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FRATTINI FUZZY SUBGROUPS OF FUZZY GROUPS

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Abstract. This paper continues the study of fuzzy group theory which has been explored over times. We propose maximal fuzzy subgroups and Frattini fuzzy subgroups of fuzzy groups as extensions of maximal subgroups and Frattini subgroups of classical groups. It is shown that every Frattini fuzzy subgroup is both characteristic and normal, respectively. Finally, some results are established in connection to level subgroups and alpha cuts of fuzzy groups.

1. Introduction

The concept of fuzzy sets proposed by Zadeh [19] generalized the theory of Cantorian sets. The theory of fuzzy sets has grown stupendously over the years giving birth to fuzzy groups proposed in [15]. Some properties of fuzzy groups were explicated as analogs to some group theoretical notions. A number of results on some properties of fuzzy groups have been discussed in [2, 3, 4, 5, 6, 14, 16]. In fact, the work by Mordeson et al. [12] provided an elaborate theory of fuzzy groups.

The notions of fuzzy subgroups and normal fuzzy subgroups have been extensively studied in literature, as can be found in [1, 9, 10, 11, 13, 18]. In the same vein, the concept of characteristic fuzzy subgroups of fuzzy groups was proposed and some of its properties were discussed [7, 8, 17].

Although researchers in fuzzy algebra are now interested in the generalizations of fuzzy subgroups, we carefully discovered that the notions of maximal subgroups and Frattini subgroups have not been established in fuzzy algebra. Hence the motivation of this work, though it seems belated but it better late than never. In this paper, we propose maximal fuzzy subgroups with illustrative example, and explicate the concept of Frattini fuzzy subgroups of fuzzy groups as an extension of Frattini subgroups. Some relevant results on Frattini fuzzy subgroups in conjunction to normal fuzzy subgroups, characteristic fuzzy subgroups, abelian fuzzy groups, alpha cuts of fuzzy groups and level sets, respectively are established. By organization, the paper is thus presented: Section 2 provides some preliminaries on fuzzy sets, fuzzy groups, fuzzy subgroups, normal fuzzy subgroups and characteristic fuzzy subgroups, respectively. In Section 3, we propose the ideas of maximal fuzzy subgroups and Frattini fuzzy subgroups with some examples. Section 4 discusses some results on maximal fuzzy subgroups and Frattini fuzzy subgroups.

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2. Preliminaries

In this section, we review some existing definitions and results which shall be used in the sequel.

Definition 2.1. [19] Let X be a nonempty set. A fuzzy set A in X is characterized by a membership function

$$\mu_A : X \to [0, 1].$$

That is,

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in X \\ 0, & \text{if } x \notin X \\ (0,1) & \text{if } x \text{ is partly in } X \end{cases}$$

Alternatively, a fuzzy set A in X is an object having the form

$$A = \{\langle x, \mu_A(x)\rangle \mid x \in X\} \text{ or } A = \{\langle \frac{\mu_A(x)}{x} \rangle \mid x \in X\},$$

where the function

$$\mu_A(x) : X \to [0, 1]$$

defines the degree of membership of the element, $x \in X$.

Definition 2.2. [19] Let A and B be two fuzzy sets of X. Then A is said to be a fuzzy subset of B written as $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x) \forall x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper fuzzy subset of B and denoted as $A \subset B$.

Definition 2.3. [15] Let X be a group. A fuzzy set A of X is said to be a fuzzy group of X if it satisfies the following two conditions:

(i) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \forall x, y \in X$,
(ii) $\mu_A(x^{-1}) = \mu_A(x) \forall x \in X$,

where $\wedge$ denotes minimum.

It can be easily verified that if A is a fuzzy group of X, then

$$\mu_A(e) \geq \mu_A(x) \, \forall x \in X.$$

The set of all fuzzy groups of X is denoted by $FG(X)$.

