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Research Article

The multiplicity of positive solutions for systems of fractional boundary value problems

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Abstract

This paper focuses on the multiple positive solutions for a coupled system of nonlinear boundary value problems of fractional order. Our approach is based on a fixed point theorem due to Bai and Ge. Also, an example is given to demonstrate the applicability of our main result.

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Keywords. multiple positive solution, fractional differential equation, fixed point theorem

1. Introduction

We concentrate on the multiple positive solutions of the following fractional differential systems with m-point integral boundary conditions:

$$\begin{cases} D^{p_1}u(t) + f_1(t, v(t), D^{q_1}v(t)) = 0, \quad t \in (0,1), \\ D^{p_2}v(t) + f_2(t, u(t), D^{q_2}u(t)) = 0, \quad t \in (0,1), \\ u'(0) = ... = u^{(n-2)}(0) = u^{(n-1)}(0) = 0, \ u(1) = \displaystyle\sum_{i=1}^{m-1} a_i \int_0^{n_i} u(s)dA(s), \\ v'(0) = ... = v^{(r-2)}(0) = v^{(r-1)}(0) = 0, \ v(1) = \displaystyle\sum_{i=1}^{m-1} b_i \int_0^{n_i} v(s)dB(s), \end{cases} \tag{1.1}$$

in which D is the standard Caputo fractional derivative, $n, r \in \mathbb{N}$, $n, r \geq 2$, $n-1 < p_1 \leq n$, $r-1 < p_2 \leq r$, $1 < q_1 < p_2 - 1$, $1 < q_2 < p_1 - 1$, $m \in \mathbb{N}$, $m \geq 2$, $0 < n_1 < n_2 < ... < n_{m-1} = 1$, $a_i, b_i \geq 0$ for $i \in \{1, 2, ..., m-1\}$. Here $f_1, f_2 \in \mathcal{C}([0,1] \times [0,\infty) \times \mathbb{R}, (0,\infty))$, $\int_0^{n_i} u(s)dA(s)$ and $\int_0^{n_i} u(s)dB(s)$ are the Riemann-Stieltjes integrals with positive measures, A and B are functions of bounded variation, $\sum_{i=1}^{m-1} a_i \int_0^{n_i} dA(s) < 1$ and $\sum_{i=1}^{m-1} b_i \int_0^{n_i} dB(s) < 1$.

In recent years, the theory of boundary value problems for fractional differential equations and coupled systems has gained a significant research area of investigation due to their applications in various research areas such as physics, biology, chemistry, control theory, economics, etc. Consequently, there are many works devoted to the existence of

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positive solutions for fractional boundary value problems by means of methods of nonlinear analysis, see [1–3, 5, 8, 9, 11, 12, 14, 16, 17] and the references therein. For instance, Su [14], investigated the following system of the form

$$\begin{cases} D^{\alpha}u(t) + f(t, v(t), D^{\mu}v(t)) = 0, & t \in (0,1), \\ D^{\beta}v(t) + g(t, u(t), D^{\vartheta}u(t)) = 0, & t \in (0,1), \\ u(0) = v(0) = u(1) = v(1) = 0, \end{cases} \tag{1.2}$$

where D is the Riemann Liouville differentiation, $\alpha, \beta \in [1,2)$, $\mu, \vartheta > 0$, $\alpha - \vartheta \geq 1$, $\beta - \mu \geq 1$, $f, g \in \mathcal{C}([0,1] \times \mathbb{R} \times \mathbb{R})$. The author established the main result of the system (1.2) by using the Schauder fixed point theorem.

In [7], the authors studied the following system of fractional boundary value problem

$$\begin{cases} D^{\alpha}u(t) + \lambda f(t, u(t), v(t)) = 0, & t \in (0,1), \\ D^{\beta}v(t) + \mu g(t, u(t), v(t)) = 0, & t \in (0,1), \\ u(0) = u'(0) = \ldots = u^{(n-2)}(0) = 0, \ u(1) = \displaystyle\sum_{i=1}^{p} a_i u(\xi_i), \\ v(0) = v'(0) = \ldots = v^{(m-2)}(0) = 0, \ v(1) = \displaystyle\sum_{i=1}^{q} b_i v(\eta_i), \end{cases} \tag{1.3}$$

in which D^{α} and D^{β} are Riemann Liouville derivatives, $n, m \in \mathbb{N}$, $n, m \geq 2$, $\alpha \in (n-1, n]$, $\beta \in (m-1, m]$, $p, q \in \mathbb{N}$, $a_i \in \mathbb{R}$ for any $i \in \{1, 2, ..., p\}$, $b_i \in \mathbb{R}$ for any $i \in \{1, 2, ..., q\}$, $0 < \xi_1 < \xi_2 < \ldots < \xi_p < 1$, $0 < \eta_1 < \eta_2 < \ldots < \eta_q < 1$ and f is a given function on $[0,1] \times (0, \infty) \times \mathbb{R}$. Here, they investigate the positive solutions of the system (1.3) via the Krasnoselskii's fixed point theorem.

