Kafkas Üniversitesi
Fen Bilimleri Enstitüsü Dergisi
Cilt 10, Sayı 2 , 85-102, 2017

Kafkas University
Institute of Natural and Applied Science Journal
Volume 10, Issue 2, 85-102, 2017

Original Article

A New Look to The Usual Norm of c_0 and Candidates to Renormings of c_0 with Fixed Point Property

Veysel NEZİR[*1]

[1]Kafkas Üniversitesi, Fen Edebiyat Fakültesi, Matematik Bölümü, 36100, Kars

(İlk Gönderim / Received: 28. 12. 2017, Kabul / Accepted: 28. 12. 2017, Online Yayın / Published Online: 31. 12. 2017)

Keywords:
Fixed Point Property,
Renorming,
Asymptotically Isometric Copy,
Banach Spaces,
Sequence Space c_0,
Nonexpansive Mappings

Abstract: In this study, we investigate some renormings of c_0 and fixed point theory related questions constructing some equivalent norms to the canonical norm of the Banach space of sequences converging to 0, c_0. Then, we show that respect to these equivalent norms, c_0 does not include any asymtoticaly isometric copy of itself with its usual norm. Dowling, Lennard and Turett proved that if a Banach space has an asymptotically isometric (ai) copy of c_0 or l^1 inside, then it fails to have the fixed point property for nonexpansive mappings (FPP(ne)). It is well-known that neither these spaces has FPP(ne) but as an intriguing work, P. K. Lin showed that l^1 can be renormed to have FPP(ne). Researchers still wonder if c_0 can be renormed to have FPP(ne). In order to work on c_0-analogue of P. K. Lin's theory, it is important to study renormings that do not have any ai copy of c_0 inside. That is why, our renormings might be candidates to answer P. K. Lin's c_0-analogue and they can be considered as the first stage to research this big open question.

c_0'ın Alışılmış Normuna Yeni Bir Bakış ve c_0'ın Sabit Nokta Teorisine Sahip Yeniden Normlamaları için Adaylar

Anahtar Kelimeler:
Sabit Nokta Teorisi,
Yeniden Normlama,
Asimtotik İzometrik Kopya,
Banach Uzayları,
c_0 Dizi Uzayı,
Genişlemeyen Fonksiyonlar

Özet: Bu çalışmamızda 0 a yakınsak dizilerin uzayı olan c_0 Banach uzayı üzerinde kendi kanonik normuna eşdeğer bazı normlar tanımlayarak c_0 uzayının yeniden normlanmışlarını sabit nokta teorisi açısından soruları inceliyoruz. Çalışmamızda gösteririz ki geliştirmiş olduğumuz eşdeğer normlara göre bu yeniden normlamalar c_0'ın alışılmış normunun asimtotik izometrik kopyasını içermez. Dowling, Lennard ve Turett ispatlamıştır ki eğer bir Banach uzayı c_0 veya l^1'in asimtotik izometrik kopyalarından birini içerirse genişlemeyen fonksiyonlar için sabit nokta teorisine (SNT(gf)) sahip olamazlar. Çok iyi bilinen bir gerçek olarak bu iki uzayın hiçbiri SNT(gf)'ye sahip

***İlgili yazar:** *veyselnezir@yahoo.com*

değildir. Çığır açıcı olarak nitelendirilen bir çalışma ile P. K. Lin göstermiştir ki l^1 uzayı SNT(gf)'ye sahip olacak şekilde yeniden normlanabilir. c_0 uzayının SNT(gf)'ye sahip olacak şekilde yeniden normlanabilip normlanamayacağı açık bir sorudur. P. K. Lin'in teorisinin c_0-analoğu üzerinde çalışabilmek için c_0'ın asimtotik izometrik kopyalarını içermeyen yeniden normlamalar üzerinde çalışmak önemlidir. Bu sebeple bizim yeniden normlamalarımız P. K. Lin'in c_0-analoğunu çözebilmek için aday olabilir ve bu büyük açık soruyu araştırmak için ilk aşama olarak kabul edilebilir.

1. INTRODUCTION

Banach space of sequences converging to 0, $(c_0, \| \,.\, \|_\infty)$ and Banach space of absolutely summable sequences $(l^1, \| \,.\, \|_1)$ have weak fixed point property; that is, every invariant nonexpansive mapping defined on any non-empty weakly compact, convex subset of the space has a fixed fixed point but both spaces fail the fixed point property; in other words, there exist a closed, bounded and convex (cbc) nonempty subset and a fixed point free invariant nonexpansive mapping defined on that set. These two spaces can be considered as the examples of nonreflexive Banach spaces failing FPP(ne) (Kirk and Sims, 2013).

The first illustrate of a non-reflexive Banach space $(X, \|\cdot\|)$ with FPP(ne) was recently given. This fact is proved for $(l^1, \| \,.\, \|_1)$ with the equivalent norm $|\|\cdot\||$ given by

$$|\| x \|| = \sup_{k\in\mathbb{N}} \frac{8^k}{1+8^k} \sum_{n=k}^{\infty} | x_n |, \text{ for all } x = (x_n)_{n\in\mathbb{N}} \in \ell^1$$

(Lin, 2008).

$(c_0, \| \,.\, \|_\infty)$ analogue of P.K. Lin's work is still unknown. Long before Lin's work, it had been showed that while l^1 fails the FPP(n.e.) with its usual norm, there exists a large class of cbc and non-weak*-compact subsets D of $(l^1, \| \,.\, \|_1)$ such that every $\| \,.\, \|_1$ −nonexpansive mappings $U: D \to D$ has a fixed point (Goebel and Kuczumow, 1979) . Thus, one can consider an analogue work of theirs for c_0 but it has to be done after renorming c_0. That is, a researcher can work on a question "do there exist any renorming of c_0 and a nonempty cbc subset C so that every nonexpansive mapping has fixed point property?".

Recently, it has been given positive answer for this question when the mapping is also affine (Nezir, 2017a; Nezir and Sade, 2017). These works are interesting because the authors invented large classes of equivalent norms and showed that the closed convex hull (cch) of some asymptotically isometric (ai) c_0-summing basis for its canonical norm has FPP(ne) when the functions are also affine whereas it was proved that if a Banach space has an ai c_0-summing basic sequence $(x_n)_{n\in\mathbb{N}}$ inside, then the cch of $(x_n)_{n\in\mathbb{N}}$, $E:=\overline{\text{co}}(\{x_n : n\in\mathbb{N}\})$, fails the fixed point property for affine nonexpansive mappings (FPP(nea)) (Lennard and Nezir, 2011; Nezir, 2012). In their works, the authors study on some specific ai c_0-summing basic sequences in c_0.

For example, they fix $b\in(0,1)$ and define the sequence $(f_n)_{n\in\mathbb{N}}$ in c_0 by setting $f_1 := be_1$, $f_2 := be_2$, and $f_n := e_n$, for every $n\geq 3$ where $(e_n)_{n\in\mathbb{N}}$ is defined to be 1 in its n th coordinate, and 0 in all other coordinates such that for both $(c_0, \|\cdot\|_\infty)$ and $(\ell^1, \|\cdot\|_1)$, the sequence $(e_n)_{n\in\mathbb{N}}$ is an unconditional basis.

Next, they define the cbc subset $C = C_b$ of c_0 by

$$C := \left\{ \sum_{n=1}^{\infty} t_n f_n : 1 = t_1 \geq t_2 \geq \ldots \geq t_n \downarrow_n 0 \right\}.$$

Then, they define the sequence $(\eta_n)_{n\in\mathbb{N}}$ in E in the following way: $\eta_1 := f_1$ and $\eta_n := f_1 + \ldots + f_n$, for every $n\geq 2$. Note that

$$C = \left\{ \sum_{n=1}^{\infty} \alpha_n \eta_n : \text{each } \alpha_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = 1 \right\}.$$

Next, they give the following theorem:

Theorem 1.1 *Assume* $b\in(0,1)$. *Then the cch of the sequence* $(\eta_n)_{n\in\mathbb{N}}$, $C=\overline{\text{co}}(\{\eta_n : n\in\mathbb{N}\})$ *is such that there exists a fixed point free affine* $\|\cdot\|_\infty$*-nonexpansive mapping* $U: C\to C$.