Example 2.4. Let $X = \{e, a, b, c\}$ be a Klein 4-group such that

$$ab = c, ac = b, bc = a, a^2 = b^2 = c^2 = e.$$

Then

$$A = \{\frac{1}{e}, \frac{0.7}{a}, \frac{0.8}{b}, \frac{0.7}{c}\}$$

is a fuzzy group of X satisfying Definition 2.3.

Remark 2.5. We notice the following from Definition 2.3:

(i) every fuzzy group is a fuzzy set but the converse is not always true.
(ii) a fuzzy set A of a group X is a fuzzy group if $\forall x, y \in X$,

$$\mu_A(xy^{-1}) \geq \mu_A(x) \wedge \mu_A(y)$$

holds.

Definition 2.6. [12] Let A be a fuzzy group of a group X. Then A^{-1} is defined by $\mu_{A^{-1}}(x) = \mu_A(x^{-1}) \, \forall x \in X$.

Thus, we notice that $A \in FG(X) \Leftrightarrow A^{-1} \in FG(X)$.

Definition 2.7. [12] Let $A \in FG(X)$. A fuzzy subset B of A is called a fuzzy subgroup of A if B is a fuzzy group. A fuzzy subgroup B of A is a proper fuzzy subgroup if $\mu_B(x) \leq \mu_A(x)$ and $\mu_A(x) \neq \mu_B(x)$ $\forall x \in X$.

Definition 2.8. [12] Let $A \in FG(X)$. Then A is said to be abelian (commutative) if for all $x, y \in X$, $\mu_A(xy) = \mu_A(yx)$.

If A is a fuzzy group of an abelian group X, then A is abelian.

Definition 2.9. [12] Let $A \in FG(X)$. Then the sets A_* and A^* are defined as

(i) $A_* = \{x \in X \mid \mu_A(x) > 0\}$ and
(ii) $A^* = \{x \in X \mid \mu_A(x) = \mu_A(e)\}$, where e is the identity element of X.

Definition 2.10. [1] Let A be a fuzzy subgroup of $B \in FG(X)$. Then A is called a normal fuzzy subgroup of B if for all $x, y \in X$, it satisfies

$$\mu_A(xyx^{-1}) \geq \mu_A(y).$$

Definition 2.11. [17] Let $A, B \in FG(X)$ such that $A \subseteq B$. Then A is called a characteristic fuzzy subgroup of B if

$$\mu_{A^\theta}(x) = \mu_A(x) \, \forall x \in X$$

for every automorphism, θ of X. That is, $\theta(A) \subseteq A$ for every $\theta \in Aut(X)$.

Definition 2.12. [12] Let $A \in FG(X)$. Then the sets $A_{[\alpha]}$ and $A_{(\alpha)}$ defined as

(i) $A_{[\alpha]} = \{x \in X \mid \mu_A(x) \geq \alpha, \ \alpha \in [0,1]\}$ and
(ii) $A_{(\alpha)} = \{x \in X \mid \mu_A(x) > \alpha, \ \alpha \in [0,1]\}$

are called strong upper alpha-cut and weak upper alpha-cut of A.

Definition 2.13. [12] Let $A \in FG(X)$. Then the sets $A^{[\alpha]}$ and $A^{(\alpha)}$ defined as

(i) $A^{[\alpha]} = \{x \in X \mid \mu_A(x) \leq \alpha, \ \alpha \in [0,1]\}$ and
(ii) $A^{(\alpha)} = \{x \in X \mid \mu_A(x) < \alpha, \ \alpha \in [0,1]\}$

are called strong lower alpha-cut and weak lower alpha-cut of A.

3. Concept of Frattini fuzzy subgroups

In this section, we propose maximal fuzzy subgroups and Frattini subgroups in fuzzy group setting by redefining some concepts in the light of fuzzy groups.

Definition 3.1. Let $B \in FG(X)$. A proper fuzzy subgroup A of B is said to be a maximal fuzzy subgroup if there exists other proper fuzzy subgroups C_i, for $i = 1, ..., n$ of B such that $\mu_{C_i}(x) \leq \mu_A(x)$ and $\mu_{C_i}(x) \neq \mu_A(x)$ $\forall x \in X$. That is, a maximal fuzzy subgroup A of B is a proper fuzzy subgroup that contains all the other proper fuzzy subgroups of B.

