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OSCILLATION OF CUSPED EULER-BERNOULLI BEAMS AND KIRCHHOFF-LOVE PLATES

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Abstract

In this paper, mathematical problems of cusped Euler-Bernoulli beams and Kirchhoff-Love plates are considered. Changes in the beam cross-section area and the plate thickness are, in general, of non-power type. The criteria of admissibility of the classical bending boundary conditions [clamped end (edge), sliding clamped end (edge), and supported end (edge)] at the cusped end of the beam and on the cusped edge of the plate have been established. The cusped end of the beam and the cusped edge of the plate can always be free independent of the character of the sharpening. A sufficient conditions for the solvability of the vibration frequency have been established. The appropriate weighted Sobolev spaces have been constructed. The well-posedness of the admissible problems has been proved by means of the Lax-Milgram theorem.

Keywords: Cusped elastic plates, Cusped elastic beams, Vibration, Degenerate elliptic equations, Weighted spaces, Hardy's inequality.

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1. Introduction

In the early fifties of the last century I. Vekua (see [23] and also [24, 25]) raised the problem of investigation of cusped elastic plates which mathematically leads to degenerate partial differential equations and systems. At that time the study of such equations and systems was in full swing and it was interesting to find a mechanical interpretation of the so-called E (i.e., Keldysh, see [8]) problem and of weighted boundary value problems (shortly: BVPs, see [1]). The first results concerning classical bending (Kirchhoff-Love

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model) of cusped elastic plates appeared in the late fifties in works of E. V. Makhover and S. G. Mikhlin (see [14]). Then there was a gap until the early seventies when the works of A. R. Khvoles [10] and G. V. Jaiani [3] devoted to this topic were published. The last and the following works (also of other authors) are within the framework either of the classical bending model or of the zero approximation of I. Vekua's hierarchical models of plates and shells (see detailed surveys [4, 5]) and are devoted to the case of power type thicknesses. Some problems for particular case of power type cusped beams are investigated by Uzumov [22] and S. Naguleswaran (see [15] and references therein).

Hierarchical models for cusped beams with rectangular cross-sections are constructed and investigated in [6] (see also [9]).

The present paper is devoted to cusped beams and plates with non-power type thicknesses.

After this Introductory Section, Section 2 will deal with the case of beams and Section 3 with the case of plates.

§ 1.1. Beams

The *vibration equation of Euler-Bernoulli beams* has the following form (see, e.g., [20]):

(1.1) $$J^b_\omega w := J^b w - \omega^2 \rho(x_2)\sigma(x_2) w = f(x_2),\ \ 0 \le x_2 \le \ell,$$

where $w = w(x_2) \in C^4(]0,\ell[)$ is the deflection of the beam,

(1.2) $$J^b w := (D^b w_{,22})_{,22},\ \ D^b := EI,$$

$\omega = \text{const}$ is the vibration frequency, $f = f(x_2) \in C([0,\ell])$ is the intensity of the load, ℓ is the length of the beam, $\rho = \rho(x_2) \in C([0,\ell])$ is the density, $\sigma = \sigma(x_2) \in C([0,\ell])$ is the area of the transversal section lying in the plane x_1, x_3, $E = E(x_2) \in C([0,\ell])$ is the Young modulus, $I = I(x_2) \in C([0,\ell])$ is the moment of inertia with respect to the barycentric axis normal to the plane x_2x_3, and the index 2 after a comma means differentiation with respect to x_2 repeated the number of times given by the following index (hence, $w_{,22} := \partial^2 w/\partial x_2^2$, etc.)

The beam is called a *cusped* (tapered) one if $I(x_2)$ vanishes at least at one of the endpoints $x_2 = 0$ and $x_2 = \ell$ of the beam. Throughout this paper we assume once and for all that

(1.3) $$D^b(x_2) > 0 \text{ for } x_2 \in\ ]0,\ell],\ \ D^b(0) \ge 0.$$

If $\omega = 0$, we obtain the *static bending equation*

(1.4) $$J^b w = f(x_2).$$

For the *bending moment* $M^b_2 w$ and the *shearing force* $Q^b_2 w$ we have the following expressions:

(1.5) $$(M^b_2 w)(x_2) = -D^b(x_2) w_{,22}(x_2),$$

(1.6) $$(Q^b_2 w)(x_2) = (M^b_2 w)_{,2}(x_2) = -(D^b(x_2) w_{,22}(x_2))_{,2}.$$

If $D^b(0) = 0$, all the above quantities will be defined as limits as $x_2 \to 0+$.

§ 1.2. Plates

The *vibration equation of Kirchhoff-Love plates* has the following form (see, e.g., [21] or [2]):

$$\begin{aligned} J^p_\omega w &:= (D^p w_{,11})_{,11} + (D^p w_{,22})_{,22} + \nu(D^p w_{,22})_{,11} + \nu(D^p w_{,11})_{,22} \\ &\quad + 2(1-\nu)(D^p w_{,12})_{,12} - \omega^2 2h(x_1,x_2)\rho(x_1,x_2)w \\ &= f(x_1,x_2) \text{ in } \Omega \subset \mathbb{R}^2, \end{aligned} \tag{1.7}$$

where $w = w(x_1,x_2)$ is the deflection, $\omega = \text{const}$ is the vibration frequency, f is the intensity of the lateral load, Ω is a bounded plane domain with Lipschitz boundary $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$ with Γ_1 lying on the x_1 axis and Γ_2 lying in the upper half plane $\{x_2 > 0\}$, $D^p \in C^2(\Omega) \cap C(\overline{\Omega})$ is the flexural rigidity of the plate,

$$D^p := \frac{2Eh^3}{3(1-\nu^2)}, \tag{1.8}$$

$2h(x_1,x_2)$ is the thickness of the plate, $E(x_1,x_2)$ is the Young modulus, ν is the Poisson ratio, $0 < \nu < 1$, $\rho(x_1,x_2) \in C(\overline{\Omega})$ is the density and indices after comma mean again differentiation with respect to the corresponding variables.

Throughout this paper we assume once and for all that

$$D^p(x_1,x_2) > 0 \text{ on } \Omega \cup \Gamma_2,\ D^p(x_1,0) \geq 0 \text{ for } (x_1,0) \in \Gamma_1. \tag{1.9}$$

If

$$2h(x_1,x_2)|_{\Gamma_1} = 0 \tag{1.10}$$

(i.e., $2h(x_1,0) = 0$ for $(x_1,0) \in \Gamma_1$) the plate is called a *cusped* one. In this case

$$D^p(x_1,x_2)|_{\Gamma_1} = 0. \tag{1.11}$$

But (1.11) may also appear if $2h|_{\Gamma_1} > 0$ but $E|_{\Gamma_1} = 0$ (or if both quantities vanish). In all these cases, the plate will be called a cusped one, although it can be even of a constant thickness but with properties of a cusped plate caused by the vanishing of the inhomogeneous Young modulus E on Γ_1.

For the *bending moments* $M^p_\alpha w$, $\alpha = 1,2$, the *twisting moments* $M^p_{12}w$, $M^p_{21}w$, the *shearing forces* $Q^p_\alpha w$, $\alpha = 1,2$, and the *generalized shearing forces* $Q^*_\alpha w$, $\alpha = 1,2$, we have the following expressions:

$$M^p_\alpha w = -D^p(w_{,\alpha\alpha} + \nu w_{,\beta\beta}),\ \alpha,\beta = 1,2;\ \alpha \neq \beta, \tag{1.12}$$

$$M^p_{12}w = -M^p_{21}w = 2(1-\nu)D^p w_{,12}, \tag{1.13}$$

$$Q^p_\alpha w = (M^p_\alpha w)_{,\alpha} + (M^p_{21}w)_{,\beta},\ \alpha = 1,2;\ \alpha \neq \beta, \tag{1.14}$$

$$Q^*_\alpha w = Q^p_\alpha w + (M^p_{21}w)_{,\beta},\ \alpha = 1,2;\ \alpha \neq \beta. \tag{1.15}$$

At points of the boundary $\partial\Omega$ where D^p vanishes, all the above quantities will be defined as limits from Ω.

§ 1.3. Cylindrical Bending

Let us remark that if all the quantities in (1.7)–(1.15) depend only on the variable x_2, we arrive at the so-called *cylindrical bending vibration* of the rectangular plate, i.e.,

$$\Omega := \{(x_1,x_2) \in \mathbb{R}^2 : a \leq x_1 \leq b,\ 0 \leq x_2 \leq \ell\},$$

with constants a, b, ℓ, and with the boundary conditions $w_{,1} = 0$, $Q^p_1 w = 0$ at the edges $\{x_1 = a, 0 < x_2 < \ell\}$ and $\{x_1 = b, 0 < x_2 < \ell\}$ (the cases $a = -\infty$, $b = \infty$ included). On the other hand, formulas (1.7), (1.12)–(1.15) turn into (1.1), (1.5), (1.6) with D^b and σ replaced by D^p and $2h$, respectively. Therefore, all the results of Section 2 concerning cusped beams can be easily reformulated for the case of cylindrical bending of cusped plates. Moreover, from the last remark it follows that the peculiarities of setting of boundary conditions at the cusped end of a beam transfer exactly to setting of boundary conditions at the cusped edge of a plate.

Obviously, if $\omega = 0$, we get the static case.

In the case of *cusped* beams and *cusped* plates, the corresponding boundary value problems (BVPs) go out of the classical setting. The present paper deals with the question of *admissible* BVPs.

2. Cusped Euler-Bernoulli Beams

In this section we omit, for the sake of simplicity, the upper index b in $J^b, J^b_\omega, D^b, M^b_2$ and Q^b_2.

Let us recall that we assume

$$w \in C^4(]0,\ell[),\ f \in C([0,\ell]),\ D \in C^2(]0,\ell[) \cap C([0,\ell]).$$

§ 2.1. The static case ($\omega = 0$). Classical solutions

Integrating successively four times the equation (1.4) and taking into account (1.5), (1.6), we obtain:

$$(Q_2 w)(x_2) = -\int_{x_0}^{x_2} f(t)dt + C_1, \tag{2.1}$$

$$\begin{aligned}(M_2 w)(x_2) &= \int_{x_0}^{x_2} (Q_2 w)(t)dt + C_2 \\ &= -\int_{x_0}^{x_2} (x_2 - t) f(t)dt + C_1(x_2 - x_0) + C_2,\end{aligned} \tag{2.2}$$

$$w_{,2}(x_2) = -\int_{x_0}^{x_2} (M_2 w)(t) D^{-1}(t)dt + C_3, \tag{2.3}$$

$$w(x_2) = -\int_{x_0}^{x_2} (x_2 - t)(M_2 w)(t) D^{-1}(t)dt + C_3(x_2 - x_0) + C_4, \tag{2.4}$$

where $x_2 \in]0,\ell[$, $x_0 \in]0,\ell]$ is fixed, and C_i, $i = 1,2,3,4$, are arbitrary constants.

Set

$$I_k := \int_0^\ell \frac{t^k}{D(t)}dt,\ k = 0,1,2,\ldots. \tag{2.5}$$

Obviously, $I_k < \infty$ implies $I_{k+1} < \infty$ and $I_{k+1} = \infty$ implies $I_j = \infty$ for $j = 0,1,2,\ldots,k$.

2.1. Statement. *Suppose* (2.0) *and let* $w \in C^4(]0,\ell[)$ *be a solution of equation* (1.4). *Then:*

1) $w \in C^2(]0,\ell])$.

2) *If* $I_1 < \infty$*, then* $w \in C([0,\ell])$.

   [*For* $I_0 = +\infty$*, we additionally assume that either* $D \in C^2([0,\ell[)$ *or the value of the first or second order derivative of* D *at the point* $x_2 = 0$ *tends to infinity*].

3) *If* $I_1 = \infty$ *and* $I_2 < \infty$*, then* $w \in C([0,\ell])$ *if and only if*

$$(M_2 w)(0) = 0. \tag{2.6}$$

(2.6*) [*We additionally assume* (*see Remark 2.2 below*) *that either* $D \in C^3([0,\ell[)$*, or that the value of the first, second or third order derivative of* D *tends to infinity as* $x_2 \to 0+$*. Further we suppose that* f *has a bounded first derivative in some right neighbourhood of the point* 0].

If (2.6) *is violated, then w is unbounded as $x_2 \to 0+$.*

4) *If $I_k = \infty$ and $I_{k+1} < \infty$ for a fixed $k \in \{2, 3, \dots\}$, then $w \in C([0, \ell])$ if and only if*

$$(2.7) \qquad (M_2 w)(0) = 0 \text{ and } (Q_2 w)(0) = 0.$$

(2.7*) *[We additionally assume that $f^{(j)}(0) = 0$, $j = 0, 1, \dots, k-2$, and that $f^{(k-1)}$ is continuous at the point* 0.]

If (2.7) *is violated, then w is unbounded as $x_2 \to 0+$.*

5) *If $I_0 < \infty$, then $w_{,2} \in C([0, \ell])$.*

6) If $I_0 = \infty$ and $I_1 < \infty$, then $w_{,2} \in C([0, \ell])$ if and only if (2.6) *holds. If* (2.6) *is violated, then $w_{,2}$ is unbounded as $x_2 \to 0+$.*

7) *If $I_k = \infty$ and $I_{k+1} < \infty$ for a fixed $k \in \{1, 2, \dots\}$, then $w_{,2} \in C([0, \ell])$ if and only if* (2.7) *holds.*

[We additionally assume that $f^{(j)}(0) = 0$, $j = 0, 1, \dots, k-2$ (if $k \geq 2$), $f^{(k-1)}(x_2)$ is continuous at the point 0 *(if $k \geq 1$)].*

If (2.7) *is violated, then $w_{,2}$ is unbounded as $x_2 \to 0+$.*

8) $(M_2 w)(x_2) \in C([0, \ell])$.

9) $(Q_2 w)(x_2) \in C([0, \ell])$.

Proof. Since $f \in C([0, \ell])$, assertions (8) and (9) obviously follow from (2.1) and (2.2). Therefore, assertions (2) (if $I_0 < \infty$) and (5) are obvious as well. Assertion (1) follows from (2.2)–(2.4), taking into account assertions (8) and (9), and $D(\ell) > 0$ (see (1.3)). For the proof of the remaining assertions (2) (if $I_0 = \infty$ but $I_1 < \infty$), (3), (4), (6) and (7) see the Appendix. □

2.2. Remark. In Statement2.1, the restrictions on f are not substantial; we could take even $f \equiv 0$ on $[0, \ell]$. On the other hand, the above restrictions can be weakened without influence on the kernel of this statement which consists in the clarification of the question of the boundedness/unboundedness of $w, w_{,2}, M_2 w$ and $Q_2 w$ as $x_2 \to 0+$ in terms of the behaviour (that is, the nature of vanishing) of $D(x_2)$ at the point $x_2 = 0$.

2.3. Remark. The unboundedness of $w_{,2}$ geometrically means that the axis of the beam is tangent to the axis x_3, which mechanically seems hard to realize, but is acceptable in some sense. The unboundedness of the deflection is not acceptable from the point of view of mechanics but can be justified as in the case of concentrated forces.

We denote the class of functions w with the properties (1)–(9) from Statement 2.1 by

$$C^4_J(]0, \ell[).$$

In order to establish the admissible boundary conditions (shortly: BCs) at the cusped end $x_2 = 0$ of the beam, we multiply both sides of equation (1.4) by $v \in C^4_J(]0, \ell[)$ and integrate twice by parts on $]x_2, \ell]$:

$$\begin{aligned}
\int_{x_2}^{\ell} f v dx_2 &= \int_{x_2}^{\ell} (D w_{,22})_{,22} v dx_2 \\
&= \int_{x_2}^{\ell} (D w_{,22}) v_{,22} dx_2 - D w_{,22} \cdot v_{,2}\Big|_{x_2}^{\ell} + (D w_{,22})_{,2} v\Big|_{x_2}^{\ell} \qquad (2.8) \\
&= \int_{x_2}^{\ell} D w_{,22} v_{,22} dx_2 + M_2 w \cdot v_{,2}\Big|_{x_2}^{\ell} - Q_2 w \cdot v\Big|_{x_2}^{\ell}.
\end{aligned}$$

Hence, if $v = w$, we have

$$\int_{x_2}^{\ell} D(w_{,22})^2 dx_2 = \int_{x_2}^{\ell} fw dx_2 + (Q_2 w)w\big|_{x_2}^{\ell} - (M_2 w)w_{,2}\big|_{x_2}^{\ell}, \tag{2.9}$$

and the last two terms in (2.9) will give all the admissible pairs of BCs provided the limits exist as $x_2 \to 0+$.

Since at the end $x_2 = \ell$ the last two terms in (2.9) are well defined, all the four bending BCs can be set. Namely, either

$$w(\ell) = w_\ell,\ w_{,2}(\ell) = w'_\ell,$$

or

$$w_{,2}(\ell) = w'_\ell,\ (Q_2 w)(\ell) = Q_2^\ell,$$

or

$$w(\ell) = w_\ell,\ (M_2 w)(\ell) = M_2^\ell,$$

or

$$(M_2 w)(\ell) = M_2^\ell,\ (Q_2 w)(\ell) = Q_2^\ell,$$

with prescribed arbitrary constants w_ℓ, w'_ℓ, M_2^ℓ and Q_2^ℓ.

All the four bending BCs can be set also at the cusped end $x_2 = 0$. The question is only when they are admissible, which depends on the geometry of the beam's sharpening or on the character of vanishing of the Young modulus which appears in the expression of $D(x_2)$.

From this point of view, according to Statement 2.1 we arrive at the following conclusion:

1. If $I_1 < \infty$, then
$$\lim_{x_2 \to 0+} (Q_2 w)(x_2) w(x_2) = (Q_2 w)(0) w(0). \tag{2.10}$$
2. If $I_1 = \infty$ and $I_2 < \infty$, then (2.10) is valid if and only if (2.6) holds. If (2.6) is violated, then $(Q_2 w)(x_2) w(x_2)$ is unbounded as $x_2 \to 0+$.
3. If $I_k = \infty$ and $I_{k+1} < \infty$ for a fixed $k \in \{2, 3, \dots,\}$, then (2.10) is valid if and only if (2.7) holds. If (2.7) is violated, then $(Q_2 w)(x_2) w(x_2)$ is unbounded as $x_2 \to 0+$.
4. If $I_0 < \infty$, then
$$\lim_{x_2 \to 0+} (M_2 w)(x_2) w_{,2}(x_2) = (M_2 w)(0) w_{,2}(0). \tag{2.11}$$
5. If $I_0 = \infty$ and $I_1 < \infty$, then (2.11) is valid if and only if (2.6) holds. If (2.6) is violated, then $(M_2 w)(x_2) w_{,2}(x_2)$ is unbounded as $x_2 \to 0+$.
6. If $I_k = \infty$ and $I_{k+1} < \infty$ for a fixed $k \in \{1, 2, \dots\}$ then (2.11) is valid if and only if (2.7) holds. If (2.7) is violated, then $(M_2 w)(x_2) w_{,2}(x_2)$ is unbounded as $x_2 \to 0+$.

Now, it is not difficult to formulate all the admissible BCs. For instance, if at the end $x_2 = \ell$ the BCs

$$w(\ell) = w_\ell,\ w_{,2}(\ell) = w'_\ell \tag{2.12}$$

appear, then at the other (cusped) end $x_2 = 0$ the following four pairs of BCs are admissible only in the cases indicated:
Either

$$w(0) = w_0,\ w_{,2}(0) = w'_0 \text{ provided } I_0 < \infty, \tag{2.13}$$

or

(2.14) $w_{,2}(0) = w'_0, \ (Q_2w)(0) = Q_2^0$ provided $I_0 < \infty$,

or

(2.15) $w(0) = w_0, \ (M_2w)(0) = M_2^0$ provided $I_1 < \infty$,

or

(2.16) $(M_2w)(0) = M_2^0, \ (Q_2w)(0) = Q_2^0$,

where w_0, w'_0, M_2^0 and Q_2^0 are prescribed arbitrary constants. We will consider the BVPs

(2.17) $(1.4), (2.13), (2.12)$;

(2.18) $(1.4), (2.14), (2.12)$;

(2.19) $(1.4), (2.15), (2.12)$

and

(2.20) $(1.4), (2.16), (2.12)$.

2.4. Theorem. *In the class* $C_J^4(]0,\ell[)$, *the BVPs* (2.17)–(2.20) *are well-posed in the Hadamard sense*[‡]. *The BVP* (2.19) *with* $M_2^0 = 0$ *is uniquely solvable even if* $I_1 = \infty$ *but* $I_2 < \infty$.