Easily, it can be seen that the sequence $(\eta_n)_{n\in\mathbb{N}}$ is an ai c_0-summing basic sequence.

In the recent works (Nezir, 2017a; Nezir and Sade, 2017); respectively, the authors define the following equivalent norms on c_0 depending on a scalar α satisfying some conditions such that they show that c_0 can be renormed so that when there exists $b\in(0,1)$, the set C given in Theorem 1.1 above and for all affine nonexpansive mappings $T: C\to C$, T has a fixed point in C.

Let $\alpha\in\Re$. For $x=(\xi_k)_k \in c_0$, define

$$\|x\| = \|x\|_\infty + \sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} Q_k \left|\xi_k - \alpha\xi_j\right| \text{ where } \sum_{k=1}^{\infty} Q_k = 1,$$

$Q_k \downarrow_k 0$ and $Q_k > Q_{k+1}$, $\forall k\in\mathbb{N}$.

$$\|x\|^{\sim} = \frac{1}{\gamma_1} \lim_{p\to\infty} \sup_{k\in\mathbb{N}} \gamma_k \left(\sum_{j=k}^{\infty} \frac{\left|\xi_j\right|^p}{j} \right)^{\frac{1}{p}} + \gamma_1 \sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} Q_k \left|\xi_k^* - \alpha\xi_j^*\right| + \gamma_1 \sqrt{\sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} {Q_k}^2 \left|\xi_k - \alpha\xi_j\right|^2}$$

where $\gamma_k \uparrow_k 1, \gamma_{k+2} > \gamma_{k+1}, \forall k\in\mathbb{N}$, $\gamma_2 = \gamma_1$, $x^\star := (\xi_j^*)_{j\in\mathbb{N}}$ is the decreasing rearrangement of x, $\sum_{k=1}^{\infty} Q_k = 1$, $Q_k \downarrow_k 0$ and $Q_k > Q_{k+1}$, $\forall k\in\mathbb{N}$ such that from the definition of decreasing rearrangement, $\exists$ a 1-1 mapping $\pi:\mathbb{N}\to\mathbb{N}$ and $\exists(\varepsilon_j)_{j\in\mathbb{N}}$ s.t.

each $\varepsilon_{\pi(j)} \in \{-1,1\}$ and then $\xi_k^* = |\xi_{\pi(k)}| = \varepsilon_{\pi(k)}\xi_{\pi(k)}, \forall k \in \mathbb{N}$.

We need to note that the denominator part "j" has to be replaced by "j^2" throughout the second work.

As it has been mentioned above, when working on renormings of c_0 (or l^1) to get large classes of non-weakly compact, cbc sets with FPP(ne), first of all, the renorming should not have an ai copy of c_0 (or l^1; respectively) inside. Indeed, it is known that if a Banach space has one of these copies inside, then it fails to have FPP(ne) (Dowling et al, 2001).

In our work, we invent some renormings of c_0 and show that with our new type of equivalent norms c_0 does not contain any ai copy of c_0. We also see interesting properties of these renormings in terms of fixed point property.

We believe that our results have great importance in terms of bringing new candidates to solve c_0 analogue of P.K. Lin's theorem. In fact, using our equivalent norms, one can obtain more equivalent norms satisfying our results and even better results.

Now, we can give preliminiaries for our work that leads us to obtain our main result.

2. PRELIMINARIES

Definition 2.1 *Let K be a non-empty cbc subset of a Banach space $(X, \|\cdot\|)$. Let $U: K \to K$ be a mapping.*

1. We say T is *affine* if for all $\lambda \in [0,1]$, for all $x, y \in K$, $U((1-\lambda)x + \lambda y) = (1-\lambda)U(x) + \lambda U(y)$.
2. We say U is *nonexpansive* if $\|U(x) - U(y)\| \le \|x - y\|$, *for all* $x, y \in K$.

Also, we say that K has the *fixed point property for nonexpansive mappings* [FPP(ne)] if for all nonexpansive mappings $U: K \to K$, there exists $z \in K$ with $U(z) = z$.

Let $(X, \|\cdot\|)$ be a Banach space and $E \subseteq X$. We will denote the cch of E by $\overline{\text{co}}(E)$. As usual, $(c_0, \|\cdot\|_\infty)$ is given by

$$c_0 := \left\{ x = (x_n)_{n\in\mathbb{N}} : \begin{array}{l}\text{each } x_n \in \mathbb{R} \\ \text{and } \lim_{n\to\infty} x_n = 0\end{array} \right\}.$$

Further, $\|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n|$, for all $x = (x_n)_{n\in\mathbb{N}} \in c_0$; and $(\ell^1, \|\cdot\|_1)$ is defined by

$$\ell^1 := \left\{ x = (x_n)_{n\in\mathbb{N}} : \text{each } x_n \in \mathbb{R} \text{ and } \|x\|_1 := \sum_{n=1}^{\infty} |x_n| < \infty \right\}.$$

We recall now the definition of an *ai c_0-summing basic sequence* in a Banach space $(X, \|\cdot\|)$, from (Lennard and Nezir, 2011).

Definition 2.2 *Let $(X, \|\cdot\|)$ be a Banach space and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X satisfying the* *following condition; then, we say* $(x_n)_{n\in\mathbb{N}}$ *is an ai* c_0 *-summing basic sequence in* $(X,\|\cdot\|)$ *: There exists a null sequence* $(\varepsilon_n)_{n\in\mathbb{N}}$ *in* $[0,\infty)$ *such that for every* $(t_n)_{n\in\mathbb{N}}\in c_{00}$,

$$\sup_{n\geq 1}\left(\frac{1}{1+\varepsilon_n}\right)\left|\sum_{j=n}^{\infty}t_j\right|\leq\left\|\sum_{j=1}^{\infty}t_j x_j\right\|\leq\sup_{n\geq 1}(1+\varepsilon_n)\left|\sum_{j=n}^{\infty}t_j\right|.$$

Note that here we can replace c_{00} by ℓ^1. Furthermore, if $L>0$ and the sequence $(z_n/L)_{n\in\mathbb{N}}$ is an *ai* c_0*-summing basic sequence*, we will call the sequence $(z_n)_{n\in\mathbb{N}}$ an L*- scaled ai* c_0 *-summing basic sequence* in $(X,\|\cdot\|)$ (Lennard and Nezir, 2011).

2.1. Ai copies of c_0, ai copies of l^1 and ai copies of $\ell^{1\boxplus 0}$

Banach Spaces containing either of asymtotically isometric copies of l^1 or asymtotically isometric copies of c_0 has rich applications on fixed point theory. In this section, we will recall the definition of Banach spaces containing asymtotically isometric copies of l^1 and theorems given by (Dowling and Lennard, 1997; Dowling et al, 2001) and the definition of Banach spaces containing asymtotically isometric copies of c_0 and theorems given by (Dowling et al, 1996; Dowling et al, 2001). Furthermore, using their ideas, we will gave an interesting definition and its application (Nezir, 2017b).

Definition 2.1.1 *Let* $(X,\|\cdot\|)$ *be a Banach space. Then, we say that* X *has an ai copy of* l^1 *inside if there exists a null sequence* $(\varepsilon_n)_n$ *in* $(0,1)$ *and a sequence* $(x_n)_n$ *in* X *such that*

$$\sum_{n=1}^{\infty}(1-\varepsilon_n)|t_n|\leq\left\|\sum_{n=1}^{\infty}t_n x_n\right\|\leq\sum_{n=1}^{\infty}|t_n|$$

for all $(t_n)_{n\in\mathbb{N}}\in l^1$ (Dowling et al, 1997).

Theorem 2.1.2 *If a Banach Space* $(X,\|\cdot\|)$ *has an ai copy of* l^1 *inside then it fails FPP(ne)* (Dowling et al, 1997).