Example 3.2. Let $X = \{e, a, b, c\}$ be a Klein 4-group and

$$B = \{\frac{1}{e}, \frac{0.3}{a}, \frac{0.4}{b}, \frac{0.3}{c}\}$$

be a fuzzy group of X. Then the following are maximal fuzzy subgroups of A:

$$A_1 = \{\frac{1}{e}, \frac{0.2}{a}, \frac{0.4}{b}, \frac{0.2}{c}\},$$

$$A_2 = \{\frac{1}{e}, \frac{0.3}{a}, \frac{0.3}{b}, \frac{0.3}{c}\},$$
$$A_3 = \{\frac{0.9}{e}, \frac{0.3}{a}, \frac{0.4}{b}, \frac{0.3}{c}\}.$$

Definition 3.3. Let $B \in FG(X)$. Suppose $A_1, A_2, ..., A_n$ (or simply A_i for $i = 1, 2, .., n$) are maximal fuzzy subgroups of B. Then the Frattini fuzzy subgroup of B denoted by $\Phi(A_i)$ is the intersection of A_i defined by

$$\mu_{\Phi(A_i)}(x) \quad = \quad \mu_{A_1}(x) \wedge ... \wedge \mu_{A_n}(x) \, \forall x \in X,$$

or simply

$$\mu_{\Phi(A_i)}(x) = \bigwedge_{i=1}^{n} \mu_{A_i}(x) \, \forall x \in X.$$

Example 3.4. From Example 3.2, the Frattini fuzzy subgroup of $B \in FG(X)$ is

$$\Phi(A_i) = \{\frac{0.9}{e}, \frac{0.2}{a}, \frac{0.3}{b}, \frac{0.2}{c}\}.$$

Remark 3.5. Every Frattini fuzzy subgroup of fuzzy group is a fuzzy group.

Foremostly, we provide a proposition that connects characteristic fuzzy subgroup and normal fuzzy subgroup.

Proposition 1. Let $B \in FG(X)$. Every characteristic fuzzy subgroup of B is a normal fuzzy subgroup.

Proof. Let $x, y \in X$. Suppose that A is a characteristic fuzzy subgroup of B. To prove that A is a normal fuzzy subgroup of B, we have to show that

$$\mu_A(xyx^{-1}) \geq \mu_A(y) \, \forall x, y \in X.$$

Now, since A is a characteristic fuzzy subgroup of B, we have

$$\begin{aligned} \mu_{A^\theta}(xyx^{-1}) = \mu_A((xyx^{-1})^\theta) &= \mu_A(x^\theta (yx^{-1})^\theta) \\ &= \mu_{(x^\theta)^{-1}A}(y^\theta (x^{-1})^\theta) \\ &= \mu_{A(x^\theta)^{-1}}(y^\theta (x^\theta)^{-1}) \\ &= \mu_A(y^\theta (x^\theta)^{-1}(x^\theta)) \\ &\geq \mu_A(y^\theta) \\ &= \mu_{A^\theta}(y). \end{aligned}$$

Hence A^θ is a normal fuzzy subgroup. □

From here henceforth, we present some results on Frattini fuzzy subgroups of fuzzy groups.

Proposition 2. If $\Phi(A_i)$ is a Frattini fuzzy subgroup of $B \in FG(X)$. Then $[\Phi(A_i)]^{-1}$ is a Frattini fuzzy subgroup of $B \in FG(X)$.

Proof. By Definitions 2.3 and 2.6, it follows that

$$\begin{aligned} \mu_{[\Phi(A_i)]^{-1}}(x) &= \mu_{\Phi(A_i)}(x^{-1}) \\ &= \mu_{\Phi(A_i)}(x) \, \forall x \in X. \end{aligned}$$

This completes the proof.