Proof. The assertions follow from Statement 2.1, Lemma A.7 and Lemma A.8 (see the Appendix). Let us define

$$I_k(x_2) = \int_{x_2}^{\ell} \frac{t^k}{D(t)} dt, \quad k = 0, 1, 2, \dots.$$

[Notice that the numbers I_k from (2.5) are in fact the values $I_k(0)$]. For the sake of simplicity, we will consider the case $f \equiv 0$ and will give the solutions of the BVPs mentioned.

(i) The unique solution of BVP (2.20) has the form

$$\begin{aligned} w(x_2) &= x_2[Q_2^0 I_1(x_2) + M_2^0 I_0(x_2)] - Q_2^0 I_2(x_2) - M_2^0 I_1(x_2) \\ &\qquad + w'_\ell(x_2 - \ell) + w_\ell, \\ w_{,2}(x_2) &= Q_2^0 I_1(x_2) + M_2^0 I_0(x_2) + w'_\ell, \\ (Q_2w)(x_2) &= Q_2^0, \ (M_2w)(x_2) = x_2 Q_2^0 + M_2^0. \end{aligned}$$

As we see from this solution of BVP (2.20), both the functions w and $w_{,2}$ are bounded as $x_2 \to 0+$ if and only if $M_2^0 = 0$ for $I_0(0) = I_0 = +\infty$ and $Q_2^0 = 0$ for $I_1(0) = I_1 = +\infty$ (in the general case, i.e., when $f \not\equiv 0$, this assertion follows from Statement 2.1). Therefore, the solution of the problem (2.20) under the additional restriction of boundedness of the solution and of its derivative exists if and only if the above conditions hold.

It is not difficult to see that

$$|w(x_2)| \le |Q_2^0|(x_2 I_1 + I_2) + 2|M_2^0| I_1 + |w'_\ell|\ell + |w_\ell|, \ x_2 \in]0,\ell], \text{ for } I_1 < +\infty$$

and

$$|I_1^{-1}(x_2) w(x_2)| \le |Q_2^0|(x_2 + \widetilde{C}) + 2M_2^0 + C^*(|w'_\ell|\ell + |w_\ell|), \ x_2 \in]0,\ell], \text{ for } I_1 = +\infty,$$

‡The BVP (2.20) for $I_1 < +\infty$ and the BVPs (2.17)–(2.19) are taken in the sense of functions bounded on $]0,\ell[$. The BVP (2.20) for $I_1 = +\infty$ is taken in the sense of functions bounded on $]0,\ell[$ and having weight $I_1^{-1}(x_2)$.

since

$$x_2 I_0(x_2) \le I_1(x_2) \ \forall\ x_2 \in]0,\ell],$$
$$I_1^{-1}(x_2) < C^* = \text{const} > 0 \ \forall\ x_2 \in]0,\ell],$$

and

$$I_2(x_2) \le \widetilde{C} I_1(x_2),\ \widetilde{C} = \text{const} > 0,\ \forall\ x_2 \in]0,\ell],$$

because of

$$\lim_{x_2\to 0+} \frac{I_2(x_2)}{I_1(x_2)} = \lim_{x_2\to 0+} \frac{I_2'(x_2)}{I_1'(x_2)} = \lim_{x_2\to 0+} x_2 = 0 \text{ if } I_1 = +\infty.$$

The continuous dependence of $w(x_2)$ and $I_1^{-1}(x_2)w(x_2)$ for $I_1 < +\infty$ and $I_1 = +\infty$, respectively, on the boundary data immediately follows from the above estimates for the solution $w(x_2)$.

(ii) The unique solution of BVP (2.19) has the form

$$w(x_2) = x_2[C_1 I_1(x_2) + M_2^0 I_0(x_2)] - C_1 I_2(x_2) - M_2^0 I_1(x_2) + w_\ell'(x_2 - \ell) + w_\ell,$$
$$w_{,2}(x_2) = C_1 I_1(x_2) + M_2^0 I_0(x_2) + w_\ell',$$
$$(Q_2 w)(x_2) = C_1,\ (M_2 w)(x_2) = C_1 x_2 + M_2^0,$$

where

$$C_1 = \frac{-I_1 M_2^0 - w_\ell' \ell + w_\ell - w_0}{I_2}.$$

As we see from this solution of the BVP (2.19) the function w is bounded, but $w_{,2}$ is bounded as $x_2 \to 0+$ if and only if $M_2^0 = 0$ for $I_0(0) = I_0 = +\infty$. Therefore, the solution of the BVP (2.19) under the additional restriction of boundedness of $w_{,2}$ exists if and only if the condition $M_2^0 = 0$ if $I_0 = +\infty$ holds. Here, it was substantial that $I_1 < \infty$ [see (2.15)]. If now $I_1 = \infty$ but $I_2 < \infty$, than for $M_2^0 = 0$, the unique solution of (2.19) has the form

$$w(x_2) = C_1[x_2 I_1(x_2) - I_2(x_2)] + w_\ell'(x_2 - \ell) + w_\ell,$$
$$w_{,2}(x_2) = C_1 I_1(x_2) + w_\ell',$$
$$(Q_2 w)(x_2) = F C_1,\ (M_2 w)(x_2) = C_1 x_2,$$

where

$$C_1 = \frac{-w_\ell' \ell + w_\ell - w_0}{I_2}.$$

Obviously, $I_2 > 0$.

If $I_1 = \infty$ but $I_2 < \infty$ and $M_2^0 \ne 0$, then the BVP (2.19) is ill-posed in the sense of nonsolvability.

(iii) The unique solution of BVP (2.18) has the form

$$w(x_2) = [x_2 I_1(x_2) - I_2(x_2)]Q_2^0 - (Q_2^0 \ell - C_2)[x_2 I_0(x_2) - I_1(x_2)] + w_\ell'(x_2 - \ell) + w_\ell,$$
$$w_{,2}(x_2) = Q_2^0 I_1(x_2) - (Q_2^0 \ell - C_2) I_0(x_2) + w_\ell',$$
$$(Q_2 w)(x_2) = Q_2^0,\ (M_2 w)(x_2) = Q_2^0(x_2 - \ell) + C_2,$$

where

$$C_2 = \frac{w'_0 - w'_\ell - Q_2^0(I_1 - \ell I_0)}{I_0}.$$

(iv) The unique solution of BVP (2.17) has the form

$$\begin{aligned} w(x_2) &= [x_2 I_1(x_2) - I_2(x_2)]C_1 - [x_2 I_0(x_2) - I_1(x_2)](C_1\ell - C_2) \\ &\qquad + w'_\ell(x_2 - \ell) + w_\ell, \\ w_{,2}(x_2) &= C_1 I_1(x_2) - (C_1\ell - C_2)I_0(x_2) + w'_\ell, \\ (Q_2 w)(x_2) &= C_1, \ (M_2 w)(x_2) = C_1(x_2 - \ell) + C_2, \end{aligned}$$

where

$$\begin{aligned} C_1 &= \Delta^{-1}[-I_1(w'_0 - w'_\ell) - I_0(w_0 - w_\ell + w'_\ell \ell], \\ C_2 &= \Delta^{-1}[(I_1 - \ell I_0)(w_0 - w_\ell + w'_\ell \ell) + (I_2 - \ell I_1)(w'_0 - w'_\ell)], \\ \Delta &= I_0 I_2 - I_1^2 > 0. \end{aligned}$$

□

2.5. Remark. Conditions on f are essential e.g. for the existence of the limit

$$\lim_{x_2 \to 0+} \int_{x_2}^{\ell} f w dx_2.$$

To this end, if w is unbounded, the additional restrictions on f [recall that as assumed, $f \in C([0, \ell])$] should be chosen appropriately. On the other hand, if $I_1 < \infty$, the above limit exists without any additional restrictions on f. This limit exists as well if $I_1 = \infty$ and $I_2 < \infty$, provided (2.6) with (2.6)* is fulfilled. If $I_k = \infty$ and $I_{k+1} < \infty$ for a fixed $k \in \{2, 3, \dots\}$, then the above limit exists, provided (2.7) with (2.7)* is fulfilled. This follows from the continuity of w on $[0, \ell]$ in all these cases (see Statement 2.1). Thus, under the assumptions (2.6), (2.6)* and (2.7), (2.7)* in the corresponding cases we get from (2.9) that

$$\int_0^\ell D(w_{,22})^2 dx_2 = \int_0^\ell f w dx_2 + (Q_2 w) \cdot w\big|_0^\ell - (M_2 w) \cdot w_{,2}\big|_0^\ell.$$

§ 2.2. The vibration problem. Weak solutions

Let $w, v \in C_J^4(]0, \ell[)$ and (2.6) and (2.7) are fulfilled in the corresponding cases. Then we can rewrite (2.8) as follows:

$$(2.21) \qquad \int_0^\ell D w_{,22} v_{,22} dx_2 = \int_0^\ell f v dx_2 + Q_2 w \cdot v\big|_0^\ell - M_2 w \cdot v_{,2}\big|_0^\ell.$$

This relation connects (in some sense) classical and weak solutions, and it is crucial in view of the definition of the latter in the sense of the expression of unstable BCs (see Remark 2.10 below). Therefore, considering weak solutions of the vibration problem, we will not be able to avoid the restrictions (2.6), (2.7) by setting the BCs.

2.6. Problem. Let us consider the vibration equation (1.1) with the following inhomogeneous BCs:

– at the non-cusped end $x_2 = \ell$ of the beam, conditions (2.12),

– at the other end $x_0 = 0$ which is a cusped one, if $D(0) = 0$, either conditions (2.13) or (2.14) or (2.15) provided $I_2 < \infty$, or (2.16) with

(2.22) $M_2^0 = 0$ if $I_0 = \infty$

and

(2.23) $Q_2^0 = 0$ if $I_1 = \infty$.

2.7. Remark. Problem 2.6 is the common formulation of the following four BVPs:

(i) (1.1), (2.12), (2.13);
(ii) (1.1), (2.12), (2.14);
(iii) (1.1), (2.12), (2.15) provided $I_2 < \infty$;
(iv) (1.1), (2.12), (2.16).

Such a formulation is convenient since it makes possible the investigation all four BVPs at the same time.

2.8. Definition. Let

(2.24) $W^{2,2}(]0,\ell[;\rho_0,\rho_2)$

be the set of all measurable functions $w = w(x_2)$ defined on $]0,\ell[$ which have on $]0,\ell[$ generalized derivatives $\partial_{x_2}^\alpha w$, $\alpha \in \{0,1,2\}$ ($\partial_{x_2}^0 w \equiv w$) such that

$$w \in L^2(]0,\ell[;\rho_0), \quad \text{i.e.,} \quad \int_0^\ell |w(x_2)|^2 \rho_0(x_2) dx_2 < \infty,$$

(2.25) $\partial_{x_2}^1 w \in L^1_{\mathrm{loc}}(]0,\ell[)$,

$$\partial_{x_2}^2 w \in L^2(]0,\ell[;\rho_2), \text{ i.e.,} \int_0^\ell |\partial_{x_2}^2 w(x_2)|^2 \rho_2(x_2) dx_2 < \infty.$$

Here ρ_0, ρ_2 are *weight functions*, i.e., functions measurable and positive a.e. in $]0,\ell[$.

The condition

$$\rho_0^{-1}(x_2),\ \rho_2^{-1}(x_2) \in L^1_{\mathrm{loc}}(]0,\ell[)$$

guarantees [13] that $W^{2,2}(]0,\ell[;\rho_0,\rho_2)$ is a Banach space and even a Hilbert space under the norm

(2.26) $$\|w\|^2_{W^{2,2}(]0,\ell[;\rho_0,\rho_2)} := \int_0^\ell [w^2\rho_0 + (\partial_{x_2}^2 w)^2 \rho_2] dx_2$$

and with the appropriate scalar product.

In what follows, we will use the notation $w_{,2}$ and $w_{,22}$ for $\partial_{x_2}^1 w$ and $\partial_{x_2}^2 w$, respectively.

First, we will consider the special case $\rho_0 \equiv 1$, $\rho_2(x_2) = D(x_2)$ with D from (1.3). In this case, we will denote the space $W^{2,2}(]0,\ell[;\rho_0,\rho_2)$ as

(2.27) $W^{2,2}(]0,\ell[;D)$.

Obviously, the last space is a Hilbert space if $\frac{1}{D(x_2)} \in L^1_{\mathrm{loc}}(]0,\ell[)$, which holds e.g. if $D \in C([0,\ell])$.

Now, we can form subspaces V_{γ_1,γ_2} of $W^{2,2}(]0,\ell[,D)$, $\gamma_1,\gamma_2 \in \{0,1\}$, as follows:

(i) In the case of the BVP (1.1), (2.12), (2.13) we define

(2.28) $$\begin{aligned} V_{0,0} := \{v \in W^{2,2}(]0,\ell[,D) : v(0) = 0,\ v_{,2}(0) = 0 \text{ and} \\ v(\ell) = 0,\ v_{,2}(\ell) = 0 \text{ in the sense of traces}\}. \end{aligned}$$

(ii) In the case of the BVP (1.1), (2.12), (2.14) we define

$$V_{0,1} := \{v \in W^{2,2}(]0,\ell[,D) : v_{,2}(0) = 0 \text{ and } v(\ell) = 0,\ v_{,2}(\ell) = 0 \text{ in the sense of traces}\}. \tag{2.29}$$

(iii) In the case of the BVP (1.1), (2.12), (2.15), provided $I_2 < \infty$, we define

$$V_{1,0} := \{v \in W^{2,2}(]0,\ell[,D) : v(0) = 0 \text{ and } v(\ell) = 0,\ v_{,2}(\ell) = 0 \text{ in the sense of traces}\}. \tag{2.30}$$

(iv) In the case of the BVP (1.1), (2.12), (2.16) we define

$$V_{1,1} := \{v \in W^{2,2}(]0,\ell[,D) : v(\ell) = 0,\ v_{,2}(\ell) = 0 \text{ in the sense of traces}\}. \tag{2.31}$$

Notice that these spaces are defined in terms of traces. If these traces exist, it is not difficult to show that all spaces V_{γ_1,γ_2} are complete. Now, the traces at the point $x_2 = \ell$ always exist since

$$W^{2,2}(]\varepsilon,\ell[,D) \subset W^{2,2}(]\varepsilon,\ell[) \quad \text{for} \quad 0 < \varepsilon < \ell \tag{2.32}$$

(where the second space is the "classical" Sobolev space).

In order to clarify the question of the existence of the traces at the point $x_2 = 0$, we make the function $D(x_2)$ subject to the following unilateral condition:

$$D(x_2) \geq D_\kappa x_2^\kappa \quad \text{for} \quad x \in]0,\ell[, \tag{2.33}$$

$D_\kappa = \text{const} > 0$, $\kappa = \text{const} \geq 0$[§]. In other words

$$0 < D_\kappa := \inf_{]0,\ell[} \frac{D(x_2)}{x_2^\kappa}. \tag{2.34}$$

It follows from (2.25) for $\rho_2(x_2) = D(x_2)$ and (2.33) that

$$\int_0^\ell x_2^\kappa [w_{,22}(x_2)]^2 dx_2 < \infty, \tag{2.35}$$

and hence, under condition (2.33),

$$W^{2,2}(]0,\ell[,D) \subset W^{2,2}(]0,\ell[,x_2^\kappa). \tag{2.36}$$

The last space is a special case of (2.24) with $\rho_0 \equiv 1$, $\rho_2(x_2) = x_2^\kappa$. The obvious inequality

$$x_2(\ell - x_2) \leq \ell x_2 \text{ for } x_2 \in [0,\ell]$$

together with (2.35) and (2.36) implies

$$W^{2,2}(]0,\ell[,D) \subset W^{2,2}(]0,\ell[,x_2^\kappa) \subset W^{2,2}(]0,\ell[,x_2^\kappa(\ell - x_2)^\kappa). \tag{2.37}$$

The last space is a special case of (2.24) with $\rho_0 \equiv 1$ and $\rho_2(x_2) = x_2^\kappa(\rho - x_2)^\kappa$. But any function

$$w \in W^{2,2}(]0,\ell[,x_2^\kappa(\ell - x_2)^\kappa)$$

has a trace at the point $x_2 = 0$ if

$$\kappa \in [0,3[\tag{2.38}$$

while its derivative $w_{,2}$ has a trace at $x_2 = 0$ if

$$\kappa \in [0,1[. \tag{2.39}$$

[§]By κ we denote the minimum of the possible exponents $\delta \geq 1$ for which $D(x_2) \geq \text{const},\ x_2^\delta$ holds. For $\kappa < 1$ it is not necessary to find the minimal possible exponent since in this case we have the same result concerning traces for all $\kappa < 1$. Let us note that $D(x_2) = D_0[\ln(\widetilde{\ell}/x_2)]^{-1}$, $\widetilde{\ell} > \ell$, satisfies (2.33) for any $\kappa > 0$. Condition (2.33) is obviously important in the neighbourhood of $x_2 = 0$.

More precisely, after a suitable change of the values of w at a set of measure zero, this functions became continuous on $[0,\ell]$, i.e., $w \in C([0,\ell])$ for (2.38) and $w_{,2} \in C([0,\ell])$ for (2.39) (see, e.g., [17]).

2.9. Remark. The obvious inequality

(2.40) $\quad x_2^4 \le \ell^{4-\kappa} x_2^\kappa$ for $x_2 \in [0,\ell]$ and $\kappa \le 4$

implies

(2.41) $\quad W^{2,2}(]0,\ell[,D) \subset W^{2,2}(]0,\ell[,x_2^\kappa) \subset W^{2,2}(]0,\ell[,x_2^4)$

for $\kappa \le 4$.

Inequality (2.33) can be rewritten as

(2.42) $$\frac{1}{D(x_2)} \le D_\kappa^{-1} x_2^{-\kappa},\ \ x_2 \in]0,\ell].$$

Whence we immediately conclude that (2.38) and (2.39) imply

(2.43) $\quad I_2 < \infty$

and¶

(2.44) $\quad I_0 < \infty,$

respectively.

Thus, the traces $v(0)$ and $v_{,2}(0)$ mentioned in (2.28)–(2.30) exist by (2.38) (i.e., by (2.43)) and by (2.39) (i.e., by (2.44)), respectively, provided (2.33) holds.

If instead of (2.33) the following inequality takes place

(2.45) $\quad D(x_2) \le D^\kappa x_2^\kappa$ for $x_2 \in]0,\ell[$,

$D^\kappa = \text{const} > 0$, $\kappa = \text{const} \ge 0$, then

$$W^{2,2}(]0,\ell[,x_2^\kappa) \subset W^{2,2}(]0,\ell[,D).$$

In this case

$$(D^\kappa)^{-1}\frac{1}{x_2^\kappa} \le \frac{1}{D(x_2)},\ \ x_2 \in]0,\ell[,$$

and (2.43) and (2.44) imply (2.38) and (2.39), respectively.

Finally, if both (2.33) and (2.45) are fulfilled, then (2.38) and (2.39) are equivalent to (2.43) and (2.44), respectively, and

$$W^{2,2}(]0,\ell[),D) = W^{2,2}(]0,\ell[,x_2^\kappa)$$

in the sense of equivalent norms.

2.10. Remark. According to the customary terminology, the BCs

(2.46) $\quad w(0) = w^0$ if $I_2 < \infty$

and

(2.47) $\quad w_{,2}(0) = w'$ if $I_0 < \infty$

with prescribed constants w^0 and w' are the *stable* (principal) BCs for the operator J_ω since they are fulfilled by functions from both sets $C_J^4(]0,\ell[)$ and $W^{2,2}(]0,\ell[,D)$. On the other hand, the BCs

(2.48) $\quad (M_2 w)(0) = M_2^0$ and $(Q_2 w)(0) = Q_2^0$

¶If $I_0 = +\infty$, then from (2.42) it follows that κ cannot be less then 1 (otherwise, i.e., if $\kappa < 1$, we have (2.44) and come to the contradiction). Thus, the conditions (2.39) and (2.44) are equivalent in this sense.

with prescribed constants M_2^0, Q_2^0 are *unstable* (natural) conditions since they are fulfilled by functions from $C_J^4(]0,\ell[)$ but *not* by functions from $W^{2,2}(]0,\ell[,D)$, due to the fact that the traces at $x_2=0$ of the second and third order derivatives of functions from the latter class *do not exist in general.*

In what follows, let $u \in W^{2,2}(]0,\ell[,D)$ and $f \in L^2(]0,\ell[)$ be given.

2.11. Definition. The function $w \in W^{2,2}(]0,\ell[,D)$ will be called a weak solution of the BVP (1.1), (2.12), (2.13) in the space $W^{2,2}(]0,\ell[,D)$ if

$$w-u \in V_{0,0} \tag{2.49}$$

and

$$J_\omega(w,v) := \int_0^\ell B_\omega(w,v)dx_2 = \int_0^\ell fvdx_2, \tag{2.50}$$

where

$$B_\omega(w,v) := Dw_{,22}v_{,22} - \omega^2\rho\sigma wv, \tag{2.51}$$

holds for every $v \in V_{0,0}$.