Definition 2.1.3 *Let* $(X,\|\cdot\|)$ *be a Banach space. Then, we say that* X *has an ai copy of* c_0 *inside if there exists a null sequence* $(\varepsilon_n)_n$ *in* $(0,1)$ *and a sequence* $(x_n)_n$ *in* X *such that*

$$\sup_{n\in\mathbb{N}}(1-\varepsilon_n)|t_n|\leq\left\|\sum_{n=1}^{\infty}t_n x_n\right\|\leq\sup_{n\in\mathbb{N}}|t_n|$$

for all $(t_n)_{n\in\mathbb{N}}\in c_0$ (Dowling et al, 1996).

Theorem 2.1.4 *If a Banach Space* $(X,\|\cdot\|)$ *contains an ai copy of* c_0 *then it fails FPP(ne)* (Dowling et al, 1996).

Definition 2.1.5 *Let* $(X,\|\cdot\|)$ *be a Banach space. Then, let's say* X *has an ai copy of* $\ell^{1\boxplus 0}$

inside if there exists a null sequence $(\varepsilon_n)_n$ *in* $(0,1)$ *and a sequence* $(x_n)_n$ *in* X *such that*

$$\frac{1}{2}\sup_{n\in\mathbb{N}}(1-\varepsilon_n)|t_n| + \frac{1}{2}\sum_{n=1}^{\infty}(1-\varepsilon_n)|t_n| \le \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \le \frac{1}{2}\sup_{n\in\mathbb{N}}|t_n| + \frac{1}{2}\sum_{n=1}^{\infty}|t_n|$$

for all $(t_n)_{n\in\mathbb{N}} \in l^1$ (Nezir, 2017b).

Theorem 2.1.6 *If a Banach Space* $(X,\|\cdot\|)$ *has an ai copy of* $\ell^1 \boxplus^0$ *inside then it fails FPP(ne)* (Nezir, 2017b).

Proof. Proof is done by combining two proofs: one is the proof of the Theorem 1.2 in (Dowling et al, 1997) and the other one is the proof of the Proposition 7 in (Dowling et al, 1996). In fact, in order for the readers to see how basic the proof is, they can see more detailed proofs of both theorems in (Dowling et al, 2001).

3. A NEW LOOK TO THE ABSOLUTE SUP NORM OF c_0

Theorem 3.1 *For any* $x=(\xi_i)_{i\in\mathbb{N}} \in c_0$ *and for any* $n,m\in\mathbb{N}$,

$$\|x\|_\infty = \lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^2}\right)^{\frac{1}{p}} = \lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{2^k}\right)^{\frac{1}{p}} \qquad (3.1)$$

Proof. Let $x=(\xi_i)_{i\in\mathbb{N}} \in c_0$. We will consider $x \ne (0,0,\cdots)$ otherwise proof of the claim is clear.

Then,

$$\exists N\in\mathbb{N} \ni \|x\|_\infty = \sup_{k\in\mathbb{N}}|\xi_k| = \max_{k\in\mathbb{N}}|\xi_k| = |\xi_N|.$$

Due to power mean inequalities formula (Hardy et all, 1952),

$$\|x\|_\infty = \max_{k\le N}|\xi_k| = \max\{|\xi_1|,|\xi_2|,\cdots,|\xi_N|\} = \lim_{p\to\infty}\left(\frac{|\xi_1|^p+|\xi_2|^p+\cdots+|\xi_N|^p}{N}\right)^{\frac{1}{p}} = \lim_{p\to\infty}\left(\sum_{k=1}^{N}\frac{|\xi_k|^p}{N}\right)^{\frac{1}{p}}.$$

Also, due to weighted power mean inequalities formula (Hardy et all, 1952),

$$\|x\|_\infty = \max_{k\le N}|\xi_k| = \max\{|\xi_1|,|\xi_2|,\cdots,|\xi_N|\} = \lim_{p\to\infty}\left(\frac{|\xi_1|^p+|\xi_2|^p+\cdots+|\xi_N|^p}{2^N}\right)^{\frac{1}{p}} = \lim_{p\to\infty}\left(\sum_{k=1}^{N}\frac{|\xi_k|^p}{2^N}\right)^{\frac{1}{p}}.$$

Claim 3.2

$$\|x\|_\infty = \lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^2}\right)^{\frac{1}{p}} = \lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{2^k}\right)^{\frac{1}{p}} \qquad (3.3)$$

Indeed,

$$\|x\|_\infty \le \lim_{p\to\infty}\left(\sum_{k=1}^{N}\frac{|\xi_k|^p}{2^k}\right)^{\frac{1}{p}} \le \lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^2}\right)^{\frac{1}{p}}.$$

On the other hand, $\exists s\in\mathbb{N}$ such that $|\xi_k| < \frac{1}{k^2}, \forall k \ge s$.

Thus,

$$\lim_{p\to\infty}\left(\sum_{k=1}^{\infty}\frac{|\xi_k|^p}{k^2}\right)^{\frac{1}{p}} = \lim_{p\to\infty}\left(\sum_{k=1}^{s-1}\frac{|\xi_k|^p}{k^2}+\sum_{k=s}^{\infty}\frac{|\xi_k|^p}{k^2}\right)^{\frac{1}{p}}$$

$$\le \lim_{p\to\infty}\left(\sum_{k=1}^{s-1}\frac{|\xi_k|^p}{k^2}+\frac{|\xi_s|^p}{s^2}+\int_s^\infty\frac{|\xi_k|^p}{k^2}\,dk\right)^{\frac{1}{p}}$$

$$= \lim_{p\to\infty}\left(\sum_{k=1}^{s}\frac{|\xi_k|^p}{k^2}+\int_s^\infty\frac{|\xi_k|^p}{k^2}\,dk\right)^{\frac{1}{p}}$$

$$\le \lim_{p\to\infty}\left(\sum_{k=1}^{s}\frac{|\xi_k|^p}{k^2}+\int_s^\infty\frac{1}{k^{2p+2}}\,dk\right)^{\frac{1}{p}}$$

$$\le \lim_{p\to\infty}\left(|\xi_N|^p\sum_{k=1}^{s}\frac{1}{k^2}-\frac{1}{(2p+1)s^{2p+1}}\right)^{\frac{1}{p}}$$

$$\le \lim_{p\to\infty}\left(|\xi_N|^p\left[1+\int_1^s\frac{1}{k^2}\,dk\right]-\frac{1}{(2p+1)s^{2p+1}}\right)^{\frac{1}{p}}$$

$$= |\xi_N|$$

$$= \|x\|_\infty .$$

3.1. An equivalent norm $\|.\|$ for c_0 such that $(c_0,\|.\|)$ does not contain an ai copy of l^1 or an ai copy of ℓ¹⊞⁰

Now, using the facts above, we will construct an equivalent norm $\|.\|$ on c_0 and we will give an unusual way to see that $(c_0,\|.\|)$ does not have an ai copy of l^1 or an ai copy of ℓ¹⊞⁰ inside. The basic method to see our result is that with any equivalent norm, c_0 cannot contain even an isomorphic copy of l^1 or a renorming of l^1 since otherwise it would have Schur property. Now, let's see our alternative proof with our equivalent norm.

Definition 4.1 *For* $x=(\xi_k)_k\in c_0$, *define*

$$\|x\| := \lim_{p\to\infty}\sup_{k\in\mathbb{N}}\gamma_k\left(\sum_{j=k}^{\infty}\frac{|\xi_j|^p}{j^2}\right)^{\frac{1}{p}} \quad \text{where } \gamma_k \uparrow_k 1,$$

γ_k is strictlyincreasing.

Then, it is easy see that $\|.\|$ is an equivalent norm on c_0 .

Now, let's see some interesting properties of this equivalent norm.