□

Proposition 3. Let $A, B \in FG(X)$. Then the following statements are equivalent:

(i) $A = \bigcap_{i=1}^{n} A_i$, where A_i are the maximal fuzzy subgroups of B.
(ii) A is a Frattini fuzzy subgroup of B.

Proof. Straightforward. □

Theorem 3.6. *Let X be a finite group. If $A \in FG(X)$, then a Frattini fuzzy subgroup $\Phi(A_i)$ of A is a normal fuzzy subgroup.*

Proof. Let $\Phi(A_i)$ be a Frattini fuzzy subgroup of A. By Remark 3.5, we get

$$\mu_{\Phi(A_i)}(xy^{-1}) \geq \mu_{\Phi(A_i)}(x) \wedge \mu_{\Phi(A_i)}(y) \; \forall x, y \in X,$$

implies $\Phi(A_i)$ is a fuzzy group of X.

Now, we prove that $\mu_{\Phi(A_i)}$ is a normal fuzzy subgroup of A. Let $x, y \in X$, then it follows that

$$\begin{aligned} \mu_{\Phi(A_i)}(yxy^{-1}) &= \mu_{\Phi(A_i)}((yx)y^{-1}) \\ &= \mu_{\Phi(A_i)}(x(yy^{-1})) \\ &= \mu_{\Phi(A_i)}(xe) \\ &\geq \mu_{\Phi(A_i)}(x). \end{aligned}$$

Hence the result by Definition 2.10.

□

Proposition 4. Every Frattini fuzzy subgroup of a fuzzy group is abelian.

Proof. Let $A \in FG(X)$ and $\Phi(A_i)$ be the Frattini fuzzy subgroup of A. It follows that $\Phi(A_i)$ is a normal fuzzy subgroup of A by Theorem 3.6. Consequently,

$$\mu_{\Phi(A_i)}(xyx^{-1}) = \mu_{\Phi(A_i)}(y) \; \forall x, y \in X.$$

Thus,

$$\mu_{\Phi(A_i)}(xy) = \mu_{\Phi(A_i)}(yx) \; \forall x, y \in X.$$

Hence the result follows by Definition 2.8. □

Theorem 3.7. *Let A_i be maximal fuzzy subgroups of $A \in FG(X)$. Then a Frattini fuzzy subgroup $\Phi(A_i)$ of A is a characteristic fuzzy subgroup.*

Proof. Suppose $\Phi(A_i)$ is a Frattini submultigroup of A. Then

$$\mu_{\Phi(A_i)}(x) = \bigwedge_{i=1}^{n} \mu_{A_i}(x) \; \forall x \in X.$$

Since $\Phi(A_i) = \bigcap_{i=1}^{n} A_i$, and A_i for $i = 1, ..., n$ are maximal fuzzy subgroups of A, then it follows that, $\Phi(A_i) \subseteq A_i$. Thus by Definition 2.11, $\Phi(A_i)$ is a characteristic fuzzy subgroup of A. □

Proposition 5. Let $A \in FG(X)$ and $\Phi(A_i)$ be a Frattini fuzzy subgroup of A. Then $[\Phi(A_i)]_*$ and $[\Phi(A_i)]^*$ are subgroups of X.

Proof. Let $x, y \in [\Phi(A_i)]_*$. Then $\mu_{\Phi(A_i)}(x) > 0$ and $\mu_{\Phi(A_i)}(y) > 0$. Now, by Definition 2.9 we have

$$\mu_{\Phi(A_i)}(xy^{-1}) \geq \mu_{\Phi(A_i)}(x) \wedge \mu_{\Phi(A_i)}(y) > 0.$$

Implies that $xy^{-1} \in [\Phi(A_i)]_*$. Hence $[\Phi(A_i)]_*$ is a subgroup of X.