2.12. Definition. The function $w \in W^{2,2}(]0,\ell[,D)$ will be called a *weak solution of the BVP* (1.1), (2.12), (2.14) *in the space* $W^{2,2}(]0,\ell[,D)$ if

$$w-u \in V_{0,1} \tag{2.52*}$$

and

$$J_\omega(w,v) = \int_0^\ell fvdx_2 + Q_2^0 v(0) \tag{2.52}$$

holds for every $v \in V_{0,1}$.

2.13. Definition. The function $w \in W^{2,2}(]0,\ell[,D)$ will be called a *weak solution of the BVP* (1.1), (2.12), (2.15) *in the space* $W^{2,2}(]0,\ell[,D)$ if

$$w-u \in V_{1,0} \tag{2.53*}$$

and

$$J_\omega(w,v) = \int_0^\ell fvdx_2 - \begin{cases} M_2^0 v_{,2}(0) & \text{if } I_0<\infty, \quad (2.53)\\ 0 & \text{if } I_0=\infty \text{ and } I_2<\infty \quad (2.54)\end{cases}$$

holds for every $v \in V_{1,0}$.

2.14. Definition. The function $w \in W^{2,2}(]0,\ell[,D)$ will be called a *weak solution of the BVP* (1.1), (2.12), (2.16) *in the space* $W^{2,2}(]0,\ell[,D)$ if

$$w-u \in V_{1,1} \tag{2.55*}$$

and

$$J_\omega(w,v) = \int_0^\ell fvdx_2 + \begin{cases} Q_2^0 v(0) - M_2^0 v_{,2}(0) & \text{if } I_0<\infty, \quad (2.55)\\ Q_2^0 v(0) & \text{if } I_0=\infty \text{ and } I_1<+\infty, \quad (2.56)\\ 0 & \text{if } I_1=\infty \text{ and } \exists k \in \{2,3,\dots\} \\ & \text{such that } I_k<+\infty \quad (2.57)\end{cases}$$

holds for every $v \in V_{1,1}$.

2.15. Remark. The conditions (2.49), (2.52*), (2.53*) and (2.55*) express the fact that the BCs (2.12), (2.13), and the first BCs in (2.14) and (2.15), are fulfilled. The BCs (2.16) and the second BCs in (2.14), (2.15) can be found directly from the identities (2.52)–(2.57). These identities are derived from the identity (2.21) for the operator J_ω instead of J. As we see from (2.54), (2.56) and (2.57), these last mentioned BCs cannot be specified in these identities if $M_2^0 \neq 0$ for $I_0 = \infty$ and $Q_2^0 \neq 0$ for $I_1 = \infty$ since for $I_0 = \infty$ and $I_1 = \infty$, the traces of $v_{,2}$ and v, respectively, at the point $x_2 = 0$ do not exist in general. Hence, the restrictions (2.22) and (2.23) are natural in this sense, too. (See also Remark 2.3 and the end of (i) and (ii) in the proof of Theorem 2.4.)

2.16. Remark. In view of (2.21), the classical solutions of the static BVPs (2.17)–(2.20) constructed in Subsection 2.1 satisfy (2.50), (2.52)–(2.57) by $\omega = 0$ in the corresponding cases. Obviously they satisfy also (2.49), (2.52*), (2.53*), (2.55*) (under conditions (2.6), (2.7) if necessary).

Besides the space (2.27) let us consider the space (2.24) with

$$\rho_0(x_2) = x_2^{\kappa-4},\ \rho_2(x) = x_2^{\kappa}. \tag{2.58}$$

We will denote this space by

$$\widetilde{W}^{2,2}(]0,\ell[, x_2^{\kappa}); \tag{2.59}$$

it is equipped with the norm

$$\|w\|_{\widetilde{W}^{2,2}(]0,\ell[,x_2^{\kappa})} := \Big(\int_0^{\ell} [x_2^{\kappa-4} w^2(x_2) + x_2^{\kappa} w_{,22}^2(x_2)] dx_2 \Big)^{1/2}. \tag{2.60}$$

The space (2.59) is a Hilbert space with the appropriate scalar product, since $x_2^{4-\kappa}$, $x_2^{-\kappa} \in L^1_{\mathrm{loc}}(]0,\ell[)$. We can easily see that

$$\widetilde{W}^{2,2}(]0,\ell[, x_2^{\kappa}) \subset W^{2,2}(]0,\ell[), x_2^{\kappa}) \text{ if } \kappa < 4, \tag{2.61}$$

$$\widetilde{W}^{2,2}(]0,\ell[, x_2^{4}) = W^{2,2}(]0,\ell[, x_2^{4}), \tag{2.62}$$

$$\widetilde{W}^{2,2}(]0,\ell[, x_2^{\kappa}) \supset W^{2,2}(]0,\ell[, x_2^{\kappa}) \text{ if } \kappa > 4. \tag{2.63}$$

Let us consider the space

$$V_\varepsilon(x_2^{\kappa}) := \big\{ v \in \widetilde{W}^{2,2}(]\varepsilon,\ell[, x_2^{\kappa}), v(\ell) = 0,\ v_{,2}(\ell) = 0 \big\}. \tag{2.64}$$

The traces $v(\ell)$ and $v_{,2}(\ell)$ are well-defined since for $\varepsilon \in]0,\ell[$

$$\widetilde{W}^{2,2}(]\varepsilon,\ell[, x_2^{\kappa}) \subset W^{2,2}(]\varepsilon,\ell[), \tag{2.65}$$

and hence, v and $v_{,2}$ are absolutely continuous on $[\varepsilon,\ell]$. Thus,

$$v_{,2}\,,\ v \in AC_R(\varepsilon,\ell) \tag{2.66}$$

(see [19, Definition 1.2]) and in view of the first boundary condition in (2.64), if $\kappa > 1$, the following Hardy inequality holds (see [19, p. 69])

$$\int_\varepsilon^\ell x_2^{\kappa-2} v^2 dx_2 \le \frac{4}{(\kappa-1)^2} \int_\varepsilon^\ell x_2^{\kappa} (v_{,2})^2 dx_2,\ \kappa > 1. \tag{2.67}$$

Therefore, taking into account the second boundary condition in (2.64), we can write

$$\int_\varepsilon^\ell x_2^{\kappa-2} (v_{,2})^2 dx_2 \le \frac{4}{(\kappa-1)^2} \int_\varepsilon^\ell x_2^{\kappa} (v_{,22})^2 dx_2,\ \kappa > 1. \tag{2.68}$$

Replacing in (2.67) κ by $\kappa - 2$, we obtain

$$\int_\varepsilon^\ell x_2^{\kappa-4} v^2 dx_2 \le \frac{4}{(\kappa-3)^2} \int_\varepsilon^\ell x_2^{\kappa-2} (v_{,2})^2 dx_2,\ \kappa > 3. \tag{2.69}$$

Combining (2.68) and (2.69), we have

$$\int_\varepsilon^\ell x_2^{\kappa-4} v^2 dx_2 \le \frac{16}{(\kappa-1)^2(\kappa-3)^2} \int_\varepsilon^\ell x_2^{\kappa} (v_{,22})^2 dx_2,\ \kappa > 3. \tag{2.70}$$

Now, considering the limit procedure as $\varepsilon \to 0+$, since the limits of the integrals in (2.70) exist for $v \in \widetilde{W}^{2,2}(]0,\ell[, x_2^{\kappa})$, we immediately get the following

2.17. Lemma. *If $v \in V_0(x_2^\kappa)$, then*

$$(2.71)\qquad \int_0^\ell x_2^{\kappa-4} v^2(x_2)dx_2 \le \frac{16}{(\kappa-1)^2(\kappa-3)^2}\int_0^\ell x_2^\kappa [v_{,22}(x_2)]^2 dx_2,\ \ \kappa > 3.$$

2.18. Corollary. *If $v \in V_0(x_2^4)$, from* (2.71) *we obtain*

$$(2.72)\qquad \int_0^\ell v^2 dx_2 \le \frac{16}{9}\int_0^\ell x_2^4 (v_{,22})^2 dx_2.$$

2.19. Remark. Obviously, all the $V_{\gamma_1\gamma_2}$ constructed by (2.28)–(2.31) are contained in $V_0(x_2^4)$ if $\kappa \le 4$ (see (2.64), (2.62), Remark 2.9, and the relations (2.41)).

First we consider the case

$$0 \le \kappa < 4,$$

i.e.

$$I_3 < +\infty.$$

2.20. Theorem. *If $0 \le \kappa < 4$ (i.e., $I_3 < +\infty$) and*

$$(2.73)\qquad \omega^2 < \frac{9D_\kappa \ell^{\kappa-4}}{16 \max\limits_{[0,\ell]} \rho\sigma},$$

then the BVPs

1. (1.1), (2.12), (2.13);
2. (1.1), (2.12), (2.14);
3. (1.1), (2.12), (2.15) *provided $I_2 < +\infty$;*
4. (1.1), (2.12), (2.16),

have unique solutions. These solutions are such that

$$\|w\|_{W^{2,2}(]0,\ell[,D)} \le C\big[\|f\|_{L_2(]0,\ell[)} + \|u\|_{W^{2,2}(]0,\ell[,D)} + \gamma_1|M_2^0| + \gamma_2|Q_2^0|\big],$$

where the constant C is independent of f, u, M_2^0, Q_2^0, and

$\gamma_1 = 0,\ \gamma_2 = 0$ *for the first problem,*

$\gamma_1 = 0,\ \gamma_2 = 1$ *for the second problem,*

$\gamma_1 = 1,\ \gamma_2 = 0$ *for the third problem,*

$\gamma_1 = 1,\ \gamma_2 = 1$ *for the fourth problem.*

Proof. It is easy to see that

$$(2.74)\qquad |J_\omega(w,v)| \le (1+T)\|v\|_{W^{2,2}(]0,\ell[,D)}\|w\|_{W^{2,2}(]0,\ell[,D)},$$

where

$$(2.75)\qquad T := \omega^2 \max_{[0,\ell]} \rho(x_2)\sigma(x_2),$$

and the functional

$$F_\omega v := \int_0^\ell f(x_2)v(x_2)dx_2 - J_\omega(u,v) + \gamma_2 v(0)Q_2^0 - \gamma_1 v_{,2}(0)M_2^0,\ v \in V_{\gamma_1,\gamma_2}$$

(see (2.28)–(2.31) and (2.50), (2.52)–(2.57)) is bounded in V_{γ_1,γ_2}:

$$(2.76)\qquad \begin{aligned} |F_\omega v| \quad &\le \quad \big[\|f\|_{L_2(]0,\ell[)} + (1+T)\|u\|_{W^{2,2}(]0,\ell[,D)} \\ &+ \quad C_0(\gamma_2|Q_2^0| + \gamma_1 M_2^0|)\big]\|v\|_{V_{\gamma_1,\gamma_2}}, \end{aligned}$$

where we have used the theorem of traces (the constant C_0 is from this theorem) and

$$(2.77)\qquad \|v\|_{V_{\gamma_1,\gamma_2}} := \|v\|_{W^{2,2}(]0,\ell[,D)}.$$

Now, taking into account (2.33), (2.75), Remark 2.9, Remark 2.19, Corollary 2.18, and introducing the notation

$$(2.78)\qquad T_0 := \frac{16\ell^{4-\kappa}}{9D_\kappa}(1+T),$$

we have

$$\begin{aligned}
\|v\|^2_{V_{\gamma_1,\gamma_2}} &= \int_0^\ell [v^2 + D(v_{,22})^2]dx_2 = \int_0^\ell v^2 dx_2 + J_\omega(v,v) + \omega^2 \int_0^\ell \rho\sigma v^2 dx_2 \\
&\le (1+T)\int_0^\ell v^2 dx_2 + J_\omega(v,v) \le \frac{16}{9}(1+T)\int_0^\ell x_2^4 (v_{,22})^2 dx_2 + J_\omega(v,v) \\
&\le \frac{16\ell^{4-\kappa}}{9D_\kappa}(1+T)\int_0^\ell D_\kappa x_2^\kappa (v_{,22})^2 dx_2 + J_\omega(v,v) \\
&\le T_0 \int_0^\ell D(v_{,22})^2 dx_2 + J_\omega(v,v) \\
&= J_\omega(v,v) + T_0\Big[J_\omega(v,v) + \omega^2\int_0^\ell \rho\sigma v^2 dx_2\Big] \\
&\le J_\omega(v,v) + T_0\Big[J_\omega(v,v) + T\int_0^\ell v^2 dx\Big] \\
&\le J_\omega(v,v) + T_0\Big[J_\omega(v,v) + \frac{16\ell^{4-\kappa}T}{9D_\kappa}\int_0^\ell D(v_{,22})^2 dx_2\Big] \\
&= J_\omega(v,v) + T_0\Big\{J_\omega(v,v) + \frac{16\ell^{4-\kappa}T}{9D_\kappa}\Big[J_\omega(v,v) + \omega^2\int_0^\ell \rho\sigma v^2 dx_2\Big]\Big\} \\
&\le J_\omega(v,v) + T_0\{J_\omega(v,v) + \frac{16\ell^{4-\kappa}T}{9D_\kappa}\Big[J_\omega(v,v) + T\int_0^\ell v^2 dx_2\Big]\Big\} \\
&\le J_\omega(v,v) + T_0\{J_\omega(v,v) + \frac{16\ell^{4-\kappa}T}{9D_\kappa}\Big[J_\omega(v,v) \\
&\qquad + \frac{16\ell^{4-\kappa}T}{9D_\kappa}\int_0^\ell D(v_{,22})^2 dx_2\Big]\Big\} \\
&= J_\omega(v,v) + T_0\Big\{J_\omega(v,v) + \frac{16\ell^{4-\kappa}T}{9D_\kappa}\Big[J_\omega(v,v) + \frac{16\ell^{4-\kappa}T}{9D_\kappa}(J_\omega(v,v) \\
&\qquad + \omega^2\int_0^\ell \rho\sigma v^2 dx_2)\Big]\Big\} \\
&= J_\omega(v,v) + T_0\Big\{J_\omega(v,v)\Big[1 + \frac{16\ell^{4-\kappa}T}{9D_\kappa} + \Big(\frac{16\ell^{4-\kappa}T}{9D_\kappa}\Big)^2\Big] \\
&\qquad + \Big(\frac{16\ell^{4-\kappa}}{9D_\kappa}\Big)^2\omega^2\int_0^\ell \rho\sigma v^2 dx_2\Big\}
\end{aligned}$$

(repeating the same $(n-2)$-times)

$$\le J_\omega(v,v) + T_0\Big[J_\omega(v,v)\frac{1-(\frac{16\ell^{4-\kappa}T}{9D_\kappa})^{n+1}}{1-\frac{16\ell^{4-\kappa}T}{9D_\kappa}} + \Big(\frac{16\ell^{4-\kappa}T}{9D_\kappa}\Big)^n\omega^2\int_0^\ell \rho\sigma v^2 dx_2\Big).$$

Now, letting n tend to infinity and taking into account that, in view of (2.73) and (2.75),

$$\frac{16\ell^{4-\kappa}T}{9D_\kappa} < 1,$$

we obtain

$$\|v\|^2_{V_{\gamma_1,\gamma_2}} \leq J_\omega(v,v) + \frac{T_0}{1-\frac{16\ell^{4-\kappa}T}{9D_\kappa}} J_\omega(v,v),$$

i.e., in view of (2.78),

$$J_\omega(v,v) \geq \frac{9D_\kappa - 16\ell^{4-\kappa}T}{9D_\kappa + 16\ell^{4-\kappa}} \|v\|^2_{V_{\gamma_1,\gamma_2}}. \tag{2.79}$$

Thus, by virtue of (2.74), (2.79), and (2.76), and according to the Lax-Milgram theorem there exists a unique $z \in V_{\gamma_1,\gamma_2}$ such that

$$J_\omega(z,v) = F_\omega v := \int_0^\ell fv dx_2 - J_\omega(u,v) + \gamma_2 v(0)Q_2^0 - \gamma_1 v_{,2}(0)M_2^0 \ \forall\ v \in V_{\gamma_1,\gamma_2},$$

whence,

$$J_\omega(w,v) = \int_0^\ell fv dx_2 + \gamma_2 v(0)Q_2^0 - \gamma_1 v_{,2}(0)M_2^0 \quad \forall\, v \in V_{\gamma_1,\gamma_2}, \tag{2.80}$$

where

$$w := u + z \in W^{2,2}(]0,\ell[), D). \tag{2.81}$$

So,

$$w - u = z \in V_{\gamma_1,\gamma_2},$$

and (2.80) means that (2.50), (2.52)–(2.57) hold in the corresponding cases.

In addition, according to the Lax-Milgram theorem

$$\|z\|_{V_{\gamma_1,\gamma_2}} \leq \frac{9D_\kappa + 16\ell^{4-\kappa}}{9D_\kappa - 16\ell^{4-\kappa}T} \|F_\omega\|_{V^*_{\gamma_1,\gamma_2}} \tag{2.82}$$

where $V^*_{\gamma_1,\gamma_2}$ is dual to V_{γ_1,γ_2}. From (2.76) it follows that

$$\|F_\omega\|_{V^*_{\gamma_1,\gamma_2}} \leq \|f\|_{L_2(]0,\ell[)} + (1+T)\|u\|_{W^{2,2}(]0,\ell[,D)} + C_0(\gamma_2|Q_2^0| + \gamma_1|M_2^0|). \tag{2.83}$$

By virtue of (2.81)–(2.83), we have

$$\begin{aligned}
&\|w\|_{W^{2,2}(]0,\ell[,D)} \leq \|u\|_{W^{2,2}(]0,\ell[,D)} + \|z\|_{V_{\gamma_1,\gamma_2}} \leq \|u\|_{W^{2,2}(]0,\ell[,D)} \\
&+ \frac{9D_\kappa + 16\ell^{4-\kappa}}{9D_\kappa - 16\ell^{4-\kappa}T}[\|f\|_{L^2(]0,\ell[)} + (1+T)\|u\|_{W^{2,2}(]0,\ell[,D)} + C_0(\gamma_2|Q_2^0| \\
&+ \gamma_1|M_2^0|)] \leq C[\|f\|_{L_2(]0,\ell[)} + \|u\|_{W^{2,2}(]0,\ell[,D)} + \gamma_2|Q_2^0| + \gamma_1|M_2^0|],
\end{aligned}$$

where

$$C := \max\left\{1 + \frac{9D_\kappa + 16\ell^{4-\kappa}}{9D_\kappa - 16\ell^{4-\kappa}T}(1+T),\ \frac{9D_\kappa + 16\ell^{4-\kappa}}{9D_\kappa - 16\ell^{4-\kappa}T}C_0\right\}.$$

□

Now, let us consider the case

$$\kappa \geq 4,$$

i.e.,

$$I_k = +\infty \text{ and } I_{k+1} < +\infty \text{ for a fixed } k \in \{3,4,\dots\}.$$

Instead of the space $W^{2,2}(]0,\ell[,D)$ with the norm (2.26) we look for a solution in the wider space

$$\widetilde{W}^{2,2}(]0,\ell[,D) \tag{2.84}$$

with the norm

$$(2.85)\qquad \|w\|^2_{\widetilde{W}^{2,2}(]0,\ell[,D)} := \int_0^\ell [x_2^{\kappa-4}w^2 + D(w_{,22})^2]dx_2.$$

More precisely,

$$(2.86)\qquad \widetilde{W}^{2,2}(]0,\ell[,D) \supset W^{2,2}(]0,\ell[,D) \text{ for } \kappa > 4$$

and

$$(2.87)\qquad \widetilde{W}^{2,2}(]0,\ell[,D) = W^{2,2}(]0,\ell[,D) \text{ for } \kappa = 4.$$

In the case under consideration, it follows from the previous arguments (see Problem 2.6, and compare (2.16) with (2.13)–(2.15)), only the BVP (1.1), (2.12), (2.16) is admissible.

Let

$$(2.88)\qquad V := \{v \in \widetilde{W}^{2,2}(]0,\ell[,D) : v(\ell) = 0,\ v_{,2}(\ell) = 0\}.$$

In view of (2.33),

$$\widetilde{W}^{2,2}(]0,\ell[,x_2^\kappa) \supset \widetilde{W}^{2,2}(]0,\ell[,D) \text{ for } \kappa \geq 4.$$

Therefore, Lemma 2.17 is also valid for $v \in V$.

