees [8].

3.2. Definition. (Discrete Logarithms) Let G be a finite cyclic group with n elements. We assume that the group is written multiplicatively. Let b be a generator of G; then every element x of G can be written in the form $x = b^k$ for some integer k. Furthermore, any two such integers representing x will be congruent modulo n. We can thus define a function

$$\log_b : G \to \mathbb{Z}_n$$

(where $\mathbb{Z}_n$ denotes the ring of integers modulo n) by assigning to x the congruence class of k modulo n. This function is a group isomorphism, called the *discrete logarithm to base b*.

The familiar base change formula for ordinary logarithms remains valid: if c is another generator of G, then we have

$$\log_c(x) = \log_c(b).\log_b(x).$$

3.3. Theorem. (Greig and Rees [2]) *Let $q = p^n$ be a prime power, with $n > 1$, let $k = p^u$ for $0 < u < n$ and let $t = p^{(n-u)}$. Define log(0)=∞. Let:*

(i) *$C_0 \equiv \{\alpha_0 = 0, \alpha_1, \dots, \alpha_{k-1}\}$ be the additive subgroup of order p^u of $(GF(q); +)$;*
(ii) *$\{Cj : j = 1, 2, \dots, (t-1)\}$ be the cosets of C_0 in $(GF(q); +)$;*
(iii) *$b^i_j = \log(C_{i+j}) \equiv \{\log(c) : c \in C_{i+j}\}$ where the discrete logarithms are taken with respect to some fixed generator of $GF(q)$.*

Then $b^i_0, b^i_1, \dots, b^i_{t-1}$ are initial blocks for a 1-rotational OBIBD$(p^n, p^u, p^u - 1; t)$ *over* $\mathbb{Z}_{q-1} \cup \{\infty\}$.

3.4. Example. Consider a design for 3^2 treatments in blocks of 3. To follow the first method of proof of Greig and Rees [2], write the elements of $GF(9)$ as x^i, where:

$$\begin{array}{ccccccccccc} x^i & = & 0 & 1 & 2 & x & x+1 & x+2 & 2x & 2x+1 & 2x+2 \\ i & = & \infty & 0 & 4 & 1 & 7 & 6 & 5 & 2 & 3 \end{array}$$

C_0 is $\{0, 1, 2\}$, with additive cosets $C_0 + x$ and C_0+2x; the subgroup fixing these cosets is evidently the group generated by $y \mapsto y + 1 \pmod 3$, isomorphic to C_0 itself. Now take a pair of the cosets $0, x;\ 1, x+1;\ 2, x+2$, and consider the difference square described by:

	x	$x+1$	$x+2$
0	x, 0	x+1, 0	x+2, 0
1	x, 1	x+1, 1	x+2, 1
2	x, 2	x+1, 2	x+2, 2

Under the subgroup $y \mapsto y+1 \pmod 3$ this difference square is fixed, the elements in the body of the square being permuted diagonally. So the full design generated by the doubly transitive group of order 72 is an OBIBD with $72/3 = 24$ blocks. The same applies for every pair of such cosets, so the design can have three sets of treatments. Following a process analogous to that of taking additive cosets gives the following blocks:

$$(0,\ x,\ 2x;\ 1,\ x+1,\ 2x+1;\ 2,\ x+2,\ 2x+2),$$
$$(x,\ 2x,\ 0;\ x+1,\ 2x+1,\ 1;\ x+2,\ 2x+2,\ 2),$$
$$(2x,\ 0,\ x;\ 2x+1,\ 1,\ x+1;\ 2x+2,\ 2,\ x+2).$$

The full design could now be generated mutiplicatively. Instead, taking discrete logarithms, the initial blocks are as follows:

$$(\infty,\ 1,\ 5;\ 0,\ 7,\ 2;\ 4,\ 6,\ 3),\ (1,\ 5,\ \infty;\ 7,\ 2,\ 0;\ 6,\ 3,\ 4),\ (5,\ \infty,\ 1;\ 2,\ 0,\ 7;\ 3,\ 4,\ 6).$$

These are developed over $Z_8 \cup \{\infty\}$, as in Examples 2.1 and 3.1.

To follow the alternative approach of Greig and Rees [2], the elements of the group $GF(q)$ are re-arranged in the following table:

	0	1	2
0	∞	0	4
x	1	7	6
$2x$	5	2	3

where the elements of the group head the columns after the first, and the representatives of the cosets appear in the first column. The discrete logarithms of the elements of the field appear in the body of the table. Now suppose that $C = 1$, i.e., consider the differences between each entry and its successor in the row. The differences are, by rows from the top left,

$$0-\infty = -\infty,\ 4-0 = 4,\ \infty-4 = \infty,\ 7-1 = 6,\ 6-7 = 7,$$
$$1-6 = 3,\ 2-5 = 5,\ 3-2 = 1 \text{ and } 5-3 = 2,$$

as required. The same would apply if $C = 2$. So that defines the BIBDs. Now consider the differences between sets of treatments at a plot (so $R = 1$ and $C = 0$). Starting at the top left as before, the differences are

$$\infty-1 = \infty,\ 0-7 = 1,\ 4-6 = 6,\ 1-5 = 4,\ 7-2 = 5,$$
$$6-3 = 3,\ 5-\infty = -\infty,\ 2-0 = 2 \text{ and } 3-4 = 7,$$

as required. The same would apply for the other block differences between sets, starting with $\infty - 7$ and $\infty - 6$, in turn.

4. Conclusion

OBIBD's are a subclass of a much wider class of generalizations of BIBD's to several sets of treatments. Mathematical terminology is used in the definition of OBIBD's, but they are theoretically analyzed using a statistical approach without involving mathematical detail.

OBIBD's can be generated from a set of initial blocks. A convenient way to represent the properties of the initial blocks is to use difference squares. This method can be useful where the numbers of treatments which are small and not prime.

Note that there are methods which use the difference squares directly - in particular, search methods. Again, such methods can be useful where the numbers of treatments are small and not prime.

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