Again, let $x, y \in [\Phi(A_i)]^*$. Then

$$\mu_{\Phi(A_i)}(x) = \mu_{\Phi(A_i)}(y) = \mu_{\Phi(A_i)}(e)$$

by Definition 2.9. It follows that

$$\begin{aligned} \mu_{\Phi(A_i)}(xy^{-1}) &\geq \mu_{\Phi(A_i)}(x) \wedge \mu_{\Phi(A_i)}(y) \\ &= \mu_{\Phi(A_i)}(e) \wedge \mu_{\Phi(A_i)}(e) \\ &= \mu_{\Phi(A_i)}(e) \\ &\geq \mu_{\Phi(A_i)}(xy^{-1}). \end{aligned}$$

Thus,

$$\mu_{\Phi(A_i)}(xy^{-1}) = \mu_{\Phi(A_i)}(e) \, \forall x, y \in X.$$

Implies $xy^{-1} \in [\Phi(A_i)]^*$. Hence $[\Phi(A_i)]^*$ is a subgroup of X. □

Proposition 6. Let $A \in FG(X)$ and $\Phi(A_i)$ be a Frattini fuzzy subgroup of A. Then $[\Phi(A_i)]_*$ and $[\Phi(A_i)]^*$ are normal subgroups of X.

Proof. By Proposition 5, $[\Phi(A_i)]_*$ and $[\Phi(A_i)]^*$ are subgroups of X. Now, let $x \in X$ and $y \in [\Phi(A_i)]_*$. By Theorem 3.6, we have

$$\mu_{\Phi(A_i)}(xyx^{-1}) = \mu_{\Phi(A_i)}(y).$$

It follows that

$$\mu_{\Phi(A_i)}(xyx^{-1}) = \mu_{\Phi(A_i)}(y) > 0.$$

Thus $xyx^{-1} \in \Phi(A_i)_*$. Hence $[\Phi(A_i)]_*$ is a normal subgroup of X.

Similarly, let $x \in X$ and $y \in [\Phi(A_i)]^*$. Then we have

$$\mu_{\Phi(A_i)}(xyx^{-1}) = \mu_{\Phi(A_i)}(y)$$

by Theorem 3.6. It implies that

$$\mu_{\Phi(A_i)}(xyx^{-1}) = \mu_{\Phi(A_i)}(y) = \mu_{\Phi(A_i)}(e).$$

Thus $xyx^{-1} \in \Phi(A_i)^*$. Hence $[\Phi(A_i)]^*$ is a normal subgroup of X. □

Proposition 7. Let $A \in FG(X)$ and $\Phi(A_i)$ be a Frattini fuzzy subgroup of A. Then $[\Phi(A_i)]_*$ and $[\Phi(A_i)]^*$ are characteristic subgroups of X.

Proof. The result follows by combining Proposition 5 and Theorem 3.7. □

Proposition 8. Let $A \in FG(X)$ and $\Phi(A_i)$ be a Frattini fuzzy subgroup of A. Then $[\Phi(A_i)]_{[\alpha]}$ is subgroup of X for $\alpha \leq \mu_{\Phi(A_i)}(e)$, where e is the identity element of $\Phi(A_i)$.

Proof. Let $x, y \in [\Phi(A_i)]_{[\alpha]}$. By Definition 2.12, we have $\mu_{\Phi(A_i)}(x) \geq \alpha$ and $\mu_{\Phi(A_i)}(y) \geq \alpha$. Since $\Phi(A_i) \in FG(X)$, it follows that

$$\mu_{\Phi(A_i)}(xy^{-1}) \geq \mu_{\Phi(A_i)}(x) \wedge \mu_{\Phi(A_i)}(y) \geq \alpha.$$

This implies that $\mu_{\Phi(A_i)}(xy^{-1}) \geq \alpha$. Thus $xy^{-1} \in [\Phi(A_i)]_{[\alpha]}$. Hence $[\Phi(A_i)]_{[\alpha]}$ is subgroup of X. □

Corollary 1. Let $A \in FG(X)$ and $\Phi(A_i)$ be a Frattini fuzzy subgroup of A. Then $[\Phi(A_i)]^{[\alpha]}$ is subgroup of X for $\alpha \geq \mu_{\Phi(A_i)}(e)$, where e is the identity element of $\Phi(A_i)$.