2.21. Definition. Let $u \in \widetilde{W}^{2,2}(]0,\ell[,D)$ be given and $x_2^{\frac{4-\kappa}{2}} f \in L_2(]0,\ell[)$. A function $w \in \widetilde{W}^{2,2}(]0,\ell[,D)$ will be called a *weak solution of the BVP* (1.1), (2.12), (2.16) *in the space* $\widetilde{W}^{2,2}(]0,\ell[,D)$ if

$$w - u \in V$$

with V defined by (2.88), and if (2.57) holds for every $v \in V$.

2.22. Theorem. *Let* $\rho(x_2)\sigma(x_2)x_2^{4-\kappa} \in C([0,\ell])$. *If* $\kappa \geq 4$ *(i.e.,* $I_k = +\infty$ *and* $I_{k+1} < +\infty$ *for a fixed* $k \in \{3,4,\dots\}$*) and*

$$(2.89)\qquad \omega^2 < \frac{(\kappa-1)^2(\kappa-3)^2 D_\kappa}{16 \max\limits_{[0,\ell]} \rho(x_2)\sigma(x_2)x_2^{4-\kappa}},$$

then the BVP (1.1), (2.12), (2.16) *with* $M_2^0 = 0$, $Q_2^0 = 0$ *has a unique weak solution in* $\widetilde{W}^{2,2}(]0,\ell[,D)$ *such that*

$$\|w\|_{\widetilde{W}^{2,2}(]0,\ell[,D)} \leq C[\|x^{\frac{4-\kappa}{2}} f(x_2)\|_{L_2(]0,\ell[)} + \|u\|_{\widetilde{W}^{2,2}(]0,\ell[,D)}],$$

where the constant C *is independent of* f *and* u.

Proof. Let

$$T_* := \omega^2 \max_{[0,\ell]} \rho\sigma x_2^{4-\kappa},$$

$$T_\kappa := \frac{16(1+T_*)}{(\kappa-1)^2(\kappa-3)^2 D_\kappa}.$$

Using Lemma 2.17 and the relations

$$\omega^2 \int_0^\ell \rho\sigma|wv|dx_2 = \omega^2 \int_0^\ell (\rho\sigma x_2^{4-\kappa})(x_2^{\frac{\kappa-4}{2}}|w|)(x_2^{\frac{\kappa-4}{2}}|v|)dx_2$$

$$\leq T_* \Big(\int_0^\ell x_2^{\kappa-4}w^2dx_2\Big)^{1/2}\Big(\int_0^\ell x_2^{\kappa-4}v^2dx_2\Big)^{1/2},$$

$$\int_0^\ell \rho\sigma v^2 dx_2 = \int_0^\ell (\rho\sigma x_2^{4-\kappa})(x_2^{\kappa-4}v^2)dx_2,$$

similarly to the proof of Theorem2.20, we get

$$|J_\omega(w,v)| \le (1+T_*)\|w\|_{\widetilde{W}^{2,2}(]0,\ell[,D)} \cdot \|v\|_{\widetilde{W}^{2,2}(]0,\ell[,D)},$$

where J_ω is defined by (2.50), (2.51),

$$|F_\omega v| \le [\|x_2^{4-\kappa} f\|_{L_2(]0,\ell[)} + (1+T_*)\|u\|_{\widetilde{W}^{2,2}(]0,\ell[,D)}]\|v\|_V,$$

where

$$F_\omega v := \int_0^\ell f(x_2)v(x_2)dx_2 - J_\omega(u,v), \ v \in V,$$

and

$$J_\omega(v,v) \ge \frac{(\kappa-1)^2(\kappa-3)^2 D_\kappa - 16T_*}{(\kappa-1)^2(\kappa-3)^2 D_\kappa + 16}\|v\|_V^2.$$

Thus, all the conditions of the Lax-Milgram theorem are fulfilled, and it is not difficult to finish the proof. □

2.23. Remark. The restriction

$$\rho(x_2)\sigma(x_2)x_2^{4-\kappa} \in C([0,\ell])$$

is not heavy because of $\sigma(0) = 0$. For instance, if we consider a beam with a rectangular cross-section, with unit width and thickness

$$(2.90) \quad 2h = h_0 x_2^{\kappa/3}, \ h_0 = \text{const} > 0,$$

then $\sigma(x_2) = h_0 x_2^{\frac{\kappa}{3}}$ and for $4 \le \kappa \le 6$

$$\rho(x_2)\sigma(x_2)x_2^{4-\kappa} = \rho(x_2)h_0 x_2^{4-\frac{2\kappa}{3}} \in C([0,\ell]).$$

2.24. Remark. In the case (2.90), $D(x_2)$ has the form

$$D(x_2) = D_* x_2^\kappa, \ D_* = \text{const} > 0,$$

provided $E = \text{const}$, $\nu = \text{const}$. If we additionally suppose that

$$\rho(x_2) = \rho_* x_2^{\frac{2\kappa}{3}-4}, \ \rho_* = \text{const} > 0,$$

then

$$\rho(x_2)\sigma(x_2)x_2^{4-\kappa} = \rho_* h_0 = \text{const}.$$

Hence, from (2.89) we have

$$\omega^2 < \frac{(\kappa-1)^2(\kappa-3)^2 D_*}{16\rho_* h_0}.$$

Whence, the greater is κ the greater is the lower bound of the eigenvalues of the operator J_ω. If now κ tends to $+\infty$, then the above bound tends to $+\infty$ as well.

2.25. Remark. Let $\ell = 1$. In the case of the homogeneous BCs for the BVP (1.1), (2.12), (2.13), from the results of [11] (see theorem 1.6_1, and Lemma 1.5_1) there follows the following sufficient condition on the vibration frequency for unique solvability:

$$\omega^2 < \min\left\{\frac{3}{\int_0^{\tau_0}(\tau_0-\tau)^3 D^{-1}(\tau)d\tau}, \ \frac{3}{\int_{\tau_0}^1(\tau-\tau_0)^3 D^{-1}(\tau)d\tau}\right\}$$

for a fixed $\tau_0 \in]0,\ell[$. Here we do not make precise the other restrictions.

§ 2.3. Vibration Problem. The General Case

Let

(2.91) $D(x_2) \in C([0,\ell]),\ D(x_2) > 0 \quad \forall\, x_2 \in]0,\ell],\ D(0) \geq 0.$

Under these assumptions, obviously,

(2.92) $\displaystyle\int_{x_2}^{\ell} D^{-1}(\tau)d\tau < +\infty,$ for every $x_2 \in]0,\ell].$

Let further

(2.93) $\displaystyle P(x_2) := D^{-1}(x_2)\Big[\int_{x_2}^{\ell} D^{-1}(\tau)d\tau\Big]^{-2},\ x_2 \in]0,\ell[,$

(2.94) $\displaystyle Q(x_2) := D(x_2)\Big[\int_{x_2}^{\ell} D^{-1}(\tau)d\tau\Big]^{2}\Big\{\int_{x_2}^{\ell} D(t)\Big[\int_{t}^{\ell} D^{-1}(\tau)d\tau\Big]^{2}dt\Big\}^{-2},$

$x_2 \in]0,\ell[.$

Evidently,

(2.95) $P(x_2), Q(x_2) \in C(]0,\ell[),$

and

(2.96) $P(x_2) > 0,\ Q(x_2) > 0 \quad \forall\, x_2 \in]0,\ell[.$

2.26. Definition. Let

(2.97) $\overset{*}{W}{}^{2,2}(]0,\ell[,D)$

be the special case of (2.24) with

$$\rho_0 = Q(x_2),\ \rho_2 = D(x_2).$$

Since

$$Q^{-1}(x_2),\ D^{-1}(x_2) \in L^1_{\mathrm{loc}}(]0,\ell[),$$

the space (2.97) is a Hilbert space.

Now, we consider Problem 2.6, where w_0, w_ℓ and w_0', w_ℓ' are the traces of a certain given function $u \in \overset{*}{W}{}^{2,2}(]0,\ell[,D)$ and of its derivative, respectively.

Let

(2.98) $\overset{*}{V} := \{v \in \overset{*}{W}{}^{2,2}(]0,\ell[,D) : v(\ell) = 0,\ v_{,2}(\ell) = 0,$ and additionally
$v(0) = 0,\ v_{,2}(0) = 0$ in the case of BCs (2.13),
$v_{,2}(0) = 0$ in the case of BCs (2.14),
$v(0) = 0$ in the case of BCs (2.15) provided $I_2 < +\infty$
(in the sense of traces)$\}$.

2.27. Remark. In (2.98) the existence of the traces in the cases indicated is assumed. But if we additionally suppose that $\int_0^{x_2} D^{-1}(t)dt < +\infty$ for $x_2 \in]0,\ell]$ (so, with (2.91), it follows that $0 < \int_0^{\ell} D^{-1}(t)dt < +\infty$), and consider the space

$$\overset{*}{W}{}^{2,2}(]0,\ell-\varepsilon[,D) \supset \overset{*}{W}{}^{2,2}(]0,\ell[,D),\ \varepsilon = \mathrm{const} > 0,$$

then, in view of (2.94),

$$Q(x_2) = D(x_2)\cdot \widetilde{D}(x_2) \quad \forall\, x_2 \in [0,\ell-\varepsilon]$$

with the positive continuous $\widetilde{D}(x_2)$ on $[0,\ell-\varepsilon]$. If we now assume (2.33), we will have

$$D(x_2) \geq D_\kappa x_2^\kappa \quad \text{and} \quad Q(x_2) \geq \widetilde{D} \cdot D_\kappa x_2^\kappa \quad \forall\, x_2 \in]0,\ell-\varepsilon[,$$

where

$$\widetilde{D} := \min_{[0,\ell-\varepsilon]} \widetilde{D}(x_2).$$

Hence,

$$D(x_2) \geq D_* x_2^\kappa, \quad Q(x_2) \geq D_* x_2^\kappa$$

with

$$D_* := \min\{D_\kappa,\ \widetilde{D} D_\kappa\}.$$

Therefore,

$$u \in \overset{*}{W}{}^{2,2}(]0,\ell-\varepsilon[,D)$$

implies

$$u \in {}_2W^{2,2}(]0,\ell-\varepsilon[,x_2^\kappa),$$

where

$$ {}_2W^{2,2}(]0,\ell-\varepsilon[,x_2^\kappa) := \Big\{u : \|u\|_{{}_2W^{2,2}(]0,\ell-\varepsilon[,x_2^\kappa)} := \int_0^{\ell-\varepsilon} [x_2^\kappa u^2 + x_2^\kappa (u_{,22})^2] dx_2 < +\infty\Big\}.$$

So,

$$\overset{*}{W}{}^{2,2}(]0,\ell-\varepsilon[,D) \subset {}_2W^{2,2}(]0,\ell-\varepsilon[,x_2^\kappa).$$

But, on the one hand,

$$ {}_2W^{2,2}(]0,\ell-\varepsilon[,x_2^\kappa) \subset {}_2W^{2,2}(]0,\ell-\varepsilon[,x_2^\kappa(\ell-x_2)^\kappa)$$

because of

$$x_2^\kappa(\ell-x_2)^\kappa \leq \ell^\kappa x_2^\kappa \quad \forall\, x_2 \in [0,\ell].$$

On the other hand (see [18, Theorem 1.1.4])

$$ {}_2W^{2,2}(]0,\ell-\varepsilon[,x_2^\kappa(\ell-x_2)^\kappa) = W^{2,2}(]0,\ell-\varepsilon[,x_2^\kappa(\ell-x_2)^\kappa) \quad \forall\, \kappa \in]-1,4].$$

Thus, if (2.33) holds, then the traces of u at $x_2=0$ in the cases mentioned in (2.98) exist (see [18, Theorem 1.1.2] or [17]).

Obviously, from

$$v \in \overset{*}{V}$$

there follows

$$v \in \overset{*}{V}_\varepsilon,$$

where

$$\overset{*}{V}_\varepsilon := \{v \in \overset{*}{W}{}^{2,2}(]\varepsilon,\ell[,D) : v(\ell)=0,\ v_{,2}(\ell)=0\} \tag{2.99}$$

with arbitrarily small $\varepsilon > 0$.

On $[\varepsilon,\ell]$:

1.

$$D(x_2) \geq \min_{[\varepsilon,\ell]} D(x_2) =: D^\varepsilon > 0 \text{ and } \frac{D(x_2)}{D^\varepsilon} \geq 1, \tag{2.100}$$

because of
(i) $D(x_2) \in C([\varepsilon, \ell])$;
(ii) $D(x_2) > 0 \; \forall \, x_2 \in [\varepsilon, \ell]$.

2.

$$\begin{aligned} &P(x_2) = D^{-1}(x_2)\Big[\int_{x_2}^{\ell} D^{-1}(\tau)d\tau\Big]^{-2} \geq \min_{[\varepsilon,\ell]} P(x_2) =: P^\varepsilon > 0, \text{ and} \\ &\frac{P(x_2)}{p^\varepsilon} \geq 1 \end{aligned} \tag{2.101}$$

because of
(i) $P(x_2) \in C([\varepsilon, \ell[)$;
(ii) $P(x_2) > 0 \; \forall \, x_2 \in [\varepsilon, \ell[$;
(iii) $\lim\limits_{x_2 \to \ell-} P(x_2) = \lim\limits_{x_2 \to \ell-} D^{-1}(x_2)\big[\int_{x_2}^{\ell} D^{-1}(\tau)d\tau\big]^{-2} = +\infty$ since $D^{-1}(\ell) > 0$.

3.

$$\begin{aligned} Q(x_2) &:= D(x_2)\Big[\int_{x_2}^{\ell} D^{-1}(\tau)d\tau\Big]^2 \Big\{\int_{x_2}^{\ell} D(t)\Big[\int_t^{\ell} D^{-1}(\tau)d\tau\Big]^2 dt\Big\}^{-2} \\ &\geq \min_{[\varepsilon,\ell]} Q(x_2) =: Q^\varepsilon > 0 \text{ and } \frac{Q(x_2)}{Q^\varepsilon} \geq 1, \end{aligned} \tag{2.102}$$

because of
(i) $Q(x_2) \in C([\varepsilon, \ell[)$;
(ii) $Q(x_2) > 0 \quad \forall \, x_2 \in [\varepsilon, \ell[$;
(iii)

$$\begin{aligned} \lim_{x_2 \to \ell-} Q(x_2) &= D(\ell) \lim_{x_2 \to \ell-} \frac{[\int_{x_2}^{\ell} D^{-1}(\tau)dt]^2}{\{\int_{x_2}^{\ell} D(t)[\int_t^{\ell} D^{-1}(\tau)d\tau]^2 dt\}^2} \\ &= D(\ell) \lim_{x_2 \to \ell-} \frac{2\int_{x_2}^{\ell} D^{-1}(\tau)d\tau \cdot [-D^{-1}(x_2)]}{2\int_{x_2}^{\ell} D(t)[\int_t^{\ell} D^{-1}(\tau)d\tau]^2 dt \{-D(x_2)[\int_{x_2}^{\ell} D^{-1}(\tau)d\tau]^2\}} \\ &= +\infty, \end{aligned}$$

since $0 < D^{-1}(\ell) < +\infty$ and $0 < D(\ell) < +\infty$.

Evidently,

$$u \in \overset{*}{W}{}^{2,2}(]0, \ell[, D)$$

implies

$$u \in \overset{*}{W}{}^{2,2}(]\varepsilon, \ell[, D). \tag{2.103}$$

But from (2.100), (2.102) we have

$$|u|^2 \leq |u|^2 \frac{Q(x_2)}{Q^\varepsilon}, \; |u_{,22}|^2 \leq |u_{,22}|^2 \frac{D(x_2)}{D^\varepsilon} \; \forall \, x_2 \in [\varepsilon, \ell[.$$

Hence, in view of (2.103), we get

$$u \in W^{2,2}(]\varepsilon, \ell[).$$

Moreover, for

$$v \in \overset{*}{W}{}^{2,2}(]0, \ell[, D)$$

with

$$v(\ell) = 0, \quad v_{,2}(\ell) = 0,$$

we have

(2.104) $v \in W^{2,2}(]\varepsilon, \ell[)$

with

$$v(\ell) = 0, \quad v_{,2}(\ell) = 0$$

in the usual sense, since by virtue of (2.104) v and its derivative are absolutely continuous in $[\varepsilon, \ell]$ (more precisely, maybe after necessary change on the set of the measure 0). Thus,

$$v \text{ and } v_{,2} \in AC_R(\varepsilon, \ell)$$

(see [19, p. 5, Definition 1.2]) and the following Hardy type inequalities hold (see [19, p. 66, Theorem 6.4]):

$$\int_\varepsilon^\ell Q(x_2)v^2(x_2)dx_2 \le 4\int_\varepsilon^\ell P(x_2)[v_{,2}(x_2)]^2 dx_2, \tag{2.105}$$

$$\int_\varepsilon^\ell P(x_2)[v_{,2}(x_2)]^2 dx_2 \le 4\int_\varepsilon^\ell D(x_2)[v_{,22}(x_2)]^2 dx_2, \tag{2.106}$$

whence,

$$\int_\varepsilon^\ell Q(x_2)v^2(x_2)dx_2 \le 16\int_\varepsilon^\ell D(x_2)[v_{,22}(x_2)]^2 dx_2. \tag{2.107}$$

Considering the limit procedure as $\varepsilon \to 0+$, since all the limit integrals exist because of $v \in \overset{*}{W}{}^{2,2}(]0, \ell[, D)$, we immediately get the following

2.28. Lemma. *If $v \in \overset{*}{W}{}^{2,2}(]0, \ell[, D)$ and $v(\ell) = 0$, $v_{,2}(\ell) = 0$, then*

$$\int_0^\ell Q(x_2)v^2(x_2)dx_2 \le 16\int_0^\ell D(x_2)[v_{,22}(x_2)]^2 dx_2.$$

2.29. Definition. Let $Q^{-\frac{1}{2}}(x_2)f(x_2) \in L_2(]0, \ell[)$. A function $w \in \overset{*}{W}{}^{2,2}(]0, \ell[, D)$ will be called a *weak solution of the Problem 2.6 in the space* $\overset{*}{W}{}^{2,2}(]0, \ell[, D)$ if it satisfies the following conditions

$$w - u \in \overset{*}{V}$$

and

$$J_\omega(w, v) := \int_0^\ell B_\omega(w, v)dx_2 = \int_0^\ell vf dx_2 + \gamma_2 v(0)Q_2^0 - \gamma_1 v_{,2}(0)M_2^0 \quad \forall\, v \in \overset{*}{V},$$

where

$$B_\omega(w, v) := Dw_{,22}v_{,22} - \omega^2\rho(x_2)\sigma(x_2)wv,$$

$\gamma_1 = 0,\ \gamma_2 = 0$ in the case of BCs (2.13),

$\gamma_1 = 0,\ \gamma_2 = 1$ in the case of BCs (2.14),

$\gamma_1 = 1,\ \gamma_2 = 0$ in the case of BCs (2.15), provided $I_2 < \infty$,

$\gamma_1 = 1,\ \gamma_2 = 1$ in the case of BCs (2.16).

2.30. Theorem. *Let $Q^{-1}(x_2)\rho(x_2)\sigma(x_2) \in C([0, \ell])$ and*

$$\omega^2 < \frac{1}{16\max\limits_{[0,\ell]}(\rho\sigma Q^{-1})}. \tag{2.108}$$

Then there exists a unique weak solution of Problem 2.6 (more precisely of the four BVPs stated there). This solution is such that

$$\|w\|_{\overset{*}{W}^{2,2}(]0,\ell[,D)} \le C\big[\|Q^{-\frac{1}{2}}f\|_{L_2(]0,\ell[)} + \|u\|_{\overset{*}{W}^{2,2}(]0,\ell[,D)} + \gamma_1|M_2^0| + \gamma_2|Q_2^0|\big],$$

where the constant C is independent of f, u, M_2^0, Q_2^0.