Theorem 3.1.2 $(c_0, \|.\|)$ *does not have an ai copy of* l^1 *inside.*

Proof. We will be using the similiar ideas in Example 10 of (Dowling et al, 2001). By contradiction, assume $(c_0, \|.\|)$ does have an ai copy of l^1 inside. That is, there exists a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in c_0 such that

$$\spadesuit \left[\begin{array}{l} \text{for every } (t_n)_{n\in\mathbb{N}} \in l^1, \text{ it follows that} \\ \sum_{n=1}^{\infty}(1-\varepsilon_n)|t_n| \le \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \le \sum_{n=1}^{\infty}|t_n|. \end{array} \right]$$

Then, for the Cesaro average of the sequence x_n, we get

$$\spadesuit\spadesuit \left[\begin{array}{l} \text{for every } (t_n)_{n\in\mathbb{N}} \in l^1, \text{ it follows that} \\ \sum_{n=1}^{\infty}\left(1-\frac{1}{n}\sum_{k=1}^{n}\varepsilon_k\right)|t_n| \le \left\| \sum_{n=1}^{\infty} t_n \frac{1}{n}\sum_{k=1}^{n} x_n \right\|. \end{array} \right]$$

Thus, replacing ε_n by $\frac{1}{n}\sum_{k=1}^{n}\varepsilon_k$, we have

$$\spadesuit\clubsuit \left[\begin{array}{l} \text{for every } (t_n)_{n\in\mathbb{N}} \in l^1, \text{ it follows that} \\ \sum_{n=1}^{\infty}(1-\varepsilon_n)|t_k| \le \left\| \sum_{n=1}^{\infty} t_n \frac{1}{n}\sum_{k=1}^{n} x_n . \right\| \end{array} \right]$$

Define $y_n := \frac{1}{n}\sum_{j=1}^{n} x_j$ for each n.

Without loss of generality we can suppose that the sequence $(y_n)_n$ is disjointly supported; i.e., that the support of y_m is disjoint from the support of y_n if $n \neq m$. This is possible since c_0 has weak Banach Saks property (Núñez, 1989), again without loss of generality, if necessary by passing to a subsequence, we can suppose its Cesarro average converges in norm to y. By replacing y_n by the $\|.\|$ -normalization of the sequence $\left(\frac{y_{2n}-y_{2n-1}}{2}\right)_n$ that satisfies $\spadesuit\clubsuit$, we may suppose that $y = 0$. By the proof of the Bessaga-Pełczyński Theorem (Bessaga and Pełczyński, 1958; Diestel, 2012), we can pass to an essentially disjointly supported subsequence of y_n. Truncating this subsequence appropriately, we get a disjointly supported sequence that satisfies $\spadesuit\clubsuit$. , when it is normalized. If necessary, by passing to subsequences, we can also suppose that $\varepsilon_n < \frac{1}{2n}$ for all $n \in \mathbb{N}$.

Let $(m(k))_{k\in\mathbb{N}_0}$ with $m(0) = 0$ and $(\xi_k)_{k\in\mathbb{N}}$ a sequence of scalars such that for each $k \in \mathbb{N}$,

$$y_k = \sum_{j=m(k-1)+1}^{m(k)} \xi_j e_j .$$

Using the triangular inequality of the norm, for each $N \in \mathbb{N}$, we get

$N+1-\varepsilon_1-N\varepsilon_N \le y_1+Ny_N$

$$\le \lim_{p\to\infty} \sup_{\substack{1\le j\le m(1)\\ m(N-1)+1\le\\ i\le m(N)}} \left\{ \begin{array}{l} \gamma_j\left(\left[\sum_{k=j}^{m(1)}\frac{|\xi_k|^p}{k^2}\right]^{\frac{1}{p}}+N\left[\sum_{k=m(N-1)+1}^{m(N)}\frac{|\xi_k|^p}{k^2}\right]^{\frac{1}{p}}\right), \\ N\gamma_i\left[\sum_{k=i}^{m(N)}\frac{|\xi_k|^p}{k^2}\right]^{\frac{1}{p}} \end{array}\right\}$$

$$\le \lim_{p\to\infty} \sup_{\substack{1\le j\le m(1)\\ m(N-1)+1\le\\ i\le m(N)}} \left\{ \begin{array}{l} \gamma_j\left[\sum_{k=j}^{m(1)}\frac{|\xi_k|^p}{k^2}\right]^{\frac{1}{p}}+N\gamma_{m(1)}\left[\sum_{k=m(N-1)+1}^{m(N)}\frac{|\xi_k|^p}{k^2}\right]^{\frac{1}{p}}, \\ N\gamma_i\left[\sum_{k=i}^{m(N)}\frac{|\xi_k|^p}{k^2}\right]^{\frac{1}{p}} \end{array}\right\}$$

$$\le \lim_{p\to\infty} \sup_{\substack{1\le j\le m(1)\\ m(N-1)+1\\ \le i\le m(N)}} \left\{ \begin{array}{l} \gamma_j\left[\sum_{k=j}^{m(1)}\frac{|\xi_k|^p}{k^2}\right]^{\frac{1}{p}} \\ +N\frac{\gamma_{m(1)}}{\gamma_{m(N-1)+1}}\left(\gamma_{m(N-1)+1}\left[\sum_{k=m(N-1)+1}^{m(N)}\frac{|\xi_k|^p}{k^2}\right]^{\frac{1}{p}}\right), \\ N\gamma_i\left[\sum_{k=i}^{m(N)}\frac{|\xi_k|^p}{k^2}\right]^{\frac{1}{p}} \end{array}\right\}$$

$$\le \max\left\{1+N\frac{\gamma_{m(1)}}{\gamma_{m(N-1)+1}}, N\right\}.$$

That is,

$$N+1-\varepsilon_1-N\varepsilon_N \le \max\left\{1+N\frac{\gamma_{m(1)}}{\gamma_{m(N-1)+1}}, N\right\}$$

for all $N\in\mathbb{N}$.

But since $\varepsilon_1<\frac{1}{2}$ and $N\varepsilon_N<\frac{1}{2}$, we have $N+1-\varepsilon_1-N\varepsilon_N>N$ and so

$$N+1-\varepsilon_1-N\varepsilon_N \le 1+N\frac{\gamma_{m(1)}}{\gamma_{m(N-1)+1}}$$

for all $N\in\mathbb{N}$.

Thus,

$$1+\frac{1}{N}-\frac{\varepsilon_1}{N}-\varepsilon_N \le \frac{1}{N}+\frac{\gamma_{m(1)}}{\gamma_{m(N-1)+1}}$$

for all $N\in\mathbb{N}$.

Therefore, we get contradiction by letting $N\to\infty$ since we would have $1\le\gamma_m(1)$.

Theorem 4.3 $(c_0,\|.\|)$ *does not have an ai copy of* $\ell^1\boxplus^0$ *inside.*

Proof. We will be using the similiar ideas to the proof of the previous theorem. By contradiction, assume $(c_0,\|.\|)$ does contain an ai copy of $l^{1\oplus 0}$. That is, there exists a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in c_0 such that

$$\heartsuit\left[\begin{array}{l} \text{for every } (t_n)_{n\in\mathbb{N}}\in l^1\text{, it follows that} \\ \frac{1}{2}\sup_{n\in\mathbb{N}}(1-\varepsilon_n)|t_n|+\frac{1}{2}\sum_{n=1}^{\infty}(1-\varepsilon_n)|t_n| \\ \le\left\|\sum_{n=1}^{\infty}t_n x_n\right\|\le\frac{1}{2}\sup_{n\in\mathbb{N}}|t_n|+\frac{1}{2}\sum_{n=1}^{\infty}|t_n| \end{array}\right].$$

Then, for the sequence of Cesaro averages,

$$\heartsuit\heartsuit\left[\begin{array}{l} \text{for every } (t_n)_{n\in\mathbb{N}}\in l^1\text{, it follows that} \\ \frac{1}{2}\sup_{n\in\mathbb{N}}(1-\varepsilon_n)\sum_{n=1}^{\infty}\frac{|t_n|}{n}+\frac{1}{2}\sum_{n=1}^{\infty}(1-\frac{1}{n}\sum_{k=1}^{n}\varepsilon_k)|t_n| \\ \le\left\|\sum_{n=1}^{\infty}t_n\frac{1}{n}\sum_{k=1}^{n}x_n.\right\| \end{array}\right]$$