On the other hand, fractional boundary value problems with Riemann Stieltjes integral boundary conditions is an interesting class of problems. Since they contain multi-point and integral boundary conditions as special cases, the problem (1.1) is more general than problems given in some literature. Also, compared with [7, 15] we consider the system whose nonlinear terms constitute the fractional derivative which gives rise to complication.

In order to establish our main theorem, the following concepts and the fixed point theorem due to Bai and Ge [4, 6] are presented.

Let ζ be a nonnegative continuous concave functional on the cone P and η, ϑ be nonnegative continuous convex functionals verifying

$$\|u\| \leq M \max\{\eta(u), \vartheta(u)\}, \text{for all } u \in P, \tag{1.4}$$

where $M > 0$ is a constant and

$$K = \{u \in P : \eta(u) < R, \vartheta(u) < k\} \neq \emptyset, \text{ for any } R > 0 \text{ and } k > 0. \tag{1.5}$$

For any $R > c > 0$ and $k > 0$, define the following convex sets

$$\begin{aligned} P(\eta, R, \vartheta, k) &= \{u \in P : \eta(u) < R, \vartheta(u) < k\}, \\ \overline{P}(\eta, R, \vartheta, p) &= \{u \in P : \eta(u) \leq R, \vartheta(u) \leq k\}, \\ P(\eta, R, \vartheta, k, \zeta, c) &= \{u \in P : \eta(u) < R, \vartheta(u) < k, \zeta(u) > c\}, \\ \overline{P}(\eta, R, \vartheta, k, \zeta, c) &= \{u \in P : \eta(u) \leq R, \vartheta(u) \leq k, \zeta(u) \geq c\}. \end{aligned}$$

Lemma 1.1 ([4]). *Suppose that $\mathbb{B}$ is a Banach space, $P \subset \mathbb{B}$ is a cone, $R_2 \geq d > e > R_1 > 0$, $k_2 \geq k_1 > 0$, and η, ϑ are nonnegative, continuous, convex functionals on P satisfying (1.4) and (1.5), ζ is a nonnegative continuous, concave functional on P verifying $\zeta(u) \leq \eta(u)$ for all $u \in \overline{P}(\eta, R_2; \vartheta, k_2)$. If $A : \overline{P}(\eta, R_2; \vartheta, k_2) \to \overline{P}(\eta, R_2; \vartheta, k_2)$ is a completely continuous operator and the following conditions are satisfied*

(i) $\left\{u \in \overline{P}(\eta, d; \vartheta, k_2, \zeta, e) : \zeta(u) > e\right\} \neq \emptyset$, *$\zeta(Au) > e$ for $u \in \overline{P}(\eta, d; \vartheta, k_2, \zeta, e)$,*

(ii) $\eta(Au) < R_1$, $\vartheta(Au) < k_1$ *for all* $u \in \overline{P}(\eta, R_1, \vartheta, k_1)$,

(iii) $\zeta(Au) > e$, *for all* $u \in \overline{P}(\eta, R_2; \vartheta, k_2, \zeta, e)$ *with* $\eta(Au) > e$,

then A *has at least three fixed points* u_1, u_2, u_3 *in* $\overline{P}(\eta, R_2, \vartheta, k_2)$ *satisfying*

$$u_1 \in P(\eta, R_1, \vartheta, k_1), \quad u_2 \in \left\{\overline{P}(\eta, R_2; \vartheta, k_2; \zeta, e) : \zeta(u) > e\right\},$$
$$u_3 \in \overline{P}(\eta, R_2, \vartheta, k_2) \setminus (\overline{P}(\eta, R_2; \vartheta, k_2; \zeta, e) \cup \overline{P}(\eta, R_1; \vartheta, k_1)).$$

2. Existence results

Many definitions on the fractional calculus have emerged over the years. In this paper, our work relies on the sense of the Caputo fractional derivative (see [10,13]) given by

$$D^p h(t) = \frac{1}{\Gamma(n-p)} \int_0^t (t-s)^{n-p-1} h^{(n)}(s)ds, \quad n-1 < p < n, \ n = [p]+1,$$

$$D^n h(t) = h^n(t), \ n \in \{1, 2, 3, ...\},$$

where $h \in \mathcal{C}^n[0, \infty)$, and $[p]$ is the integer part of p.