Proof. It is easy to see that

$$\begin{aligned}
|J_\omega(w,v)| &= \Big|\int_0^\ell D^{\frac{1}{2}}w_{,22}D^{\frac{1}{2}}v_{,22}dx_2 + \omega^2\int_0^\ell \rho(x_2)\sigma(x_2)Q^{-1}(x_2)Q^{\frac{1}{2}}w\cdot Q^{\frac{1}{2}}vdx_2\Big| \\
&\le \Big[\int_0^\ell D(w_{,22})^2dx_2\Big]^{1/2}\Big[\int_0^\ell D(v_{,22})^2dx_2\Big]^{1/2} \\
&\qquad + \overset{*}{T}\Big[\int_0^\ell Qw^2dx_2\Big]^{1/2}\Big[\int_0^\ell Qv^2dx_2\Big]^{1/2} \\
&\le (1+\overset{*}{T})\|v\|_{\overset{*}{W}^{2,2}(]0,\ell[,D)}\|v\|_{\overset{*}{W}^{2,2}(]0,\ell[,D)},
\end{aligned}$$

where

(2.109) $\overset{*}{T} := \omega^2\max\limits_{[0,\ell]}(\rho\sigma Q^{-1}).$

Hence, the functional

$$F_\omega v := \int_0^\ell v(x_2)f(x_2)dx_2 - J_\omega(u,v) + \gamma_2 v(0)Q_2^0 - \gamma_1 v_{,2}(0)M_2^0,\ \ v\in\overset{*}{V}$$

is bounded in $\overset{*}{V}$:

$$|F_\omega v| \le \big[\|Q^{-1}f\|_{L_2(\Omega)} + (1+\overset{*}{T})\|u\|_{\overset{*}{W}^{2,2}(]0,\ell[,D)} + C_0(\gamma_2|Q_2^0| + \gamma_1|M_2^0|)\big]\|v\|_{\overset{*}{V}},$$

where C_0 is the constant from the trace theorem. In order to use the Lax-Milgram theorem it remains to show the $\overset{*}{V}$-ellipticity of $J_\omega(w,v)$. Indeed, using Lemma 2.28 and introducing the notation

(2.110) $T_0 := 16(1+\overset{*}{T}),$

we have

$$\begin{aligned}
\|v\|^2_{\overset{*}{V}} &:= \int_0^\ell Q(x_2)v^2dx_2 + \int_0^\ell D(x_2)(v_{,22})^2dx_2 \\
&= \int_0^\ell Q(x_2)v^2dx_2 + J_\omega(v,v) + \omega^2\int_0^\ell \rho\sigma Q^{-1}Qv^2dx_2 \\
&\le (1+\overset{*}{T})\int_0^\ell Q(x_2)v^2dx_2 + J_\omega(v,v) \\
&\le 16(1+\overset{*}{T})\int_0^\ell D(x_2)(v_{,22})^2dx_2 + J_\omega(v,v) \\
&= T_0\Big[J_\omega(v,v) + \omega^2\int_0^\ell \rho\sigma Q^{-1}Qv^2dx_2\Big] + J_\omega(v,v) \\
&\le J_\omega(v,v) + T_0\Big[J_\omega(v,v) + \overset{*}{T}\int_0^\ell Qv^2dx_2\Big] \\
&\le J_\omega(v,v) + T_0\Big[J_\omega(v,v) + 16\overset{*}{T}\int_0^\ell D(v_{,22})^2dx_2\Big]
\end{aligned}$$

$$= J_\omega(v,v) + T_0\Big[J_\omega(v,v) + 16\overset{*}{T}J_\omega(v,v) + 16\overset{*}{T}\omega^2\int_0^\ell \rho\sigma Q^{-1}Qv^2dx_2\Big]$$

$$\leq J_\omega(v,v) + T_0\Big[J_\omega(v,v) + 16\overset{*}{T}J_\omega(v,v) + (16\overset{*}{T})^2\int_0^\ell D(v_{,22})^2dx_2\Big]$$

$$= J_\omega(v,v) + T_0\Big[J_\omega(v,v) + 16\overset{*}{T}J_\omega(v,v) + (16\overset{*}{T})^2J_\omega(v,v)$$

$$+ (16\overset{*}{T})^2\overset{*}{T}\int_0^\ell Qv^2dx_2\Big] \leq J_\omega(v,v)$$

$$+ T_0\Big\{J_\omega(v,v)\Big[1 + 16\overset{*}{T} + (16\overset{*}{T})^2 + (16\overset{*}{T})^3\int_0^\ell D(v_{,22})^2dx_2\Big]\Big\}$$

(repeating the same $(n-2)$-times more)

$$\leq J_\omega(v,v) + T_0\Big[J_\omega(v,v)\frac{1-(16\overset{*}{T})^{n+1}}{1-16\overset{*}{T}} + (16\overset{*}{T})^{n+1}\int_0^\ell D(v_{,22})^2dx_2\Big].$$

Now, letting n tend to infinity and taking into account that

$$16\overset{*}{T} < 1$$

because of (2.108), (2.109), we obtain

$$\|v\|^2_{\overset{*}{V}} \leq J_\omega(v,v) + \frac{T_0}{1-16\overset{*}{T}}J_\omega(v,v).$$

Whence,

$$J_\omega(v,v) \geq \frac{1-16\overset{*}{T}}{1-16\overset{*}{T}+T_0}\|v\|^2_{\overset{*}{V}} = \frac{1-16\overset{*}{T}}{17}\|v\|^2_{\overset{*}{V}},$$

since, in view of (2.110),

$$1 - 16\overset{*}{T} + T_0 = 17.$$

Now, we can use the Lax-Milgram theorem and complete the proof similarly to the proof of Theorem 2.20. □

3. Cusped Kirchhoff-Love Plates

In this section we omit, for the sake of simplicity, the upper index p in J^p_ω, D^p, M^p_α, M^p_{12}, M^p_{21}, Q^p_α and $\alpha = 1, 2$.

Similarly to Subsections 2.2 and 2.3 concerning cusped beams, we set problems with admissible BCs at the cusped edge of the plates, and investigate weak solutions.

§ 3.1. The Vibration Problem

3.1. Problem. Let us consider for equation (1.7) the following inhomogeneous BCs:

– On Γ_2

(3.1) $$w = g_1,\ \frac{\partial w}{\partial n} = g_2,$$

– On Γ_1

either

(3.2) $$w = w_0(x_1),\ w_{,2} = w_0^1(x_1) \quad \text{iff } I_0(x_1) < +\infty,$$

or

$$w_{,2} = w_0^1(x_1),\ Q_2^* = Q_2^0(x_1) \text{ iff } I_0(x_1) < +\infty, \tag{3.3}$$

or

$$w = w_0(x_1),\ M_2 = M_2^0(x_1) \begin{cases} \not\equiv 0 \text{ when } I_0(x_1) < +\infty \\ \equiv 0 \text{ when } I_0(x_1) = +\infty \end{cases} \text{ iff } I_2 < +\infty, \tag{3.4}$$

or

$$\begin{aligned} M_2 = M_2^0(x_1) &\begin{cases} \not\equiv 0 \text{ when } I_0(x_1) < +\infty, \\ \equiv 0 \text{ when } I_0(x_1) = +\infty, \end{cases} \\ Q_2^* = Q_2^0(x_1) &\begin{cases} \not\equiv 0 \text{ when } I_1(x_1) < +\infty, \\ \equiv 0 \text{ when } I_1(x_1) = +\infty, \end{cases} \end{aligned} \tag{3.5}$$

where g_1, g_2 and w_0, w_0^1, Q_2^0, M_2^0 are prescribed functions on Γ_2 and Γ_1, respectively,

$$\begin{aligned} &I_k(x_1) := \int_0^{\ell(x_1)} \tau^k D^{-1}(x_1, \tau) d\tau, \\ &k \in \{0, 1, \dots\},\ (x_1, \ell(x_1)) \in \Omega \text{ for } (x_1, 0) \in \Gamma_2. \end{aligned} \tag{3.6}$$

Let us now introduce some function spaces.

3.2. Definition. Let

$$W^{2,2}(\Omega, p) \text{ and } \widetilde{W}^{2,2}(\Omega, p) \tag{3.7}$$

be the sets of all measurable functions $w(x_1, x_2)$ defined on Ω which have on Ω locally summable generalized derivatives $\partial_{x_1,x_2}^{(\alpha_1,\alpha_2)} w$ for $\alpha_1 + \alpha_2 \in \{0, 2\}$, $\alpha_1, \alpha_2 \in \{0, 1, 2\}$, such that

$$\int_\Omega \rho_{\alpha_1,\alpha_2}(x_1, x_2) |\partial_{x_1,x_2}^{(\alpha_1,\alpha_2)} w|^2 d\Omega < +\infty,\ \partial_{x_1,x_2}^{(0,0)} w = w, \tag{3.8}$$

for

$$\rho_{0,0} := 1,\ \rho_{2,0} = \rho_{1,1} = \rho_{0,2} := p(x_1, x_2)$$

and

$$\rho_{0,0} := x_2^{\kappa-4},\ \rho_{2,0} = \rho_{1,1} = \rho_{0,2} := p(x_1, x_2),$$

respectively, with $p(x_1, x_2)$ a bounded function measurable on Ω.

Let us further consider the following sets for different cases of the function $p(x_1, x_2)$:

$$W^{2,2}(\Omega, D) \text{ and } \widetilde{W}^{2,2}(\Omega, D) \tag{3.9}$$

with $p(x_1, x_2) = D(x_1, x_2)$ satisfying (1.9), and

$$V^{2,2}(\Omega, x_2^\kappa) := W^{2,2}(\Omega, x_2^\kappa), \quad (p(x_1, x_2) = x_2^\kappa), \tag{3.10}$$

$$\widetilde{V}^{2,2}(\Omega, x_2^\kappa) := \widetilde{W}^{2,2}(\Omega, x_2^\kappa), \quad (p(x_1, x_2) = x_2^\kappa), \tag{3.11}$$

and

$$V^{2,2}(\Omega, d^\kappa) := W^{2,2}(\Omega, d^\kappa) \quad (p(x_1, x_2) = d(x_1, x_2)), \tag{3.12}$$

where

$$d(x_1, x_2) := \operatorname{dist}\{(x_1, x_2) \in \Omega,\ \partial\Omega\}.$$

Further, let us introduce the following norms:

$$\|w\|^2_{W^{2,2}(\Omega,D)} := \int_\Omega [w^2 + \nu D(w_{,11}+w_{,22})^2 + (1-\nu)D(w_{,11})^2 + 2(1-\nu)D(w_{,12})^2 + (1-\nu)D(w_{,22})^2]d\Omega, \tag{3.13}$$

$$\|w\|^2_{\widetilde{W}^{2,2}(\Omega,D)} := \int_\Omega [x_2^{\kappa-4}w^2 + \nu D(w_{,11}+w_{,22})^2 + (1-\nu)D(w_{,11})^2 + 2(1-\nu)D(w_{,12})^2 + (1-\nu)D(w_{,22})^2]d\Omega, \tag{3.14}$$

$$\|w\|^2_{\widetilde{V}^{2,2}(\Omega,x_2^\kappa)} := \int_\Omega \{x_2^{\kappa-4}w^2 + x_2^\kappa[(w_{,11})^2(w_{,12})^2 + (w_{,22})^2]\}d\Omega, \tag{3.15}$$

$$\|w\|^2_{V^{2,2}(\Omega,x_2^\kappa)} := \int_\Omega \{w^2 + x_2^\kappa[(w_{,11})^2 + (w_{,12})^2 + (w_{,22})^2]\}d\Omega, \tag{3.16}$$

$$\|w\|^2_{V^{2,2}(\Omega,d^\kappa)} := \int_\Omega \{w^2 + d^\kappa[(w_{,11})^2 + (w_{,12})^2 + (w_{,22})^2]\}d\Omega. \tag{3.17}$$

From (1.9) it is clear that in our cases if $D \in C(\overline{\Omega})$, then

$$\rho^{-1}_{\alpha_1,\alpha_2} \in L_1^{\mathrm{loc}}(\Omega).$$

Therefore, according to [KO], the spaces (3.9)–(3.12) with the norms (3.13)–(3.17), respectively, will be Banach spaces, and moreover, Hilbert spaces under the appropriate scalar products.

3.3. Lemma.

$$V^{2,2}(\Omega, x_2^\kappa) \subset V^{2,2}(\Omega, d^\kappa(x_1,x_2)) \ \forall\, \kappa \geq 0. \tag{3.18}$$

Proof. This follows from the obvious inequality

$$d(x_1,x_2) \leq x_2 \text{ for } (x_1,x_2) \in \Omega \tag{3.19}$$

(if $d(x_1,x_2)$ is a regularized distance, then in the inequality (3.19) arises a constant factor). □

Further, without loss of generality, we suppose that the domain Ω lies inside the rectangle:

$$\Pi := \{(x_1,x_2) \in \mathbb{R}^2 : a < x_1 < b,\ 0 < x_2 < \ell\}, \tag{3.20}$$

with a constant $\ell > \max\limits_{(x_1,x_2)\in\overline{\Omega}}\{x_2\}$.

3.4. Lemma.

$$V^{2,2}(\Omega, x_2^\kappa) \subset V^{2,2}(\Omega, x_2^4) \ \textit{for}\ 0 \leq \kappa < 4. \tag{3.21}$$

$$V^{2,2}(\Omega, x_2^4) = \widetilde{V}^{2,2}(\Omega, x_2^4). \tag{3.22}$$

Proof. The proof of (3.21) follows from $\ell^{4-\kappa}x_2^\kappa \geq x_2^4$ for $0 \leq x_2 < \ell$, (3.22) is evident. □

Let

$$\Omega_\delta := \{(x_1,x_2) \in \Omega : x_2 > \delta,\ \delta = \text{const } > 0\}.$$

Evidently,

$$\widetilde{V}^{2,2}(\Omega, x_2^\kappa) \subset \widetilde{V}^{2,2}(\Omega_\delta, x_2^\kappa) \subset W^{2,2}(\Omega_\delta), \tag{3.23}$$

where $W^{2,2}(\Omega_\delta)$ is the usual (i.e., non-weighted) Sobolev space. Hence, there exist the traces

$$w|_{\Gamma_2} \in W^{\frac{3}{2},2}(\Gamma_2),\ \left.\frac{\partial w}{\partial n}\right|_2 \in W^{\frac{1}{2},2}(\Gamma_2) \quad \forall\ v \in \widetilde{V}^{2,2}(\Omega, x_2^\kappa).$$

3.5. Lemma. *If* $v \in \widetilde{V}^{2,2}(\Omega, x_2^\kappa)$ *and*

$$v|_{\Gamma_2} = 0, \ \frac{\partial v}{\partial n}\Big|_{\Gamma_2} = 0, \tag{3.24}$$

then

$$\int_\Omega x_2^{\kappa-4} v^2 d\Omega \le \frac{16}{(\kappa-1)^2(\kappa-3)^2} \int_\Omega x_2^\kappa (v_{,22})^2 d\Omega, \ \kappa > 3. \tag{3.25}$$

Proof. Let us complete the definition of the function v in $\Pi \setminus \Omega$ by assuming v to be equal to zero there. Then evidently,

$$v \in \widetilde{V}^{2,2}(\Omega, x_2^\kappa)$$

implies

$$\int_\Pi [x_2^{\kappa-4} v^2 + x_2^\kappa (v_{,22})^2] dx_1 dx_2 < +\infty,$$

i.e.,

$$v(x_1, \cdot) \in \widetilde{W}^{2,2}(]0, \ell[, x_2^\kappa)$$

(see (2.59)) and

$$v(x_1, \ell) = 0, \ v_{,2}(x_1, \ell) = 0$$

for almost every $x_1 \in]a, b[$. We can now apply Lemma 2.17, i.e.,

$$\int_0^\ell x_2^{\kappa-4} v^2(x_1, x_2) dx_2 \le \frac{16}{(\kappa-1)^2(\kappa-3)^2} \int_0^\ell x_2^\kappa [v_{,2}(x_1, x_2)]^2 dx_2, \ \kappa > 3, \tag{3.26}$$

for almost every $x_1 \in]a, b[$. Integrating both sides of (3.26) with respect to x_1 over $]a, b[$, we get

$$\begin{aligned} \int_\Omega x_2^{\kappa-4} v^2 d\Omega &= \int_\Pi x_2^{\kappa-4} v^2 dx_1 dx_2 \\ &\le \frac{16}{(\kappa-1)^2(\kappa-3)^2} \int_\Pi x_2^\kappa (v_{,2})^2 dx_1 dx_2 \\ &= \frac{16}{(\kappa-1)^2(\kappa-3)^2} \int_\Omega x_2^\kappa (v_{,22}) d\Omega \text{ for } \kappa > 3. \end{aligned}$$

□

3.6. Corollary. *If* $v \in \widetilde{V}^{2,2}(\Omega, x_2^4) = V^{2,2}(\Omega, x_2^4)$ *and* (3.24) *is fulfilled, then*

$$\int_\Omega v^2 d\Omega \le \frac{16}{9} \int_\Omega x_2^4 (v_{,22})^2 d\Omega.$$

Let

$$D(x_1, x_2) \ge D_\kappa x_2^\kappa \quad \forall \ (x_1, x_2) \in \Omega, \ \textit{i.e.,} \ 0 < D_\kappa := \inf_\Omega \frac{D(x_1, x_2)}{x_2^\kappa}. \tag{3.27}$$

If $\kappa \ge 1$, then by κ we denote the minimal among all the exponents $\delta \ge \kappa \ge 1$ for which (3.27) holds. This means that if we have the inequality (3.27) for $\kappa \ge 1$, we have to check whether or not there exists a smaller exponent for which (3.27) is valid. If so, then we have to continue this procedure until we arrive at the minimal one.

If (3.27) holds for $\kappa < 1$, then we need no additional revision since for all $\kappa < 1$ we have the same result concerning the traces.

The condition (3.27) is essential in a right neighbourhood of Γ_1. Then it can be easily extended for the whole domain Ω.

Let us note that, when $\omega = 0$, Problem 3.1 is considered in [Jai5] under a condition different from the condition (3.27), namely

$$0 \le D_0 := \inf_{\Omega} \frac{D(x_1, x_2)}{x_2^4},$$

which does not make it possible to discuss the traces of solutions when the thickness is of the non-power type. Besides, in [7] the spaces, when $D_0 = 0$ (i.e., in the particular case of the power type thickness, when $\kappa \ge 4$), are not transparent (in the sense of the so called ideal elements) even in the case of the power type thickness.

3.7. Lemma. *If* (3.27) *holds, then*

$$W^{2,2}(\Omega, D) \subset V^{2,2}(\Omega, x_2^\kappa) \subset V^{2,2}(\Omega, d^\kappa(x_1, x_2)) \ \forall \kappa \ge 0, \tag{3.28}$$

and

$$W^{2,2}(\Omega, D) \subset \widetilde{W}^{2,2}(\Omega, D) \subset \widetilde{V}^{2,2}(\Omega, x_2^\kappa) \ \text{for } \kappa \ge 4. \tag{3.29}$$

Proof. The proof of (3.28) follows from (3.27) and (3.18). Formula (3.29) follows from (3.27), since

$$\int_\Omega w^2 d\Omega < +\infty$$

implies

$$\int_\Omega x_2^{\kappa-4} w^2 d\Omega < +\infty$$

for $\kappa \ge 4$. □

3.8. Lemma. *If $w \in W^{2,2}(\Omega, D)$, and (3.27) is valid, then there exist traces*

$$w|_{\Gamma_1} \in B_2^{\frac{3-\kappa}{2}}(\Gamma_1) \subset L_2(\Gamma_1) \ \text{if } 0 \le \kappa < 3 \ (\text{i.e., } I_2(x_1) < +\infty), \tag{3.30}$$

$$w_{,2}|_{\Gamma_1} \in B^{\frac{1-\kappa}{2}}(\Gamma_1) \subset L_2(\Gamma_1) \ \text{if } 0 \le \kappa < 1 \ (\text{i.e., } I_0(x_1) < +\infty), \tag{3.31}$$

where $B_2^{\frac{3-\kappa}{2}}(\Gamma_1)$ and $B_2^{\frac{1-\kappa}{2}}(\Gamma_1)$ are Besov spaces.

Proof. Since (3.27) is valid, according to Lemma 3.7 (see (3.28)), $w \in W^{2,2}(\Omega, D)$ implies

$$w \in V^{2,2}(\Omega, d^\kappa(x_1, x_2)).$$

But functions from this space (see [18, Theorem 1.1.2] or [16, Subsection 10.1], and also [12, p. 74]) have properties (3.30) and (3.31) if $\partial\Omega \in C^{1+\varepsilon}$ and $\partial\Omega \in C^{2+\varepsilon}$ (which means that the boundary is locally described by functions whose first and second derivatives, correspondingly, satisfy the Hölder condition with a Hölder exponent $\varepsilon \in]0, 1[$, respectively). Since in our case Γ_1 is a part of a straight line, these local conditions are fulfilled all the more. □

Now, we constitute the spaces V and $\widetilde{V}$ from the spaces $W^{2,2}(\Omega, D)$ and $\widetilde{W}^{2,2}(\Omega, D)$, respectively, as follows:

$$\begin{aligned} V := \Big\{ & v \in W^{2,2}(\Omega, D) : v|_{\Gamma_2} = 0, \ \frac{\partial v}{\partial n}\Big|_{\Gamma_2} = 0, \text{ and additionally} \\ & \text{either } v|_{\Gamma_1} = 0, \ v_{,2}|_{\Gamma_1} = 0 \text{ (if we consider BCs (3.2))} \\ & \text{or } v_{,2}|_{\Gamma_1} = 0 \text{ (if we consider BCs (3.3))} \\ & \text{or } v|_{\Gamma_1} = 0 \text{ (if we consider BCs (3.4))} \\ & \text{in the sense of traces} \Big\} \end{aligned} \tag{3.32}$$

and

(3.33) $\widetilde{V} := \left\{ v \in \widetilde{W}^{2,2}(\Omega, D) : v|_{\Gamma_2} = 0,\ \frac{\partial v}{\partial n}\Big|_{\Gamma_2} = 0 \text{ in the sense of traces} \right\}.$

Using the trace theorem, it is not difficult to prove the completeness of V and $\widetilde{V}$.