Then, without loss of generality, by passing to a subsequence if necessary, we can suppose that a null sequence $(\varepsilon_n)_n$ in $(0,1)$ can be found such that

$$♥♣\left[\begin{array}{l}\text{for every } (t_n)_{n\in\mathbb{N}} \in l^1 \text{, it follows that} \\ \frac{1}{2}\sup_{n\in\mathbb{N}}(1-\frac{1}{n}\sum_{k=1}^{n}\varepsilon_k)|t_n| + \frac{1}{2}\sum_{n=1}^{\infty}(1-\frac{1}{n}\sum_{k=1}^{n}\varepsilon_k)|t_n| \\ \le \left\| \sum_{n=1}^{\infty} t_n \frac{1}{n}\sum_{k=1}^{n} x_n . \right\| \end{array}\right]$$

Indeed we can do this. For example, in ♥♥, instead of ε_n, we could consider $1-\frac{1-\varepsilon_n}{(1+\varepsilon_n)^k}$ for any $k\in\mathbb{N}$ that still satisfies ♥ (also note that [♣] $\frac{1-\varepsilon_n}{(1+\varepsilon_n)^k}|t_n|$ is in c_0 (so it reaches to its maximum)), then for k large enough, without loss of generality, by passing to a subsequence if necessary and taking the fact [♣] into consideration, we could suppose that there exists $N\in\mathbb{N}$ such that $\sup_{n\in\mathbb{N}}\frac{1-\varepsilon_n}{(1+\varepsilon_n)^k}|t_n| = \frac{1-\varepsilon_N}{(1+\varepsilon_N)^k}|t_N| \le \frac{|t_N|}{N}$ and there exists $s\in\mathbb{N}$ with $\frac{1-\varepsilon_s}{(1+\varepsilon_s)^k}\le\frac{1}{N}$.

Then, passing to a subsequence if necessary, we could reorder $\frac{1-\varepsilon_n}{(1+\varepsilon_n)^k}$ so that $s=1$ and $1-\frac{1-\varepsilon_n}{(1+\varepsilon_n)^k}$ is decreasing. Thus, after all these assumptions that we could have, in in ♥♥, replacing ε_n by the last sequence suggested; i.e., $1-\frac{1-\varepsilon_n}{(1+\varepsilon_n)^k}$ that satisfies $\frac{1-\varepsilon_1}{(1+\varepsilon_1)^k}\le\frac{1}{N}$ and $\sup_{n\in\mathbb{N}}(1-\varepsilon_n)\sum_{n=1}^{\infty}\frac{|t_n|}{n} = \max_{1\le n\le N}(1-\varepsilon_n)\sum_{n=1}^{\infty}\frac{|t_n|}{n}$, we would have $\frac{1-\varepsilon_n}{n}\ge(1-\varepsilon_1)(1-\varepsilon_n)$ and so for every $(t_n)_{n\in\mathbb{N}}\in l^1$ it follows that

$$\begin{aligned}&\frac{1}{2}\sup_{n\in\mathbb{N}}\frac{(1-\varepsilon_1)(1-\varepsilon_n)+(1-\frac{1}{n}\sum_{k=1}^{n}\varepsilon_k)}{2}|t_n| \\ &+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(1-\varepsilon_1)(1-\varepsilon_n)+(1-\frac{1}{n}\sum_{k=1}^{n}\varepsilon_k)}{2}|t_n| \\ &\le \frac{1}{2}\sup_{n\in\mathbb{N}}(1-\varepsilon_1)(1-\varepsilon_n)|t_n| + \frac{1}{2}\sum_{n=1}^{\infty}(1-\frac{1}{n}\sum_{k=1}^{n}\varepsilon_k)|t_n| \\ &\le \frac{1}{2}\sup_{n\in\mathbb{N}}\frac{(1-\varepsilon_n)}{n}|t_n| + \frac{1}{2}\sum_{n=1}^{\infty}(1-\frac{1}{n}\sum_{k=1}^{n}\varepsilon_k)|t_n| \\ &\le \frac{1}{2}\sup_{n\in\mathbb{N}}(1-\varepsilon_n)\sum_{n=1}^{\infty}\frac{|t_n|}{n} + \frac{1}{2}\sum_{n=1}^{\infty}(1-\frac{1}{n}\sum_{k=1}^{n}\varepsilon_k)|t_n| \\ &\le \left\|\sum_{n=1}^{\infty}t_n\frac{1}{n}\sum_{k=1}^{n}x_n\right\|\end{aligned}.$$

Thus, finally replacing $(1-\frac{1}{n}\sum_{k=1}^{n}\varepsilon_k)$ by $\frac{(1-\varepsilon_1)(1-\varepsilon_n)+(1-\frac{1}{n}\sum_{k=1}^{n}\varepsilon_k)}{2}$ we could have ♥♣.

Now that we have ♥♣, we can also say the following inequality ♣♣ by replacing ε_n by $1-\frac{1}{n}\sum_{k=1}^{n}\varepsilon_k$ and we will have similar proof steps to the proof of previous theorem.

$$♣♣\left[\begin{array}{l}\text{there exists a null sequence } (\varepsilon_n)_n \in (0,1) \\ \text{such that for every } (t_n)_{n\in\mathbb{N}} \in l^1\text{, it follows that} \\ \frac{1}{2}\sup_{n\in\mathbb{N}}(1-\varepsilon_n)|t_n| + \frac{1}{2}\sum_{n=1}^{\infty}(1-\varepsilon_n)|t_n| \le \left\|\sum_{n=1}^{\infty}t_n\frac{1}{n}\sum_{k=1}^{n}x_n .\right\| \end{array}\right]$$

Thus, firstly, define $y_n := \frac{1}{n}\sum_{j=1}^{n}x_j$ for each n. Without loss of generality we can assume that the sequence $(y_n)_n$ is disjointly supported; i.e., that the support of y_m is disjoint from the support of y_n if $n \neq m$ (using the Weak Banach Saks property of c_0 again). By replacing y_n by the $\|.\|$ -normalization of the sequence $\left(\frac{y_{2n} - y_{2n-1}}{2}\right)_n$ that satisfies ♣⁑, we may suppose that $y = 0$. Using the similar ideas in the previous theorem, we can pass to a normalized essentially disjointly supported subsequence of y_n that satisfies ♣⁑. By passing to subsequences if necessary, we can also assume that $\varepsilon_n < \frac{1}{3n}$ for all $n \in \mathbb{N}$.

Let $(m(k))_{k\in\mathbb{N}_0}$ with $m(0) = 0$ and $(\xi_k)_{k\in\mathbb{N}}$ a sequence of scalars such that for each $k \in \mathbb{N}$, $y_k = \sum_{j=m(k-1)+1}^{m(k)} \xi_j e_j$. Using the triangular inequality of the norm, for each $K \in \mathbb{N}$, we get

$$\frac{K - K\varepsilon_K}{2} + \frac{K + 1 - \varepsilon_1 - K\varepsilon_K}{2} \le y_1 + Ky_K$$

$$\le \lim_{p\to\infty} \sup_{\substack{1\le j\le m(1) \\ m(K-1)+1\le \\ i\le m(K)}} \left\{ \begin{array}{l} \gamma_j \left(\left[\sum_{k=j}^{m(1)} \frac{|\xi_k|^p}{k^2} \right]^{\frac{1}{p}} + K \left[\sum_{k=m(K-1)+1}^{m(K)} \frac{|\xi_k|^p}{k^2} \right]^{\frac{1}{p}} \right), \\ K\gamma_i \left[\sum_{k=i}^{m(K)} \frac{|\xi_k|^p}{k^2} \right]^{\frac{1}{p}} \end{array} \right\}$$