Proof. Combining Definition 2.13 and Proposition 8, the result follows. □

Proposition 9. Let $A \in FG(X)$ and $\Phi(A_i)$ be a Frattini fuzzy subgroup of A. Then $[\Phi(A_i)]_{[\alpha]}$ is a normal subgroup of X for $\alpha \leq \mu_{\Phi(A_i)}(e)$, where e is the identity element of $\Phi(A_i)$.

Proof. By Theorem 3.6, it follows that $\Phi(A_i)$ is a normal fuzzy subgroup of A. By Proposition 8, $[\Phi(A_i)]_{[\alpha]}$ is subgroup of X. Now let $x \in X$ and $y \in [\Phi(A_i)]_{[\alpha]}$. By Theorem 3.6,

$$\mu_{\Phi(A_i)}(xyx^{-1}) = \mu_{\Phi(A_i)}(y).$$

It follows that

$$\mu_{\Phi(A_i)}(xyx^{-1}) = \mu_{\Phi(A_i)}(y) \geq \alpha.$$

Thus $xyx^{-1} \in [\Phi(A_i)]_{[\alpha]}$. This completes the proof. □

Corollary 2. Let $A \in FG(X)$ and $\Phi(A_i)$ be a Frattini fuzzy subgroup of A. Then $[\Phi(A_i)]^{[\alpha]}$ is a normal subgroup of X for $\alpha \geq \mu_{\Phi(A_i)}(e)$, where e is the identity element of $\Phi(A_i)$.

Proof. Combining Theorem 3.6, Corollary 1 and Proposition 9, the proof follows. □

Proposition 10. Let $A \in FG(X)$ and $\Phi(A_i)$ be a Frattini fuzzy subgroup of A. Then $[\Phi(A_i)]_{[\alpha]}$ is a characteristic subgroup of X for $\alpha \leq \mu_{\Phi(A_i)}(e)$, where e is the identity element of $\Phi(A_i)$.

Proof. Combining Theorem 3.7 and Proposition 8, the proof follows. □

Corollary 3. Let $A \in FG(X)$ and $\Phi(A_i)$ be a Frattini fuzzy subgroup of A. Then $[\Phi(A_i)]^{[\alpha]}$ is a characteristic subgroup of X for $\alpha \geq \mu_{\Phi(A_i)}(e)$, where e is the identity element of $\Phi(A_i)$.

Proof. Combining Theorem 3.7, Corollary 1 and Proposition 10, the proof follows. □

Proposition 11. Let $A \in FG(X)$. If A is a Frattini fuzzy subgroup of a fuzzy group B, and B is a Frattini fuzzy subgroup of a fuzzy group C, then A is a Frattini fuzzy subgroup of a fuzzy group C.

Proof. Suppose A is a Frattini fuzzy subgroup of a fuzzy group B, and B is a Frattini fuzzy subgroup of a fuzzy group C. Then, by transitivity, it follows that A is a Frattini fuzzy subgroup of C. □

Corollary 4. With the same hypothesis as in Proposition 11, A is a characteristic fuzzy subgroup of a fuzzy group C.

Proof. By Proposition 11, it follows that A is a Frattini fuzzy subgroup of C. Synthesizing Theorem 3.7, the proof is complete. □

Corollary 5. With the same hypothesis as in Proposition 11, A is a normal fuzzy subgroup of a fuzzy group C.

Proof. By Proposition 11, it follows that A is a Frattini fuzzy subgroup of C. Synthesizing Theorem 3.6, the result follows. □

4. Conclusion

In continuation to the study of fuzzy group theory, we have proposed maximal fuzzy subgroups and Frattini fuzzy subgroups of fuzzy groups as extensions of maximal subgroups and Frattini subgroups, respectively. Some related results were presented. For further study, more groups theoretic results of maximal subgroups and Frattini subgroups could be analogously presented in fuzzy group setting.

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