In view of Lemma 3.8, we can suppose that the functions g_1, g_2, w_0, w_0^1 from Problem 3.1 are traces of a prescribed function

(3.34) $u \in W^{2,2}(\Omega, D).$

Let further $Q_2^0, M_2^0 \in L_2(\Gamma_1)$.

3.9. Definition. Let $f \in L_2(\Omega)$ and $\kappa < 4$ (i.e., $I_3(x_1)|_{\Gamma_1} < +\infty$). A function $w \in W^{2,2}(\Omega, D)$ will be called a *weak solution of Problem 3.1*, when $I_3(x_1)|_{\Gamma_1} < +\infty$, in the space $W^{2,2}(\Omega, D)$ if it satisfies the following conditions:

(3.35) $w - u \in V$

and

$$\begin{aligned} J_\omega(w,v) := \int_\Omega B_\omega(w,v)\,d\Omega = \int_\Omega fv\,d\Omega + \gamma_2 \int_{\Gamma_1} Q_2^0 v\,dx_1 \\ - \gamma_1 \int_{\Gamma_1} M_2^0 v_{,2}\,dx_1 \ \forall v \in V, \end{aligned} \tag{3.36}$$

where

$\gamma_1 = 0,\ \gamma_2 = 0$ for the BCs (3.2),
$\gamma_1 = 0,\ \gamma_2 = 1$ for the BCs (3.3),
$\gamma_1 = 1,\ \gamma_2 = 0$ for the BCs (3.4),
$\gamma_1 = 1,\ \gamma_2 = 1$ for the BCs (3.5),

and

$$\begin{aligned} B_\omega(w,v) := \nu D(w_{,11} + w_{,22})(v_{,11} + v_{,22}) + (1-\nu) D w_{,11} v_{,11} \\ + 2(1-\nu) D w_{,12} v_{,12} + (1-\nu) D w_{,22} v_{,22} - \omega^2 2h\rho w v. \end{aligned} \tag{3.37}$$

3.10. Definition. Let g_1 and g_2 be traces of a prescribed $u \in \widetilde{W}(\Omega, D)$, $x_2^{\frac{4-\kappa}{2}} f \in L_2(\Omega)$, and $\kappa \geq 4$ (i.e., $I_k(x_1)|_{\Gamma_1} = +\infty$ for a fixed $k \geq 3$). A function $w \in \widetilde{W}^{2,2}(\Omega, D)$ will be called a *weak solution of the problem* (1.7), (3.1), (3.5) (*i.e., of the last BVP of Problem 3.1 when* $I_k(x_1)|_{\Gamma_1} = +\infty$ *for a fixed* $k \geq 3$) *in the space* $\widetilde{W}^{2,2}(\Omega, D)$ if it satisfies the following conditions:

(3.38) $w - u \in \widetilde{V}$

and

(3.39) $J_\omega(w,v) := \int_\Omega B_\omega(w,v)\,d\Omega = \int_\Omega fv\,d\Omega \ \forall v \in \widetilde{V},$

where $B_\omega(w,v)$ is defined by (3.37).

3.11. Theorem. *Let*

(3.40) $\omega^2 < \dfrac{9(1-\nu) D_\kappa \ell^{\kappa-4}}{16 \max\limits_{\overline{\Omega}} 2h\rho}.$

There exists a unique weak solution of Problem 3.1, when $I_3 < +\infty$ (more precisely, of each of all the four BVPs stated in Problem 3.1). This solution is such that

$$\|w\|_{W^{2,2}(\Omega,D)} \le C[\|f\|_{L_2(\Omega)} + \|u\|_{W^{2,2}(\Omega,D)} + \gamma_1\|M_2^0\|_{L_2(\Gamma_1)} + \gamma_2\|Q_2^0\|_{L_2(\Gamma_1)}], \tag{3.41}$$

where the constant C is independent of f, u, M_2^0, and Q_2^0.

3.12. Theorem. *Let $2h\rho x_2^{4-\kappa} \in C(\overline{\Omega})$ and*

$$\omega^2 < \frac{(\kappa-1)^2(\kappa-3)^2(1-\nu)D_\kappa}{16\max\limits_{\overline{\Omega}} 2h\rho x_2^{4-\kappa}}. \tag{3.42}$$

There exists a unique weak solution of Problem 3.1, when $I_k(x_1)|_{\Gamma_1} = +\infty$ for a fixed $k \ge 3$. This solution is such that

$$\|w\|_{\widetilde{W}^{2,2}(\Omega,D)} \le C[\|x_2^{\frac{4-\kappa}{2}} f\|_{L_2(\Omega)} + \|u\|_{\widetilde{W}^{2,2}(\Omega,D)}], \tag{3.43}$$

where the constant C is independent of f and u.

Proof of Theorem 3.11. This is similar to that of Theorem 2.20, and is based on the Lax-Milgram theorem. It is easy to show the following three inequalities (see (3.47), (3.50), (3.52) below which imply the proof).

In view of (3.36), (3.37), (3.13), we have

$$\begin{aligned}
|J_\omega(w,v)| &\le \int_\Omega (\nu D)^{\frac12}|w_{,11}+w_{,22}|\cdot(\nu D)^{\frac12}|v_{,11}+v_{,22}|d\Omega \\
&\quad + \int_\Omega [(1-\nu)D]^{\frac12}|w_{,11}|\cdot[(1-\nu)D]^{\frac12}|v_{,11}|d\Omega \\
&\quad + \int_\Omega [2(1-\nu)D]^{\frac12}|w_{,12}|\cdot[2(1-\nu)D]^{\frac12}|v_{,12}|d\Omega \\
&\quad + \int_\Omega [(1-\nu)D]^{\frac12}|w_{,22}|\cdot[(1-\nu)D]^{\frac12}|v_{,22}|d\Omega + T\int_\Omega |w||v|d\Omega \\
&\le \Big[\int_\Omega \nu D(w_{,11}+w_{,22})^2 d\Omega\Big]^{\frac12}\Big[\int_\Omega \nu D(v_{,11}+v_{,22})^2 d\Omega\Big]^{\frac12} \\
&\quad + \Big[\int_\Omega (1-\nu)D(w_{,11})^2 d\Omega\Big]^{\frac12}\Big[\int_\Omega (1-\nu)D(v_{,11})^2 d\Omega\Big]^{\frac12} \\
&\quad + \Big[\int_\Omega 2(1-\nu)D(w_{,12})^2 d\Omega\Big]^{\frac12}\Big[\int_\Omega 2(1-\nu)D(w_{,12})^2 d\Omega\Big]^{\frac12} \\
&\quad + \Big[\int_\Omega (1-\nu)D(w_{,22})^2 d\Omega\Big]^{\frac12}\Big[\int_\Omega (1-\nu)D(v_{,22})^2 d\Omega\Big]^{\frac12} \\
&\quad + T\Big[\int_\Omega w^2 d\Omega\Big]^{\frac12}\Big[\int_\Omega v^2 d\Omega\Big]^{\frac12} \\
&\le (4+T)\|w\|_{W^{2,2}(\Omega,D)}\|v\|_{W^{2,2}(\Omega,D)},
\end{aligned} \tag{3.44}$$

where

$$T := 2\omega^2 \max_{\overline{\Omega}}[h(x_1,x_2)\rho(x_1,x_2)]. \tag{3.45}$$

In particular,

$$|J_\omega(w,v)| \le (4+T)\|w\|_{W^{2,2}(\Omega,D)}\|v\|_V \quad \forall\, w \in W^{2,2}(\Omega,D) \text{ and } \forall\, v \in V \tag{3.46}$$

and

$$|J_\omega(w,v)| \le (4+T)\|w\|_V\|v\|_V \ \forall\, w,v \in V. \tag{3.47}$$

Taking into account (3.46), and

$$\Big|\int_{\Gamma_1} vQ_2^0 dx_1\Big| \le \|v\|_{L_2(\Gamma_1)}\|Q_2^0\|_{L_2(\Gamma_1)} \le C_0\|v\|_V\|Q_2^0\|_{L_2(\Gamma_1)}, \tag{3.48}$$

$$\begin{aligned}\Big|\int_{\Gamma_1} v_{,2}M_2^0 dx_1\Big| &\le \|v_{,2}\|_{L_2(\Gamma_1)}\|M_2^0\|_{L_2(\Gamma_1)}\\ &\le C_0\|v\|_V\|M_2^0\|_{L_2(\Gamma_1)}\end{aligned} \tag{3.49}$$

with the positive constant C_0 from the trace theorem, it is not difficult to see, that the functional

$$F_\omega v := \int_\Omega fv d\Omega - J_\omega(u,v) + \gamma_2\int_{\Gamma_1} Q_2^0 v dx_1 - \gamma_1\int_{\Gamma_1} M_2^0 v_{,2} dx_1,\ v\in V,$$

is bounded in V:

$$\begin{aligned}|F_\omega v| \le \{\|f\|_{L_2(\Omega)} + (4+T)\|u\|_{W^{2,2}(\Omega,D)} + C_0 + [\|Q_2^0\|_{L_2(\Gamma_1)}&\\ + \|M_2^0\|_{L_2(\Gamma_1)}]\}\|v\|_V&.\end{aligned} \tag{3.50}$$

Let

$$T_0 := \frac{16\ell^{4-\kappa}(1+T)}{9(1-\nu)D_\kappa},\quad T_1 := \frac{16\ell^{4-\kappa}T}{9(1-\nu)D_\kappa}. \tag{3.51}$$

Analogously to (2.79), in view of Corollary 3.6, we get

$$\begin{aligned}\|v\|_V^2 &= \int_\Omega\{v^2 + D[\nu(v_{,11}+v_{,22})^2 + (1-\nu)(v_{,11})^2 + 2(1-\nu)(v_{,12})^2\\ &\qquad + (1-\nu)(v_{,22})^2]\}d\Omega\\ &= \int_\Omega v^2 d\Omega + J_\omega(v,v) + 2\omega^2\int_\Omega h\rho v^2 d\Omega\\ &\le J_\omega(v,v) + (1+T)\int_\Omega v^2 d\Omega\\ &\le J_\omega(v,v) + T_0\int_\Omega (1-\nu)D_\kappa x_2^\kappa(v_{,22})d\Omega\\ &\le J_\omega(v,v) + T_0\int_\Omega (1-\nu)D(v_{,22})^2 d\Omega\\ &\le J_\omega(v,v) + T_0\int_\Omega D[(1-\nu)(v_{,22})^2 + \nu(v_{,11}+v_{,22})^2 + (1-\nu)(v_{,11})^2\\ &\qquad + 2(1-\nu)(v_{,12})^2]d\Omega\\ &= J_\omega(v,v) + T_0\Big[J_\omega(v,v) + 2\omega^2\int_\Omega h\rho v^2 d\Omega\Big]\\ &\le J_\omega(v,v) + T_0\Big[J_\omega(v,v) + T\int_\Omega v^2 d\Omega\Big]\\ &\le J_\omega(v,v) + T_0\Big[J_\omega(v,v) + T_1\int_\Omega (1-\nu)D(v_{,22})^2 d\Omega\Big]\\ &\le J_\omega(v,v) + T_0\Big\{J_\omega(v,v) + T_1\Big[J_\omega(v,v) + T_1\int_\Omega(1-\nu)D(v_{,22})^2 d\Omega\Big]\Big\}\end{aligned}$$

$$\leq J_\omega(v,v) + T_0\Big\{J_\omega(v,v) + T_1 J_\omega(v,v) + (T_1)^2\Big[J_\omega(v,v) + T_1\int_\Omega (1-\nu)D(v_{,22})^2 d\Omega\Big]\Big\}$$

$$= J_\omega(v,v) + T_0\Big\{J_\omega(v,v)[1+T_1+(T_1)^2] + (T_1)^3\int_\Omega (1-\nu)D(v_{,22})^2 d\Omega\Big\}$$

(repeating the same $(n-2)$-times more)

$$\leq J_\omega(v,v) + T_0\Big[J_\omega(v,v)\frac{1-(T_1)^{n+1}}{1-T_1} + (T_1)^{n+1}\int_\Omega (1-\nu)D(v_{,22})^2 d\Omega\Big].$$

Now, letting n tend to infinity and taking into account that by virtue of (3.51), (3.45) and (3.40),

$$T_1 < 1,$$

we obtain

$$\|v\|_V^2 \leq J_\omega(v,v) + \frac{T_0}{1-T_1} J_\omega(v,v) = \frac{1-T_1+T_0}{1-T_1} J_\omega(v,v),$$

whence, in view of (3.51),

$$J_\omega(v,v) \geq \frac{9(1-\nu)D_\kappa - 16\ell^{4-\kappa}T}{9(1-\nu)D_\kappa + 16\ell^{4-\kappa}}\|v\|_V^2. \tag{3.52}$$

□

Proof of Theorem 3.12. This is similar to that of Theorem 2.22 and Theorem 3.11. Obviously,

$$\begin{aligned}
|J_\omega(w,v)| \leq & \int_\Omega (\nu D)^{\frac{1}{2}} |w_{,11} + w_{,22}| \cdot (\nu D)^{\frac{1}{2}} |v_{,11} + v_{,22}| d\Omega \\
& + \int_\Omega [(1-\nu)D]^{\frac{1}{2}} |w_{,11}| \cdot [(1-\nu)D]^{\frac{1}{2}} |v_{,11}| d\Omega \\
& + \int_\Omega [2(1-\nu)D]^{\frac{1}{2}} |w_{,12}| \cdot [2(1-\nu)D]^{\frac{1}{2}} |v_{,12}| d\Omega \\
& + \int_\Omega [(1-\nu)D]^{\frac{1}{2}} |w_{,22}| \cdot [(1-\nu)D]^{\frac{1}{2}} |v_{,22}| d\Omega \\
& + 2\omega^2 \int_\Omega h\rho x_2^{4-\kappa} x_2^{\frac{\kappa-4}{2}} |w| x_2^{\frac{\kappa-4}{2}} |v| d\Omega \\
\leq & \Big[\int_\Omega \nu D(w_{,11}+w_{,22})^2 d\Omega\Big]^{\frac{1}{2}} \Big[\int_\Omega \nu D(v_{,11}+v_{,22})^2 d\Omega\Big]^{\frac{1}{2}} \\
& + \Big[\int_\Omega (1-\nu)D(w_{,11})^2 d\Omega\Big]^{\frac{1}{2}} \Big[\int_\Omega (1-\nu)D(v_{,11})^2 d\Omega\Big]^{\frac{1}{2}} \\
& + \Big[\int_\Omega 2(1-\nu)D(w_{,12})^2 d\Omega\Big]^{\frac{1}{2}} \Big[\int_\Omega 2(1-\nu)D(w_{,12})^2 d\Omega\Big]^{\frac{1}{2}} \\
& + \Big[\int_\Omega (1-\nu)D(w_{,22})^2 d\Omega\Big]^{\frac{1}{2}} \Big[\int_\Omega (1-\nu)D(v_{,22})^2 d\Omega\Big]^{\frac{1}{2}} \\
& + T_*\Big[\int_\Omega x_2^{\kappa-4} w^2 d\Omega\Big]^{\frac{1}{2}} \Big[\int_\Omega x_2^{\kappa-4} v^2 d\Omega\Big]^{\frac{1}{2}} \\
\leq & (4+T_*)\|w\|_{\widetilde{W}^{2,2}(\Omega,D)} \|v\|_{\widetilde{W}^{2,2}(\Omega,D)},
\end{aligned}$$

where

$$T_* := 2\omega^2 \max_{\overline{\Omega}}[h\rho x_2^{4-\kappa}]. \tag{3.53}$$

Further, for

$$F_\omega v := \int_\Omega f v d\Omega - J_\omega(w, v), \quad v \in \widetilde{V},$$

we have

$$\begin{aligned}|F_\omega v| &\le \{\|f\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)} + (4+T_*)\|w\|_{\widetilde{W}^{2,2}(\Omega,D)}\|v\|_{\widetilde{W}^{2,2}(\Omega,D)}\}\\ &\le \{\|f\|_{L_2(\Omega)} + (4+T_*)\|w\|_{\widetilde{W}^{2,2}(\Omega,D)}\}\|v\|_{\widetilde{V}},\end{aligned}$$

since $\|v\|_{L_2(\Omega)} \le \|v\|_{\widetilde{V}}$ and $\|v\|_{\widetilde{W}^{2,2}(\Omega,D)} = \|v\|_{\widetilde{V}}$.

Let

$$T_0^* := \frac{16(1+T_*)}{(\kappa-1)^2(\kappa-3)^2 D_\kappa(1-\nu)}, \quad T_1^* := \frac{16T_*}{(\kappa-1)^2(\kappa-3)^2 D_\kappa(1-\nu)}. \tag{3.54}$$

Evidently, taking into account the second imbedding of (3.29), Lemma 3.5, (3.27) and (3.37),

$$\begin{aligned}\|v\|^2_{\widetilde{V}} &:= \int_\Omega \{x_2^{\kappa-4}v^2 + D[\nu(v_{,11}+v_{,22})^2 + (1-\nu)(v_{,11})^2\\ &\qquad + 2(1-\nu)(v_{,12})^2 + (1-\nu)(v_{,22})^2]\}d\Omega\\ &= \int_\Omega x_2^{\kappa-4}v^2 d\Omega + J_\omega(v,v) + 2\omega^2\int_\Omega h\rho x_2^{4-\kappa}x_2^{\kappa-4}v^2 d\Omega\\ &\le J_\omega(v,v) + (1+T_*)\int_\Omega x_2^{\kappa-4}v^2 d\Omega\\ &\le J_\omega(v,v) + T_0^*\int_\Omega (1-\nu)D_\kappa x_2^\kappa (v_{,22})^2 d\Omega\\ &\le J_w(v,v) + T_0^*\int_\Omega (1-\nu)D(v_{,22})^2 d\Omega\\ &\le J_\omega(v,v) + T_0^*\int_\Omega D[\nu(v_{,11}+v_{,22})^2 + (1-\nu)(v_{,11})^2\\ &\qquad + 2(1-\nu)(v_{,12})^2 + (1-\nu)(v_{,22})^2]d\Omega\\ &= J_\omega(v,v) + T_0^*[J_w(v,v) + 2\omega^2\int_\Omega h\rho x_2^{4-\kappa}x_2^{\kappa-4}v^2 d\Omega]\\ &\le J_\omega(v,v) + T_0^*[J_\omega(v,v) + T_*\int_\Omega x_2^{\kappa-4}v^2 d\Omega]\\ &\le J_\omega(v,v) + T_0^*[J_\omega(v,v) + T_1^*\int_\Omega (1-\nu)D(v_{,22})^2 d\Omega]\\ &\le J_\omega(v,v) + T_0^*\{J_\omega(v,v) + T_1^*[J_\omega(v,v) + T_1^*\int_\Omega (1-\nu)D(v_{,22})^2 d\Omega]\}\\ &\le J_\omega(v,v) + T_0^*\{J_\omega(v,v) + T_1^* J_\omega(v,v) + (T_1^*)^2\\ &\qquad \times [J_\omega(v,v) + T_1^*\int_\Omega (1-\nu)D(v_{,22})^2 d\Omega]\}\\ &= J_\omega(v,v) + T_0^*\{J_\omega(v,v)[1 + T_1^* + (T_1^*)^2] + (T_1^*)^3\int_\Omega (1-\nu)D(v_{,22})^2 d\Omega\}\\ &\text{(repeating the same } (n-2)\text{-times more)}\\ &= J_\omega(v,v) + T_0^*[J_\omega(v,v)\frac{1-(T_1^*)^{n+1}}{1-T_1^*} + (T_1^*)^{n+1}\int_\Omega (1-\nu)D(v_{,22})^2 d\Omega].\end{aligned}$$

Now, letting n tend to infinity and taking into account that, by virtue of (3.54), (3.53) and (3.42), obviously

$$T_1^* < 1,$$

we obtain

$$\|v\|^2_{\widetilde{V}} \le J_\omega(v,v) + \frac{T_0^*}{(1-T_1^*)} J_\omega(v,v) = \frac{1-T_1^*+T_0^*}{(1-T_1^*)} J_\omega(v,v).$$

But, in view of (3.54),

$$\frac{(1-T_1^*)+T_0^*}{(1-T_1^*)} = \frac{(\kappa-1)^2(\kappa-3)^2 D_\kappa(1-\nu) - 16T_* + 16(1+T_*)}{(\kappa-1)^2(\kappa-3)^2 D_\kappa(1-\nu) - 16T_*}$$
$$= \frac{(\kappa-1)^2(\kappa-3)^2(1-\nu)D_\kappa + 16}{(\kappa-1)^2(\kappa-3)^2(1-\nu)D_\kappa - 16T_*}.$$

Thus,

$$J_\omega(v,v) \ge \frac{(\kappa-1)^2(\kappa-3)^2(1-\nu)D_\kappa - 16T_*}{(\kappa-1)^2(\kappa-3)^2(1-\nu)D_\kappa + 16}\|v\|^2_{\widetilde{V}} \ \ \forall v \in \widetilde{V}.$$

□

§ 3.2. The General Case

3.13. Definition. Let

(3.55) $$\overset{*}{W}{}^{2,2}(\Omega, D)$$

be the set of all measurable functions w defined on Ω which have on Ω locally summable generalized derivatives up to the order 2 such that (3.8) is valid for

$$\rho_{0,0} := Q(x_1,x_2), \ \rho_{2,0} = \rho_{1,1} = \rho_{0,2} := D(x_1,x_2),$$

where

(3.56) $$Q(x_1,x_2) := D(x_1,x_2)\Big[\int_{x_2}^{\ell} D^{-1}(x_1,\tau)d\tau\Big]^2 \times \Big\{\int_{x_2}^{\ell} D(x_1,t)\Big[\int_t^{\ell} D^{-1}(x_1,\tau)d\tau\Big]^2 dt\Big\}^{-2},$$

with $D \in C(\overline{\Omega})$ and

(3.57) $$\int_{x_2}^{\ell(x_1)} D^{-1}(x_1,\tau)d\tau < +\infty \quad \text{for} \ \ (x_1,0) \in \Gamma_1.$$

We recall that $\ell(x_1) := \max\limits_{(x_1,x_2)\in\overline{\Omega}}\{x_2\}$ for $(x_1,0) \in \Gamma_1$.