$$\le \lim_{p\to\infty} \sup_{\substack{1\le j\le m(1) \\ m(K-1)+1\le \\ i\le m(K)}} \left\{ \begin{array}{l} \gamma_j \left[\sum_{k=j}^{m(1)} \frac{|\xi_k|^p}{k^2} \right]^{\frac{1}{p}} + K\gamma_{m(1)} \left[\sum_{k=m(K-1)+1}^{m(K)} \frac{|\xi_k|^p}{k^2} \right]^{\frac{1}{p}}, \\ K\gamma_i \left[\sum_{k=i}^{m(K)} \frac{|\xi_k|^p}{k^2} \right]^{\frac{1}{p}} \end{array} \right\}$$

$$\le \lim_{p\to\infty} \sup_{\substack{1\le j\le m(1) \\ m(K-1)+1\le \\ i\le m(K)}} \left\{ \begin{array}{l} \gamma_j \left[\sum_{k=j}^{m(1)} \frac{|\xi_k|^p}{k^2} \right]^{\frac{1}{p}} \\ + K \frac{\gamma_{m(1)}}{\gamma_{m(K-1)+1}} \left(\gamma_{m(K-1)+1} \left[\sum_{k=m(K-1)+1}^{m(K)} \frac{|\xi_k|^p}{k^2} \right]^{\frac{1}{p}} \right), \\ K\gamma_i \left[\sum_{k=i}^{m(K)} \frac{|\xi_k|^p}{k^2} \right]^{\frac{1}{p}} \end{array} \right\}.$$

Thus,

$$\frac{K - K\varepsilon_K}{2} + \frac{K + 1 - \varepsilon_1 - K\varepsilon_K}{2} \le \max\left\{ 1 + K\frac{\gamma_{m(1)}}{\gamma_{m(K-1)+1}}, K \right\}.$$

That is,

$$K + \frac{1-\varepsilon_1}{2} - K\varepsilon_K \le \max\left\{ 1 + K\frac{\gamma_{m(1)}}{\gamma_{m(K-1)+1}}, K \right\}$$

for all $K \in \mathbb{N}$.

But since $\varepsilon_1 < \frac{1}{3}$ and $K\varepsilon_K < \frac{1}{3}$, we have $K + \frac{1-\varepsilon_1}{2} - K\varepsilon_K > K$ and so

$$K + \frac{1-\varepsilon_1}{2} - K\varepsilon_K \le 1 + K\frac{\gamma_{m(1)}}{\gamma_{m(K-1)+1}}$$

for all $K \in \mathbb{N}$.

Thus,

$$1 + \frac{1}{2K} - \frac{\varepsilon_1}{2K} - \varepsilon_K \le \frac{1}{K} + \frac{\gamma_{m(1)}}{\gamma_{m(K-1)+1}}$$

for all $K \in \mathbb{N}$.

Therefore, we get contradiction by letting $K \to \infty$ since we would have $1 \le \gamma_m(1)$.

4. MORE ON NEZIR'S EQUİVALENT NORM ON c_0 AND SOME GENERALİZATIONS FOR HIS IDEAS

In this section, we will be working on Nezir's equivalent norm in (Nezir, 2017a) that we introduced in our first section. We will see some more properties for his norm and we will obtain some other equivalent norms on c_0 giving similar results to his such that these new types of equivalent norms are generilazations of his.

First, we would like to recall his norm and its results.

Definition 4.1. *Let* $\alpha \in \Re$. *For* $x = (\xi_k)_k \in c_0$, *define*

$$\|x\| = \|x\|_\infty + \sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} Q_k \left|\xi_k - \alpha\xi_j\right|$$

where $\sum_{k=1}^{\infty} Q_k = 1,\ Q_k \downarrow_k 0$ and $Q_k > Q_{k+1},\ \forall k \in \mathbb{N}$ (Nezir, 2017a).

Theorem 4.2 if $\alpha = 0$ or if $Q_1 > \dfrac{2|\alpha|}{1+2|\alpha|}$ when $|\alpha| > 1$, then $(c_0, \|.\|)$ does not contain an ai copy of c_0 where the norm $\|.\|$ is defined as in Definition 4.1 (Nezir, 2017a).

Example 4.3 *Fix* $b \in (0,1)$. *We define the sequence* $(f_n)_{n\in\mathbb{N}}$ *in* c_0 *by setting* $f_1 := be_1$, $f_2 := be_2$, *and* $f_n := e_n$, *for all integers* $n \geq 3$. *Next, define the cbc subset* $E = E_b$ *of* c_0 *by*

$$E := \left\{ \sum_{n=1}^{\infty} t_n f_n : 1 = t_1 \geq t_2 \geq \ldots \geq t_n \downarrow_n 0 \right\}.$$

Let us define the sequence $(\eta_n)_{n\in\mathbb{N}}$ in E in the following way. Let $\eta_1 := f_1$ and $\eta_n := f_1 + \ldots + f_n$, for all integers $n \geq 2$. It is straightforward to check that

$$E = \left\{ \sum_{n=1}^{\infty} \alpha_n \eta_n : \text{each } \alpha_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \alpha_n = 1 \right\}.$$

Then, in (Lennard and Nezir, 2011), they show that $E = E_b$ is the cch of $(\eta_n)_{n\in\mathbb{N}}$ which is an ai c_0-summing basis respect to $\|\cdot\|_\infty$ and that there exists an affine $\|\cdot\|_\infty$-nonexpansive mapping $U : E \to E$ that is fixed point free.

Theorem 4.4 *There exist constants* $\alpha \geq \dfrac{1}{2}$ *and* $b \in (0,1)$ *the set* E *defined as in the example above has FPP(nea) where the used norm* $\|\cdot\|$ *on* c_0 *is given as in Definition 4.1 such that* $Q_1 > \dfrac{2\alpha}{1+2\alpha}$ (Nezir, 2017a).

Now, we will provide an interesting property of this equivalent norm which shows how nice it is in terms of fixed point property. We know that researchers working on sequence spaces first check what is the behaviour of the right shift mapping since they see usually that the right shift mapping or a power of that is mostly nonexpansive or asymptotically nonexpansive on their choosen cbc subsets, e.g. the convex hull of the summing basis, and they are fixed point free. Thus, one can say that the right shift mapping or any power of that is the usual test mapping to see if the space or the set fails the fixed point property for nonexpansive mappings. Therefore, the following result will be about an investigation for the behaviour of the right shift mapping on some well-known cbc subsets of c_0.

Proposition 4.5 *For the equivalent norm* $\|\cdot\|$, *if* $\alpha = 0$, *the right shift mapping defined on the cch of the usual summing basis is nonexpansive and fixed point free. But if* $\alpha \geq \frac{1}{2}$ *and* $Q_1 > \frac{2\alpha}{1+2\alpha}$, *then right shift mapping or any power of that is not nonexpansive for our norm. Also, for specific choices of the sequence* $(Q_n)_{n\in\mathbb{N}}$ *the previous statement is still true for any* $\alpha > 0$ *where* $Q_1 > \frac{2\alpha}{1+2\alpha}$.

Proof. When $\alpha = 0$, for $x = (\xi_k)_k \in c_0$, define

$$\|x\| = \|x\|_\infty + \sum_{k=1}^{\infty} Q_k |\xi_k| \text{ where } \sum_{k=1}^{\infty} Q_k = 1, Q_k \downarrow_k 0$$

and $Q_k > Q_{k+1}, \forall k \in \mathbb{N}$.

Then, define

$$E := \left\{ \sum_{n=1}^{\infty} t_n e_n : 1 = t_1 \geq t_2 \geq \ldots \geq t_n \downarrow_n 0 \right\}.$$

and $T\left(\sum_{n=1}^{\infty} t_n e_n\right) = e_1 + \sum_{n=1}^{\infty} t_n e_{n+1}$

for all $x = \sum_{n=1}^{\infty} t_n e_n \in C$.