Let us introduce the following norm:

(3.58) $$\|w\|^2_{\overset{*}{W}{}^{2,2}(\Omega,D)} := \int_\Omega \{Qw^2 + D[\nu(w_{,11}+w_{,22})^2 + (1-\nu)(w_{,11})^2 + 2(1-\nu)(w_{,12})^2 + (1-\nu)(w_{,22})^2]\}d\Omega.$$

Since

$$Q^{-1}, D^{-1} \in L_1^{\mathrm{loc}}(\Omega),$$

the space (3.55) with the norm (3.58) is a Banach space, and moreover, a Hilbert space with the scalar product

(3.59) $$(w,v)_{\overset{*}{W}{}^{2,2}(\Omega,D)} := \int_\Omega \{Qwv + D[\nu(w_{,11}+w_{,22})(v_{,11}+v_{,22}) + (1-\nu)w_{,11}v_{,11} + 2(1-\nu)w_{,12}v_{,12} + (1-\nu)w_{,22}v_{,22}]\}d\Omega.$$

3.14. Lemma. *If*

$$v \in \overset{*}{W}{}^{2,2}(\Omega, D) \tag{3.60}$$

and

$$v|_{\Gamma_2} = 0, \quad \frac{\partial v}{\partial n}\Big|_{\Gamma_2} = 0, \tag{3.61}$$

in the sense of traces, then

$$\int_\Omega Q(x_1,x_2)v^2(x_1,x_2)d\Omega \le 16\int_\Omega D(x_1,x_2)[v_{,22}(x_1,x_2)]^2 d\Omega. \tag{3.62}$$

Proof. Without loss of generality, we suppose that the domain Ω lies inside of the rectangle Π from (3.20) and complete the definition of the function v in $\Pi \setminus \Omega$, assuming v equal to zero there.

Evidently, (3.60) implies

$$\int_\Pi [Qv^2 + D(v_{,22})^2]d\Omega < +\infty,$$

i.e., for almost every fixed x_1, we have

$$v(x_1,\cdot) \in \overset{*}{W}{}^{2,2}(]0,\ell[, D)$$

(see (2.97)) and

$$v(x_1,\ell) = 0, \quad v_{,2}(x_1,\ell) = 0.$$

We recall that $\ell > \max_{(x_1,x_2)\in\bar\Omega}\{x_2\}$. Now, we can apply Lemma 2.28, i.e.,

$$\int_0^\ell Q(x_1,x_2)v^2(x_1,x_2)dx_2 \le 16\int_0^\ell D(x_1,x_2)[v_{,22}(x_1,x_2)]^2 dx_2 \tag{3.63}$$

for almost every $x_1 \in]a,b[$. Integrating both the sides of (3.63) over $]a,b[$, we get (3.62). □

Let

$$\begin{aligned} \overset{*}{V} := \Big\{ v \in \overset{*}{W}{}^{2,2}(\Omega, D) : v|_{\Gamma_2} = 0,\ \frac{\partial v}{\partial u}\Big|_{\Gamma_2} = 0,\ &\text{and additionally} \\ &\text{either } v|_{\Gamma_1} = 0,\ v_{,2}|_{\Gamma_1} = 0 \text{ (if we consider BCs (3.2))} \\ &\text{or } v_{,2}|_{\Gamma_1} = 0 \text{ (if we consider BCs (3.3))} \\ &\text{or } v|_{\Gamma_1} = 0 \text{ (if we consider BCs (3.4))} \\ &\text{in the sense of traces}\Big\}. \end{aligned} \tag{3.64}$$

3.15. Definition. Let $Q^{-1/2}f \in L_2(\Omega)$ and g_1, g_2, w_0, w_0^1 be the traces of a prescribed function $u \in \overset{*}{W}{}^{2,2}(\Omega, D)$ and of its first derivatives. Let further M_2^0, $Q_2^0 \in L_2(\Gamma_1)$ be also prescribed. A function $w \in \overset{*}{W}{}^{2,2}(\Omega, D)$ will be called a *weak solution of Problem 3.1 in the space* $\overset{*}{W}{}^{2,2}(\Omega, D)$ if it satisfies the following conditions:

$$w - u \in \overset{*}{V}$$

and (3.36) is valid for all $v \in \overset{*}{V}$.

3.16. Theorem. *Let* $2h\rho Q^{-1} \in C(\overline{\Omega})$ *and*

$$\omega^2 < \frac{1-\nu}{16 \max\limits_{\overline{\Omega}} 2h\rho Q^{-1}}.$$

Then there exists a unique weak solution w *of Problem 3.1 which satisfies*

$$\|w\|_{\overset{*}{W}{}^{2,2}(\Omega,D)} \le C[\|Q^{-1/2} f\|_{L_2(\Omega)} + \|u\|_{\overset{*}{W}{}^{2,2}(\Omega,D)} + \gamma_1 \|M_2^0\|_{L_2(\Gamma_1)} + \gamma_2 \|Q_2^0\|_{L^2(\Gamma_1)}]$$

with a constant C *independent of* f, u, M_2^0 *and* Q_2^0.

Proof. This is similar to the proof of Theorem 3.12 (taking into account Lemma 3.14). □

Appendix

This appendix is devoted to a **Proof of Statement 2.1**. This proof is based on the following lemmas.

A.1. Lemma. *If for a fixed* $k \ge 0$

(A.1) $I_k = +\infty$ *and* $I_{k+1} < +\infty$;

(A.2) $f^{(j)}(0) = 0,\ j = 0, 1, \dots, k-2$ *(for the case* $k \ge 2$*)*;

(A.3) $f^{(k-1)}(x_2)$ *is continuous at* $x_2 = 0$ *(for the case* $k \ge 1$*)*,

then

(A.4) $$\int_{x_2}^{x_0} (M_2 w)(\tau) D^{-1}(\tau) d\tau \in C([0,\ell])$$

iff

(A.5) $(M_2 w)(0) = 0,$

when $k = 0$, *and iff*

(A.6) $(M_2 w)(0) = 0, \quad (Q_2 w)(0) = 0,$

when $k \ge 1$.

Proof. Obviously, in the case $k = 0$,

(A.7) $$|(M_2 w)(\tau) D^{-1}(\tau)| = \Big|\frac{(M_2 w)(\tau)}{\tau}\Big| \tau D^{-1}(\tau) \le C\tau D^{-1}(\tau) \quad \forall \tau \in\,]0, x_0],\ C = \text{const},$$

since, by virtue of $(M_2 w)(0) = 0$ and $\frac{d}{d\tau}(M_2 w)(\tau) = Q(\tau)$, we have

$$\lim_{\tau\to 0+} \frac{(M_2 w)(\tau)}{\tau} = Q(0) < +\infty,$$

i.e.,

$$\Big|\frac{(M_2 w)(\tau)}{\tau}\Big| < C \ \forall\, \tau \in\,]0, x_0].$$

But in the case under consideration, $I_1 < +\infty$. Hence, (A.4) follows from (A.7) because of

(A.8) $$\int_{x_2}^{x_0} |(M_2 w)(\tau)| D^{-1}(\tau) d\tau \le C \int_{x_2}^{x_0} \tau D^{-1}(\tau) d\tau < +\infty \ \forall\, x_2 \in [0, x_0].$$

Similarly in the case $k \geq 1$, we have

(A.9) $$|(M_2w)(\tau)|D^{-1}(\tau) = \Big|\frac{(M_2w)(\tau)}{\tau^{k+1}}\Big|\tau^{k+1}D^{-1}(\tau) \leq C\tau^{k+1}D^{-1}(\tau)\ \forall x_2 \in]0, x_0],$$

since, in view of (A.6), (A.2) and (A.3), and taking into account that

(A.10) $M_{2,2} = Q_2$ and $Q_{2,2} = -f$

we have

$$\lim_{\tau\to 0+} \frac{(M_2w)(\tau)}{\tau^{k+1}} = \lim_{\tau\to 0} \frac{(Q_2w)(\tau)}{(k+1)\tau^k} = \lim_{\tau\to 0+} \frac{-f^{(k-1)}(\tau)}{(k+1)!} = \frac{-f^{(k-1)}(0)}{(k+1)!}$$

i.e.,

$$\Big|\frac{(M_2w)(\tau)}{\tau^{k+1}}\Big| \leq C \quad \forall\tau \in]0, x_0].$$

But in this case $I_{k+1} < +\infty$. Therefore, (A.4) is evident from (A.9).

Let us show that in the case $k = 0$ the condition (A.5) is also necessary for (A.4). Indeed, if we assume that (A.4) holds and at the same time, without loss of generality, suppose that $(M_2w)(0) > 0$, then $(M_2w)(x_2) > \widetilde{C} = \text{const} > 0$ in some neighbourhood $[0, \varepsilon]$ of 0, and

(A.11) $$+\infty > \int_0^\varepsilon (M_2w)(\tau)D^{-1}(\tau)d\tau > \widetilde{C}\int_0^\varepsilon D^{-1}(\tau)d\tau,$$

whence, $I_0 < +\infty$. But the last contradicts $I_0 = +\infty$ (see (A.1) for $k = 0$). Thus, $(M_2w)(0) = 0$.

Analogously, we can show the necessity of the conditions (A.6) for the case $k \geq 1$. The necessity of (A.5) follows from the previous reasoning. Now, let (A.4) and (A.5) be valid, but $(Q_2w)(0) > 0$. Taking into account (2.1), (2.2), it is easy to show that

(12) $x_2(Q_2w)(x_2) = (M_2w)(x_2) + \mathcal{K}(x_2)$,

where

$$\mathcal{K}(x_2) := C_1x_0 - C_2 - \int_{x_0}^{x_2} f(t)tdt.$$

From (12) we obtain

(A.13) $\mathcal{K}(0) = -(M_2w)(0) = 0$

since (A.5) is fulfilled.

Taking into account (A.13) and $\mathcal{K}'(x_2) = -f(x_2)\cdot x_2$, we have

$$|\mathcal{K}(\tau)|D^{-1}(\tau) = \Big|\frac{\mathcal{K}(\tau)}{\tau^{k+1}}\Big|\tau^{k+1}D^{-1}(\tau) \leq C\tau^{k+1}D^{-1}(\tau)\ \forall x_2 \in]0, x_0],$$

since

$$\lim_{\tau\to 0+} \frac{\mathcal{K}(\tau)}{\tau^{k+1}} = \lim_{\tau\to 0+} \frac{\mathcal{K}'(\tau)}{(k+1)\tau^k} = \lim_{\tau\to 0+} \frac{-f(\tau)\tau}{(k+1)\tau^k}$$
$$= -\lim_{\tau\to 0+} \frac{f(\tau)}{(k+1)\tau^{k-1}} = -\frac{f^{k-1}(0)}{(k+1)\cdot(k-1)!},$$

i.e.,

$$\Big|\frac{\mathcal{K}(\tau)}{\tau^{k+1}}\Big| \leq C\ \forall\tau \in]0, x_0].$$

Hence,

$$\int_{x_2}^{x_0} \mathcal{K}(\tau)D^{-1}(\tau)d\tau \in C([0, \ell])$$

because of $I_{k+1} < +\infty$. Thus,

$$\text{(A.14)} \quad \Big| \int_0^{x_0} \mathcal{K}(\tau) D^{-1}(\tau) d\tau \Big| \le \int_0^{x_0} |\mathcal{K}(\tau)| D^{-1}(\tau) d\tau < +\infty.$$

Then, in view of (A.12), (A.4) and (A.14), we have

$$\text{(A.15)} \quad \begin{aligned} \int_0^{x_0} \tau |(Q_2 w)(\tau)| D^{-1}(\tau) d\tau \le \int_0^{x_0} |(M_2 w)(\tau)| D^{-1}(\tau) d\tau \\ + \int_0^{x_0} |\mathcal{K}(\tau)| D^{-1}(\tau) d\tau < +\infty. \end{aligned}$$

But a necessary condition for (A.15) (see the left hand side) is the condition $(Q_2 w)(0) = 0$. In fact, if $(Q_2 w)(0) > 0$, then similarly to (A.11) we get

$$\int_0^{\varepsilon} \tau D^{-1}(\tau) d\tau < +\infty,$$

which contradicts $I_1 = +\infty$ (see (1) for $k \ge 1$). Thus,

$$(Q_2 w)(0) = 0.$$

□

A.2. Lemma. *If (A.5) and (A.6) are violated in the cases $k = 0$ (i.e., $I_0 = +\infty$ and $I_1 < +\infty$) and $k \ge 1$ (i.e., $I_k = +\infty$ and $I_{k+1} < +\infty$), respectively, then*

$$\text{(A.16)} \quad \lim_{x_2 \to 0+} \int_{x_2}^{x_0} (M_2 w)(\tau) D^{-1}(\tau) d\tau = +\infty.$$

Proof. Let first $(M_2 w)(0) \ne 0$. Without loss of generality we assume $(M_2 w)(0) > 0$, then there exists an $\varepsilon = \text{const} > 0$ such that

$$\text{(A.17)} \quad (M_2 w)(x_2) \ge \widetilde{C} > 0 \quad \forall x_2 \in [0, \varepsilon].$$

Therefore,

$$\begin{aligned} \lim_{x_2 \to 0+} \int_{x_2}^{x_0} (M_2 w)(\tau) D^{-1}(\tau) d\tau = \lim_{x_2 \to 0+} \int_{x_2}^{\varepsilon} (M_2 w)(\tau) D^{-1}(\tau) d\tau \\ + \int_{\varepsilon}^{x_0} (M_2 w)(\tau) D^{-1}(\tau) d\tau = +\infty \end{aligned}$$

since

$$\lim_{x_2 \to 0+} \int_{x_2}^{\varepsilon} (M_2 w)(\tau) D^{-1}(\tau) d\tau \ge \widetilde{C} \lim_{x_2 \to 0+} \int_{x_2}^{\varepsilon} D^{-1}(\tau) d\tau = +\infty$$

because of $I_0 = +\infty$. So, when (A.5) is violated and, hence, (A.6) is violated as well, (A.16) is proved for $k \ge 0$.

Let now $k \ge 1$ and

$$(M_2 w)(0) = 0 \quad \text{but} \quad (Q_2 w)(0) \ne 0,$$

i.e., (A.6) is violated. We assume,

$$\text{(A.18)} \quad (Q_2 w)(x_2) \ge \widetilde{C} > 0 \ \forall x_2 \in [0, \varepsilon].$$

Then, taking into account (A.12), we have

$$\begin{aligned}\lim_{x_2\to 0+}\int_{x_2}^{\varepsilon}[(M_2w)(\tau)+\mathcal{K}(\tau)]D^{-1}(\tau)d\tau &= \lim_{x_2\to 0+}\int_{x_2}^{\varepsilon}\tau(Q_2w)(\tau)D^{-1}(\tau)d\tau\\ &\geq \widetilde{C}\lim_{x_2\to 0+}\int_{x_2}^{\varepsilon}\tau D^{-1}(\tau)d\tau\\ &= +\infty,\end{aligned}$$

because of $I_1 = +\infty$ (we recall that from $I_k = +\infty$ for a fixed $k \in \{1, 2, \dots\}$ it follows that $I_1 = +\infty$). Therefore, by virtue of (A.14),

$$\lim_{x_2\to 0+}\int_{x_2}^{\varepsilon}(M_2w)(\tau)D^{-1}(\tau)d\tau = +\infty.$$

Thus, (A.16) is proved in all the cases stated in Lemma A.2. □

A.3. Lemma. *If either $I_1 = +\infty$ and $I_2 < +\infty$, and (A.5) is violated or $I_k = +\infty$ and $I_{k+1} < +\infty$, $k \in \{2, 3, \dots\}$, and (A.6) is violated, then*

$$\lim_{x_2\to 0+}\int_{x_2}^{x_0}(\tau - x_2)(M_2w)(\tau)D^{-1}(\tau)d\tau = \infty. \tag{A.19}$$

Proof. Let first $(M_2w)(0) > 0$, then both (A.5) and (A.6) are violated for $k \geq 1$. After the substitution $\tau - x_2 = t$, from the left hand side of (19), and taking into account (17), we get

$$\begin{aligned}&\lim_{x_2\to 0+}\int_{0}^{x_0-x_2}t(M_2w)(x_2+t)D^{-1}(x_2+t)dt\\ &\qquad= \lim_{x_2\to 0+}\int_{0}^{\varepsilon/2}t(M_2w)(x_2+t)D^{-1}(x_2+t)dt\\ &\qquad\qquad+\int_{\varepsilon/2}^{x_0}t(M_2w)(t)D^{-1}(t)dt = +\infty\end{aligned}$$

since

$$\begin{aligned}\lim_{x_2\to 0+}\int_{0}^{\varepsilon/2}t(M_2w)(x_2+t)D^{-1}(x_2+t)dt &\geq \widetilde{C}\lim_{x_2\to 0+}\int_{0}^{\varepsilon/2}tD^{-1}(x_2+t)dt\\ &= +\infty,\ x_2, t \in \Big[0, \frac{\varepsilon}{2}\Big[,\end{aligned}$$

because of $x_2 + t < \varepsilon$ and $I_1 = +\infty$. So, (19) is proved in this case.

Let now $(M_2w)(0) = 0$ but $(Q_2w)(0) > 0$, i.e., (A.6) is violated for $k \geq 2$ and there exists an $\varepsilon = \text{const} > 0$ such that $(Q_2w)(x_2) \geq \widetilde{C} > 0\ \forall x_2 \in [0, \varepsilon]$. Similarly, in view of

(A.12), we obtain

$$\begin{aligned}
&\lim_{x_2\to 0+}\int_0^{x_0-x_2} t[(t+x_2)(Q_2w)(x_2+t)-\mathcal{K}(x_2+t)]D^{-1}(x_2+t)dt\\
&=\lim_{x_2\to 0+}\int_0^{x_0-x_2} t(t+x_2)(Q_2w)(x_2+t)D^{-1}(x_2+t)dt\\
&\qquad\qquad-\lim_{x_2\to 0+}\int_0^{x_0-x_2} t\mathcal{K}(x_2+t)D^{-1}(x_2+t)dt\\
&=\lim_{x_2\to 0+}\int_0^{\varepsilon/2} t(t+x_2)(Q_2w)(x_2+t)D^{-1}(x_2+t)dt\\
&\qquad\qquad+\int_{\varepsilon/2}^{x_0} t^2(Q_2w)(t)D^{-1}(t)dt-\int_0^{x_0} t\mathcal{K}(t)D^{-1}(t)dt\\
&=+\infty,
\end{aligned}$$

since

$$\begin{aligned}
&\lim_{x_2\to 0+}\int_0^{\varepsilon/2} t(t+x_2)(Q_2w)(x_2+t)D^{-1}(x_2+t)dt\\
&\geq \widetilde{C}\lim_{x_2\to 0+}\int_0^{\varepsilon/2} t(t+x_2)D^{-1}(x_2+t)dt=+\infty,
\end{aligned}$$

because of $I_2=\infty$ and

$$\Big|\int_{\varepsilon/2}^{x_2} t^2(Q_2w)(t)D^{-1}(t)dt\Big|<+\infty,$$

$$\Big|\int_0^{x_2} t\mathcal{K}(t)D^{-1}(t)dt\Big|<+\infty$$

(the finiteness of the last term readily follows from (A.14)). Thus, Lemma A.3 is completely proved. □

From Lemma A.1 we immediately have

A.4. Corollary. *Under the assumptions of Lemma A.1,*

$$\lim_{x_2\to 0+} x_2\int_{x_2}^{x_0}(M_2w)(\tau)D^{-1}(\tau)d\tau=0, \tag{A.20}$$

$$\int_{x_2}^{x_0}(M_2w)\tau D^{-1}(\tau)d\tau\in C([0,\ell]). \tag{A.21}$$

Let us note that for (A.20), (A.21) and the conditions (A.5), (A.6) (in corresponding cases) are sufficient but not necessary (see Lemmas A.7–A.9 below).

A.5. Lemma. *If $I_0=+\infty$ and $I_1<+\infty$, then*

$$\Big|x_2\int_{x_2}^{x_0}(M_2w)(\tau)D^{-1}(\tau)d\tau\Big|<const<+\infty\ \forall\, x_2\in\,]0,x_0]. \tag{A.22}$$

Proof. Evidently, by virtue of $I_1<+\infty$,

$$\begin{aligned}
\Big|x_2\int_{x_2}^{x_0}(M_2w)(\tau)D^{-1}(\tau)d\tau\Big|&=\Big|\int_{x_2}^{x_0}(M_2w)(\tau)\frac{x_2}{\tau}\tau D^{-1}(\tau)d\tau\Big|\\
&\leq C\int_0^{x_0}\tau D^{-1}(\tau)d\tau\\
&=\text{const}<+\infty\ \forall\, x_2\in\,]0,x_0],
\end{aligned}$$

because of

$$|(M_2w)(t)| \le C = \text{const}\ \forall\, t \in [0, x_0],$$

and

$$0 < \frac{x_2}{\tau} \le 1$$

since $0 < x_2 \le \tau \le x_0$. □

A.6. Lemma. *If $(M_2w)(0) = 0$, $I_1 = +\infty$ and $I_2 < +\infty$, then*

$$\Big| x_2 \int_{x_2}^{x_0} (M_2w)(\tau) D^{-1}(\tau) d\tau \Big| < const < +\infty\ \forall\, x_2 \in]0, x_0]. \tag{A.23}$$

Proof. Evidently, by virtue of $I_2 < +\infty$,

$$\begin{aligned} \Big| x_2 \int_{x_2}^{x_0} (M_2w)(\tau) D^{-1}(\tau) d\tau \Big| &= \Big| \int_{x_2}^{x_0} \frac{(M_2w)(\tau)}{\tau} \frac{x_2}{\tau} \tau^2 D^{-1}(\tau) d\tau \Big| \\ &\le C \int_0^{x_0} \tau^2 D^{-1}(\tau) d\tau \\ &= \text{const} < +\infty\ \forall\, x_2 \in]0, x_0] \end{aligned}$$

because of

$$0 < \frac{x_2}{\tau} \le 1$$

(since $0 < x_2 \le \tau \le x_0$) and

$$\Big| \frac{(M_2w)(\tau)}{\tau} \Big| < C\ \forall\, \tau \in]0, x_0] \tag{A.24}$$

(since $\lim\limits_{\tau \to 0+} \frac{(M_2w)(\tau)}{\tau} = (Q_2w)(0) < +\infty$). □

A.7. Lemma. *Let either $D \in C^2([0, \ell[)$ or the value of the first or second order derivative of D at the point $x_2 = 0$ tend to infinity. If $I_0 = +\infty$ and $I_1 < +\infty$, then*

$$\lim_{x_2 \to 0+} x_2 \int_{x_2}^{x_0} (M_2w)(\tau) D^{-1}(\tau) d\tau$$

$$= 0 \begin{cases} \text{if } (M_2w)(0) = 0; \\ \text{if } (M_2w)(0) \ne 0 \text{ and either } D'(0) \ne 0, \text{ or } D'(0) = \infty, \\ \text{or } D'(0) = 0 \text{ and } D''(0) = \infty. \end{cases}$$

The case $D'(0) = 0$, $D''(0) = 0$, and $(M_2w)(0) \ne 0$ (at the same time) and the case $D'(0) = 0$ and $D''(0) \ne 0$ cannot occur.

Proof. By virtue of (A.10), we have

$$\begin{aligned} &\lim_{x_2 \to 0+} x_2 \int_{x_2}^{x_0} (M_2w)(\tau) D^{-1}(\tau) d\tau \\ &= \lim_{x_2 \to 0+} \frac{x_2^2 (M_2w)(x_2)}{D(x_2)} \\ &= \lim_{x_2 \to 0+} \frac{2x_2 (M_2w)(x_2) + x_2^2 (Q_2w)(x_2)}{D'(x_2)} \\ &= \begin{cases} 0 & \text{if } D'(0) \ne 0 \text{ or } D'(0) = \infty; \\ \lim\limits_{x_2 \to 0+} \dfrac{2(M_2w)(x_2) + 4x_2(Q_2w)(x_2) - x_2^2 f(x_2)}{D''(x_2)} & \text{if } D'(0) = 0. \end{cases} \end{aligned} \tag{A.25}$$

Therefore, when $D'(0) = 0$, we obtain

$$\lim_{x_2 \to 0} x_2 \int_{x_2}^{x_0} (M_2 w)(\tau) D^{-1}(\tau) d\tau = \begin{cases} 0 & \text{if } D''(0) = \infty; \\ \dfrac{2(M_2 w)(0)}{D''(0)} & \text{if } D''(0) \neq 0, \end{cases}$$

and

$$\lim_{x_2 \to 0+} x_2 \int_{x_2}^{x_0} (M_2 w)(\tau) D^{-1}(\tau) d\tau = \infty \text{ if } D''(0) = 0 \text{ and } (M_2 w)(0) \neq 0. \tag{A.26}$$

But $D''(0) = 0$ and $(M_2 w)(0) \neq 0$ cannot take place at the same time, otherwise (A.22) (which has been proved under assumptions $I_0 = +\infty$, $I_1 < +\infty$ without any requirement of differentiability of $D(x_2)$) and (A.26) will contradict each other. Hence, (A.26) is excluded. Also the case $D'(0) = 0$, $D''(0) \neq 0$ cannot occur since, otherwise,

$$\lim_{\tau \to 0_+} \tau^\gamma [\tau D^{-1}(\tau)] = \lim_{\tau \to 0_+} \frac{(\gamma + 1)\tau^\gamma}{D'(\tau)} = \lim_{\tau \to 0_+} \frac{(\gamma + 1)\gamma\tau^{\gamma - 1}}{D''(\tau)} = \frac{2}{D''(0)} > 0$$

for $\gamma = 1$. Hence, $I_1^0 = +\infty$. But the latter contradicts the assumption $I_1^0 < +\infty$.

When $(M_2 w)(0) = 0$, according to Lemma A.1 for $k = 0$, (A.4) holds since in our case (A.5) is valid. Therefore,

$$\lim_{x_2 \to 0+} x_2 \int_{x_2}^{x_0} (M_2 w)(\tau) D^{-1}(\tau) d\tau = 0.$$

□

A.8. Lemma. *If $I_1 = +\infty$ and $I_2 < +\infty$, then*

1.

$$\lim_{x_2 \to 0+} x_2 \int_{x_2}^{x_0} (M_2 w)(\tau) D^{-1}(\tau) d\tau =$$

$$= \begin{cases} 0 & \textit{when } D'(0) \neq 0 \textit{ or} D'(0) = \infty \\ & \textit{or } D'(0) = 0 \textit{ and } D''(0) = \infty; \\ \dfrac{2(M_2 w)(0)}{D''(0)} & \textit{when } D'(0) = 0 \textit{ and } D''(0) \neq 0, \\ \infty & \textit{when } D'(0) = 0 \textit{ and } D''(0) = 0 \end{cases} \tag{A.27}$$

if

$$(M_2 w)(0) \neq 0$$

and in addition either $D \in C^2([0, \ell[)$ or the value of the first and second order derivatives of D tends to infinity as $x_2 \to 0$;

2.

$$\lim_{x_2 \to 0+} x_2 \int_{x_2}^{x_0} (M_2 w)(\tau) D^{-1}(\tau) d\tau =$$

$$= 0 \begin{cases} \textit{when } (Q_2 w)(0) = 0; \\ \textit{when } (Q_2 w)(0) \neq 0 \textit{ and either } D'(0) \neq 0 \textit{ or } D'(0) = \infty, \\ \textit{or } D'(0) = 0 \textit{ and } D''(0) = \infty, \\ \textit{or } D'(0) = 0 \textit{ and } D''(0) \neq 0, \\ \textit{or } D'(0) = 0, D''(0) = 0 \textit{ and } D'''(0) = \infty \end{cases} \tag{A.28}$$

[the case $D'''(0) = 0$ and $(Q_2w)(0) \neq 0$ (at the same time) and the case $D'(0) = 0$, $D''(0) = 0$, and $D'''(0) \neq 0$ cannot occur] if

$$(M_2w)(0) = 0$$

and in addition either $D \in C^3([0, \ell[)$ or the value of the first, second or third order derivatives of D at the point $x_2 = 0$ tends to infinity and f has a bounded derivative in a neighbourhood $]0, \varepsilon[$ of the point $x_2 = 0$.

Proof. In both cases the reasoning used for (A.25) is valid. Therefore, (A.27) easily follows if $(M_2w)(0) \neq 0$. If $(M_2w)(0) = 0$, when $D'(0) = 0$, from (A.25) we get

$$\lim_{x_2\to 0+} x_2 \int_{x_2}^{x_0} (M_2w)(\tau)D^{-1}(\tau)d\tau = \\ = \begin{cases} 0 & \text{if } D''(0) \neq 0; \\ \lim\limits_{x_2\to 0+} \dfrac{6(Q_2w)(x_2) - 6x_2 f(x_2) - x_2^2 f'(x_2)}{D'''(x_2)} & \text{if } D''(0) = 0. \end{cases}$$

Hence, when $D'(0) = 0$, $D''(0) = 0$, we have

$$\lim_{x_2\to 0+} x_2 \int_{x_2}^{x_0} (M_2w)(\tau)D^{-1}(\tau)d\tau = \begin{cases} 0 & \text{if } D'''(0) = \infty; \\ \dfrac{6(Q_2w)(0)}{D'''(0)} & \text{if } D'''(0) \neq 0, \end{cases}$$

$$\text{(A.29)} \qquad \lim_{x_2\to 0+} x_2 \int_{x_2}^{x_0} (M_2w)(\tau)D^{-1}(\tau)d\tau = \infty \ \text{ if } D'''(0) = 0,\ (Q_2w)(0) \neq 0.$$

But $D'''(0) = 0$ and $(Q_2w)(0) \neq 0$ cannot take place at the same time, otherwise (A.29) and (A.23) (see Lemma A.6 which has been proved under the assumptions $I_1 = +\infty$, $I_2 < +\infty$, without any requirement of differentiability of $D(x_2)$), will contradict each other. Thus, (A.29) is excluded. Also the case $D'(0) = 0$, $D''(0) = 0$, $D'''(0) \neq 0$ cannot occur since in this case $I_2^0 = +\infty$, which is in contradiction with our assumption $I_2^0 < +\infty$. Indeed,

$$\begin{aligned} \lim_{\tau\to 0_+} \tau^\gamma[\tau^2 D^{-1}(\tau)] &= \lim_{\tau\to 0_+} \frac{(\gamma+2)\tau^{\gamma+1}}{D'(\tau)} \\ &= \lim_{\tau\to 0_+} \frac{(\gamma+2)(\gamma+1)\gamma\tau^{\gamma-1}}{D'''(\tau)} \\ &= \frac{6}{D'''(0)} \\ &> 0 \text{ for } \gamma = 1. \end{aligned}$$

But this means that $I_2^0 = +\infty$. When $(Q_2w)(0) = 0$, according to Lemma A.1 for $k = 1$, (A.4) holds iff (A.6) is valid. Therefore,

$$\lim_{x_2\to 0+} x_2 \int_{x_2}^{x_0} (M_2w)(\tau)D^{-1}(\tau)d\tau = 0 \text{ if } D'(0) = 0,\ D''(0) = 0,\ D'''(0) = 0.$$

So, (A.28) is proved. □

A.9. Lemma. *If $I_1 = +\infty$ and $I_2 < +\infty$, then*

$$\text{(A.30)} \qquad \lim_{x_2\to 0+} \int_{x_2}^{x_0} (M_2w)(\tau)\tau D^{-1}(\tau)d\tau = \int_0^{x_0} (M_2w)(\tau)\tau D^{-1}(\tau)d\tau < \infty$$

iff (A.5) holds.

Proof. For every $\tau \in]0, x_0]$ we have

$$|(M_2w)(\tau)\tau D^{-1}(\tau)d\tau| = \Big|\frac{(M_2w)(\tau)}{\tau}\Big| |\tau^2 D^{-1}(\tau)| \le C|\tau^2 D^{-1}(\tau)|, \tag{A.31}$$

by virtue of (A.24). But the right hand side of (A.31) is integrable on $]0, x_0[$, because of $I_2 < +\infty$. Therefore, the left hand side of (A.31) will be also integrable on $]0, x_0[$, and so, we arrive at (A.30).

The necessity of (A.5) we can show with the help of (A.31) in a usual way by a contradiction (see e.g., (A.11)). □

A.10. Remark. All the above lemmas remain true if we replace the original restriction

$$f \in C([0, \ell])$$

by

$$f \in L_1(]0, \ell[),$$

provided that in the cases of Lemmas A.7 and A.8 (provided $M(w)(0) = 0$) we additionally assume that f is bounded in some neighbourhood $]0, \varepsilon]$ of the point $x_2 = 0$, and that f is continuous at the point $x_2 = 0$, respectively.

Now we continue with the Proof of Statement 2.1.

Assertion 2) (when $I_0 = +\infty$ but $I_1 < +\infty$). On the one hand

$$\int_{x_2}^{x_0} \tau(M_2w)(\tau)D^{-1}(\tau)d\tau \in C([0, \ell]),$$

because of $I_1 < +\infty$. On the other hand, from Lemma A.7 it follows that

$$x_2 \int_{x_2}^{x_o} (M_2w)(\tau)D^{-1}(\tau)d\tau \in C([0, \ell]).$$

Thus, taking into account (2.4), we get $w \in C([0, \ell])$.

Assertion 3). The assertion $w \in C([0, \ell])$ immediately follows from (2.4) and Lemma A.8 (the second point) and Lemma A.9. Let now, $(M_2w)(0) \neq 0$, then from Lemma A.3 (case $I_1 = +\infty$ and $I_2 < +\infty$) there follows the unboundedness of $w(x_2)$.

Assertion 4). The assertion $w \in C([0, \ell])$ immediately follows from (2.4) and Corollary A.4 for $k \ge 2$. If (2.7) is violated the unboundedness of $w(x_2)$ follows from Lemma A.3 (case $I_k = +\infty$ and $I_{k+1} = +\infty$, $k \in \{2, 3 \dots\}$).

Assertion 6). The proof follows from (2.3), Lemma A.1 for $k = 0$ and Lemma A.2 for $k = 0$.

Assertion 7). The proof follows from (2.3), Lemma A.1 for $k \ge 1$ and Lemma A.2 for $k \ge 1$.

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References

[1] Bitsadze, A. V. *Mixed-Type Equations* ((in Russian) Moscow, 1959).

[2] Donnel, L. H. *Beams, Plates, and Shells* (McGraw-Hill Book Company, 1976).

[3] Jaiani, G. V. *On the deflections of thin wedge-shapped shells*, Bulletin of the Academy of Sciences of the Georgian Republic **65** (3) (in Russian), 543–546, 1972.

[4] Jaiani, G. V. *Elastic bodies with non-smooth boundaries-cusped plates and shells*, ZAMM **76** Suppl. 2, 117–120, 1996.

[5] Jaiani, G. V. *Initial and Boundary Value Problems for Singular Differential Equations and Applications to the Theory of Cusped Bars and Plates*, ISAAC Serials Vol. 6, Eds.: H. Begehr, O. Celebi, W. Tutschke, Kluwer, Dordrecht, 113–149, 1999.

[6] Jaiani, G. V. *On a mathematical model of bars with variable rectangular cross-sections*, ZAMM-Zeitschrift fuer Angewandte Mathematik und Mechanik **81** (3), 147–173, 2001.

[7] Jaiani, G. V. *Bending of an orthotropic cusped plate*, Applied Mathematics and Informationcs **4** (1), 29–65, 1999.

[8] Keldysh, M. V. *On Some Cases of Degeneration of an Equation of Elliptic Type on the Domain Boundary*, DAN SSSR **77** (2), 181–183, 1951.

[9] Kharibegashvili, S. S. and Jaiani, G. V. *Dynamical Problems in the (0,0) and (1,0) Approximations of a Mathematical Model of Cusped Bars*, Proceedings of the International Graz Workshop, Functional-Analytic and Complex Methods, Interactions, and Applications to Partial Differential Equations, World Scientific, 188–248, 2001.

[10] Khvoles, A. R. *The general representation for solutions of equilibrium equations of prismatic shell with variable thickness*, Seminar of the Institute of Applied Mathematics of Tbilisi State University, Annot. of Reports 5 (in Russian), 19–21, 1971.

[11] Kiguradze, I. T. and Shekhter, B. L. *Singular Boundary Value Problems for Ordinary Differential Equations of the Second Order* (Itogi Nauki i Tekhniki, Ser. Sovrem. Probl. Mat., Nov. Dost., **30** (in Russian), (Moscow, 1987).

[12] Kufner, A. *Weighted Sobolev Spaces* (John Wiley & Sons, 1985).

[13] Kufner, A. and Opic, B. *How to define reasonably weighted sobolev spaces*, Commentations Mathematicae Universitatis Carolinae **25** (3), 537–554, 1984.

[14] Mikhlin, S. G. *Variational Methods in Mathematical Physics* ((in Russian) Nauka, Moscow, 1970).

[15] Naguleswaran, S. *The vibration of a "complete" Euler-Bernoulli beam of constant depth and breadth proportional to axial coordinate raised to a positive exponent.* J. Sound Vib. **187** (2), 311–327, 1995.

[16] Nikolskii, S. M. *Approximation of Functions of Several Variables and Embedding theorems* ((in Russian) Moscow, 1977).

[17] Nikolskii, S. M. and Lizorkin, P. I. *On some inequalities for functions from weighted classes and on boundary value problems with strong degeneration of the boundary*, DAN, SSSR **195** (3), 512–515, 1964.

[18] Nikolskii, S. M., Lizorkin, P. I. and Miroshin, N. V. *Weighted functional spaces and their applications to the investigation of boundary value problems for degenerate elliptic equations*, Izvestia Vysshikh Uchebnykh Zavedenii **8** (315) (in Russian), 4–30, 1988.

[19] Opic, B. and Kufner, A. *Hardy-Type Inequalities*, 1992.

[20] Tricomi, F. G. *Ricerche di Ingeneria* (Marzo-Aprile 1936-XIV).

[21] Timoshenko, S. and Woinowsky-Krieger, S. *Theory of Plates and Shells* (McGraw-Hill, Book Company, New York-Toronto-London, 1959).

[22] Uzunov, S. G. *Estimate of convergence of the finite element method for a degenerate differential equation*, (in collection: Boundary Value Problems for Non-linear Equations, Novosibirsk (in Russian), 68–74, 1982).

[23] Vekua, I. N. *On a way of calculating of prismatic shells.* Proceedings of A. Razmadze Institute of Mathematics of Georgian Academy of Sciences **21** (in Russian), 191–259, 1955.

[24] Vekua, I. N. *The theory of thin shallow shells of variable thickness*, Proceedings of A. Razmadze Institute of Mathematics of Georgian Academy of Sciences **30** (in Russian), 5–103, 1965.

[25] Vekua, I. N. Shell Theory: General Methods of Construction (Pitman Advanced Publishing Program, Boston-London-Melbourne, 1985).