Now, write

$$x = \sum_{n=1}^{\infty} t_n e_n \text{ and } y = \sum_{n=1}^{\infty} s_n e_n.$$

Then,

$$\begin{aligned}\|Tx - Ty\| &= \|Tx - Ty\|_\infty + \sum_{k=3}^{\infty} Q_k |t_k - s_k| \\ &= \|x - y\|_\infty + \sum_{k=3}^{\infty} Q_k |t_k - s_k| \\ &\leq \|x - y\|_\infty + \sum_{k=1}^{\infty} Q_k |t_k - s_k| \\ &= \|x - y\|.\end{aligned}$$

But, for $\alpha \neq 0$, when $Q_1 > \frac{2|\alpha|}{1+2|\alpha|}$,

define; for $x = (\xi_k)_k \in c_0$,

$$\|x\| = \|x\|_\infty + \sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} Q_k |\xi_k - \alpha\xi_j| \text{ where } \sum_{k=1}^{\infty} Q_k = 1,$$

$Q_k \downarrow_k 0$ and $Q_k > Q_{k+1}, \forall k \in \mathbb{N}$

Consider the cch of the usual summing basis and define the right shift mapping on this set; i.e

$$E := \left\{ \sum_{n=1}^{\infty} t_n e_n : 1 = t_1 \geq t_2 \geq \ldots \geq t_n \downarrow_n 0 \right\}.$$

$$T\left(\sum_{n=1}^{\infty} t_n e_n\right) = e_1 + \sum_{n=1}^{\infty} t_n e_{n+1} \quad \textit{for all } x = \sum_{n=1}^{\infty} t_n e_n \in C$$

First of all, define for each $j \in \mathbb{N}$,

$$\|x\|_{(j)} = \|x\|_\infty + \sum_{k=1}^{\infty} Q_k |\xi_k - \alpha\xi_j|.$$

Then,

$$\|x\| = \sup_{j\in\mathbb{N}} \|x\|_{(j)}.$$

Now,

$$for\ x = \sum_{n=1}^{\infty} t_n e_n\ and\ y = \sum_{n=1}^{\infty} s_n e_n\ in\ C$$

$$Tx - Ty = 0e_1 + \sum_{n=1}^{\infty}(t_n - s_n)e_{n+1}$$
$$= \sum_{n=1}^{\infty}(t_n - s_n)e_{n+1}$$

Hence, (because the first and the second terms of the sequence $Tx - Ty$ are 0)

$$\|Tx-Ty\|_{(1)} = \|Tx-Ty\|_\infty + \sum_{k=2}^{\infty} Q_{k+1}|t_k - s_k|$$

Note that $\|Tx-Ty\|_\infty = \|x-y\|_\infty$ since $t_1 = s_1 = 1\ and\ so\ |t_1 - s_1| = 0.$

Thus,

$$\|Tx-Ty\|_{(1)} = \|x-y\|_\infty + \sum_{k=1}^{\infty} Q_{k+1}|t_k - s_k|$$

Moreover, (because the first term of the sequence $x - y$ is 0)

$$\|x-y\|_{(1)} = \|x-y\|_\infty + \sum_{k=1}^{\infty} Q_k|t_k - s_k|$$

Case 1: Let $1 > \alpha > \frac{1}{2}$, then $Q_1 > \frac{1}{2}$.

Since $\sum_{k=1}^{\infty} Q_k = 1$, there exists $M \in \mathbb{N}$ s.t. $2(1-\sum_{k=1}^{M} Q_k) > 0$ and $Q_2 - Q_{M+1} > 0.$ then $1-\sum_{k=1}^{M} Q_k > \sum_{k=1}^{M} Q_k - 1.$ Also, since $Q_1 > \frac{1}{2}$, $1-\sum_{k=1}^{M} Q_k > \sum_{k=1}^{M} Q_k - 2Q_1$ so $1-2\sum_{k=2}^{M} Q_k > 0$.

Hence, for $x = (1,1,\dots,\underbrace{1}_{M^{th}\ place},0,0,\dots,0,\dots)$,

$y = (1,0,0,0,0,\dots,0,\dots)$,

$Tx = (1,1,1,\dots,\underbrace{1}_{(M+1)^{th}\ place},0,0,\dots,0,\dots)$ and

$Ty = (1,1,0,0,0,\dots,0,\dots)$

$x - y = (0,1,\dots,\underbrace{1}_{M^{th}\ place},0,0,\dots,0,\dots)$ and

$Tx - Ty = (0,0,1,\dots,\underbrace{1}_{(M+1)^{th}\ place},0,0,\dots,0,\dots).$

Then,

$$\|Tx-Ty\| = 1 + \left(1-2\sum_{k=3}^{M+1} Q_k\right)\alpha + \sum_{k=3}^{M+1} Q_k \text{ and}$$

$$\|x-y\| = 1 + \left(1-2\sum_{k=2}^{M} Q_k\right)\alpha + \sum_{k=2}^{M} Q_k.$$

Hence,

$$\|Tx-Ty\| - \|x-y\| = \left(\sum_{k=3}^{M+1} Q_k - \sum_{k=2}^{M} Q_k\right)(1-2\alpha)$$
$$= (2\alpha-1)(Q_2 - Q_{M+1}) > 0.$$

Case 2: Let $\alpha \geq 1$. Then, we could again write $Q_1 > \frac{1}{2}$, then still $1-2\sum_{k=2}^{M} Q_k > 0$ and get the same results as in the above case but here simply we could just consider $x = (1,1,0,\dots,0,\dots)$ and $y = (1,0,\dots,0,\dots)$, and then,

$\|x-y\| = 1 + \alpha - Q_2$.

Hence, $\|Tx-Ty\| - \|x-y\| = Q_2 - Q_3 > 0.$

Case 3: Let $\alpha = \frac{1}{2}$, then $Q_1 > \frac{1}{2}$ and $\sum_{k=2}^{\infty} Q_k < \frac{1}{2}$.

Pick $x = \left(1, \frac{1}{8}, \frac{1}{16}, 0, .., 0, \dots\right)$ and $y = (1, 0, \dots, 0, \dots)$ so $x - y = \left(0, \frac{1}{8}, \frac{1}{16}, 0, .., 0, \dots\right)$ and $Tx - Ty = \left(0, 0, \frac{1}{8}, \frac{1}{16}, 0, .., 0, \dots\right)$.

Then,

$$\|x-y\| = \|x-y\|_{(2)} = \frac{1}{8} + (1 - Q_2 - Q_3)\frac{1}{8}\alpha + |1-\alpha| Q_2 \frac{1}{8} + |1-2\alpha| Q_3 \frac{1}{16}$$

and

$$\|Tx-Ty\| = \|Tx-Ty\|_{(3)} = \frac{1}{8} + (1 - Q_3 - Q_4)\frac{1}{8}\alpha + |1-\alpha| Q_3 \frac{1}{8} + |1-2\alpha| Q_4 \frac{1}{16}.$$

Thus, $\|Tx-Ty\| - \|x-y\| = \frac{1}{16}(Q_3 - Q_4) > 0.$

Also, for specific choices of the sequence $(Q_n)_{n\in\mathbb{N}}$, the previous statement is still true for any $\alpha > 0$ where $Q_1 > \frac{2|\alpha|}{1+2|\alpha|}$.

Indeed, extending the following example, it is possible to show this. Let's see a simple example for smaller α.

Let $\alpha = \frac{1}{4}$, then $Q_1 > \frac{1}{3}$ and $\sum_{k=2}^{\infty} Q_k < \frac{2}{3}$.

Pick $x = \left(1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, 0, .., 0, \dots\right)$ and $y = (1, 0, \dots, 0, \dots)$ so $x - y = \left(0, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, 0, .., 0, \dots\right)$ and $Tx - Ty = \left(0, 0, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, 0, .., 0, \dots\right)$.

Then,

$$\|x-y\| = \frac{1}{4} + (1 - Q_2 - Q_3 - Q_4)\frac{1}{4}\alpha + |1-\alpha| Q_2 \frac{1}{4} + |1-4\alpha| Q_3 \frac{1}{16} + |1-16\alpha| Q_4 \frac{1}{64}$$

and

$$\|Tx-Ty\| = \frac{1}{4} + (1 - Q_3 - Q_4 - Q_5)\frac{1}{4}\alpha + |1-\alpha| Q_3 \frac{1}{4} + |1-4\alpha| Q_4 \frac{1}{16} + |1-16\alpha| Q_5 \frac{1}{64}.$$

Then,

$$\|Tx-Ty\| - \|x-y\| = \frac{1}{64}(12Q_3 - 8Q_2 - Q_5 - 3Q_4).$$

Hence, specific choice of $(Q_n)_{n\in\mathbb{N}}$ would tell us the right shift would not be nonexpansive. We can leave the rest to the reader who can see any power of the right shift would not be nonexpansive on the cch of the usual summing basis or even any subsequence of that.

Now, we can define some more equivalent norms satisfying the properties of the one given in Definition 4.1 and present the following corollary which can be considered as a generalization.

Corollary 3.5 *Let* $\alpha > 0$ *and* $\alpha_n \downarrow_n \alpha$ *and let* $Q_1 > \frac{2\alpha_1}{1+2\alpha_1}$. *Then, define*

$$\|x\|_{\alpha}^{\sim} := \|x\|_{\infty} + \sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} Q_k \left|\xi_k - \alpha_j \xi_j\right|$$

where $\sum_{k=1}^{\infty} Q_k = 1$, $Q_k \downarrow_k 0$ and $Q_k > Q_{k+1}$, $\forall k \in \mathbb{N}$

and define

$$\|x\|_{\alpha'}^{\sim} := \|x\|_{\infty} + \sup_{j,s\in\mathbb{N}} \sum_{k=1}^{\infty} Q_k \left|\xi_k - \alpha_s \xi_j\right|.$$

where $\sum_{k=1}^{\infty} Q_k = 1$, $Q_k \downarrow_k 0$ and $Q_k > Q_{k+1}$, $\forall k \in \mathbb{N}$

Then, $\left(c_0, \|.\|_{\alpha}^{\sim}\right)$ or $\left(c_0, \|.\|_{\alpha'}^{\sim}\right)$ do not have any ai copy of c_0 inside.

Furthermore, let $\beta > 0$ and $\beta_n \uparrow_n \beta$ and let $Q_1 > \frac{2\beta}{1+2\beta}$.

Then, define

$$\|x\|_{\beta}^{\sim} := \|x\|_{\infty} + \sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} Q_k \left|\xi_k - \beta_j \xi_j\right|$$

where $\sum_{k=1}^{\infty} Q_k = 1$, $Q_k \downarrow_k 0$ and $Q_k > Q_{k+1}$, $\forall k \in \mathbb{N}$

and define

$$\|x\|_{\beta'}^{\sim} := \|x\|_{\infty} + \sup_{j,s\in\mathbb{N}} \sum_{k=1}^{\infty} Q_k \left|\xi_k - \beta_s \xi_j\right|.$$

where $\sum_{k=1}^{\infty} Q_k = 1$, $Q_k \downarrow_k 0$ and $Q_k > Q_{k+1}$, $\forall k \in \mathbb{N}$

Then, $\left(c_0, \|.\|_{\beta}^{\sim}\right)$ or $\left(c_0, \|.\|_{\beta'}^{\sim}\right)$ do not have any ai copy of c_0 inside.

Proof. We would like to skip the details of the proof but we can give a quick idea about it since the proof uses the method given in (Nezir, 2017a).

Firstly, to show $\left(c_0, \|.\|_{\alpha}^{\sim}\right)$ does not have any ai copy of c_0 inside, we repeat arguments in the previous theorem by considering the sequence α_n is decreasing and so each term does not exceed α_1 and so we would imitate the proof of the theorem taking α_1 instead of α. Next, showing $\left(c_0, \|.\|_{\alpha'}^{\sim}\right)$ does not have any ai copy of c_0 inside is trivial since if we assume by contradiction that it contains an ai copy of c_0 then we would say there exists a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in c_0 such that for every $n \in \mathbb{N}$ and every choice of scalars $t_1, t_2, \ldots, t_n$, it follows that

$$\max_{1\le k\le n} (1-\varepsilon_k)|t_k| \le \left\|\sum_{k=1}^{n} t_k x_k\right\|_{\alpha'}^{\sim} \le \max_{1\le k\le n} |t_k|$$ but

then there exists $m_0 \in \mathbb{N}$ such that

$$\max_{1\le k\le n} (1-\varepsilon_k)|t_k| \le \left\|\sum_{k=1}^{n} t_k x_k\right\|_{\alpha_{m_0}}^{\sim} \le \max_{1\le k\le n} |t_k|$$ where

$$\|x\|_{\alpha_{m_0}}^{\sim} = \|x\|_\infty + \sup_{j\in\mathbb{N}} \sum_{k=1}^{\infty} Q_k \left| \xi_k - \alpha_{m_0} \xi_j \right|$$

for $x = (\xi_k)_k \in c_0$

and this would be a contradiction due to our previous theorem.

For the norm $\|.\|_{\beta'}^{\sim}$, again we can use the method above getting β_{m_0} for some $m_0 \in \mathbb{N}$ and for the other norm; i.e., for the norm $\|.\|_{\beta}^{\sim}$ use the method for $\|.\|_{\alpha}^{\sim}$ but just consider $\beta_n \le \beta$ for each $n \in \mathbb{N}$ and so use β instead of α_1 where it is needed.

REFERENCES

Bessaga, C., Pełczyński, A. (1958). On Bases and Unconditional Convergence of Series in Banach Spaces. *Studia Mathematica*, 17(2), 151-164.

Diestel J. (2012). Sequences and Series in Banach Spaces. Vol. 92. Springer Science & Business Media, New York, 263.

Goebel K., Kuczumow T. (1979). Irregular Convex Sets with Fixed-Point Property for Nonexpansive Mappings. *Colloquium Mathematicae*, 40 (2), 259–264.

Hardy, G. H., Littlewood, J. E., Pólya, G. (1952). Inequalities. Cambridge University press. 324.

Dowling, P., Lennard, C. (1997). Every Nonreflexive Subspace of $L_1[0, 1]$ Fails The Fixed Point Property. *Proceedings of the American Mathematical Society*, 125(2), 443-446.

Dowling, P.N., Lennard, C.J., Turett, B. (1996). Reflexivity and The Fixed-Point Property for Nonexpansive Maps. *Journal of Mathematical Analysis and Applications*, 200(3), 653-662.

Dowling, P.N., Lennard, C.J., Turett, B. (2001). Renormings of ℓ^1 and c_0 and Fixed Point Properties. In: Handbook of metric fixed point theory. Kirk W. and Sims B. (eds), Springer Netherlands, 269-297.

Kirk, W. and Sims, B. (2013). Handbook of Metric Fixed Point Theory. Springer Science & Business Media.

Lennard, C., Nezir, V. (2011). The Closed, Convex Hull of Every AI c_0-Summing Basic Sequence Fails The FPP for Affine Nonexpansive Mappings. *Journal of Mathematical Analysis and Applications*, 381(2), 678-688.

Lin, P.K. (2008). There Is An Equivalent Norm on ℓ^1 That Has The Fixed Point Property. *Nonlinear Analysis: Theory, Methods & Applications*, 68(8), 2303-2308.

Nezir, V. (2012). Fixed Point Properties for c_0-Like Spaces. Ph.D. Thesis, University of Pittsburgh, Pittsburgh, PA, USA.

Nezir, V. (2017a). Renorming c_0 and Affine Fixed Point Property. *Submitted*.

Nezir, V. (2017b). Asymptotically Isometric Copies of ℓ^1⊞^{0}. *Submitted*.

Nezir, V., Sade, S. (2017). Abundance of Equivalent Norms on c_0 with Fixed Point Property for Affine Nonexpansive Mappings. *Communications Faculty of Sciences University of Ankara Series A1*, 67(1), 1-28.

Núñez, C. (1989). Characterization of Banach Spaces of Continuous Vector Valued Functions with The Weak Banach-Saks Property. *Illinois Journal of Mathematics*, 33(1), 27–41.