Consider the following boundary value problem of fractional differential equation

$$\begin{cases} -D^{p_1} u(t) = h(t), \quad t \in (0,1), \\ u'(0) = ... = u^{(n-2)}(0) = u^{(n-1)}(0) = 0, \\ u(1) = \sum_{i=1}^{m-1} a_i \int_0^{n_i} u(s)dA(s). \end{cases} \tag{2.1}$$

In order to establish the positive solutions of the boundary value problem (2.1), we give the following lemma:

Lemma 2.1. *Let* $h \in \mathcal{C}([0,1], \mathbb{R})$, *then the BVP* (2.1) *has an integral expression*

$$u(t) = \int_0^1 H_1(t,s)h(s)ds,$$

where

$$H_1(t,s) = G_1(t,s) + \frac{1}{D_1} \sum_{i=1}^{m-1} a_i \int_0^{n_i} G_1(t,s)dA(t), \tag{2.2}$$

$$G_1(t,s) = \frac{1}{\Gamma(p_1)} \begin{cases} (1-s)^{p_1-1}, & 0 \le t \le s \le 1, \\ (1-s)^{p_1-1} - (t-s)^{p_1-1}, & 0 \le s \le t \le 1, \end{cases} \tag{2.3}$$

and $D_1 = 1 - \sum_{i=1}^{m-1} a_i \int_0^{n_i} dA(s)$.

Proof. Let u verify (2.1). Then the general solution of (2.1) is given by

$$u(t) = -\frac{1}{\Gamma(p_1)} \int_0^t (t-s)^{p_1-1} h(s)ds + c_1 + c_2 t + ... + c_{n-1} t^{n-1}.$$

By the boundary conditions of (2.1), an easy calculation yields that $c_2 = c_3 = ... = c_{n-1} = 0$, and

$$c_1 = \frac{1}{\Gamma(p_1)} \int_0^1 (1-s)^{p_1-1} h(s)ds + \sum_{i=1}^{m-1} a_i \int_0^{n_i} u(s)dA(s).$$

Thus, the unique solution of the BVP (2.1) is of the form

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(p_1)} \int_0^t (t-s)^{p_1-1} h(s)ds + \sum_{i=1}^{m-1} a_i \int_0^{n_i} u(s)dA(s) + \frac{1}{\Gamma(p_1)} \int_0^1 (1-s)^{p_1-1} h(s)ds \\ &= \int_0^1 G_1(t,s)h(s)ds + \sum_{i=1}^{m-1} a_i \int_0^{n_i} u(s)dA(s), \end{aligned} \tag{2.4}$$

where

$$\sum_{i=1}^{m-1} a_i \int_0^{n_i} u(s)dA(s) = \sum_{i=1}^{m-1} a_i \int_0^{n_i} \Big[\int_0^1 G_1(s,\tau)h(\tau)d\tau + \sum_{i=1}^{m-1} a_i \int_0^{n_i} u(\tau)dA(\tau)\Big] dA(s).$$

So,

$$\sum_{i=1}^{m-1} a_i \int_0^{n_i} u(s)dA(s) = \frac{1}{D_1} \sum_{i=1}^{m-1} a_i \int_0^1 \int_0^{n_i} G_1(s,\tau)dA(s)h(\tau)d\tau. \tag{2.5}$$

Putting (2.5) into (2.4), we get that

$$\begin{aligned} u(t) &= \int_0^1 G_1(t,s)h(s)ds + \int_0^1 \Big[\frac{1}{D_1} \sum_{i=1}^{m-1} a_i \int_0^{n_i} G_1(t,s)dA(t)\Big] h(s)ds \\ &= \int_0^1 H_1(t,s)h(s)ds. \end{aligned}$$

□

By (2.2) *and* (2.3)*, one can easily see that* $G_1(t,s)$ *and* $H_1(t,s)$ *have the following properties:*

Lemma 2.2. *The function* $G_1(t,s)$ *given by* (2.3) *satisfies*

(i) $G_1(t,s) \geq 0$ *for any* $t, s \in [0,1]$.
(ii) $G_1(t,s) \leq G_1(s,s)$ *for any* $t, s \in [0,1]$.
(iii) *For any* $\mu \in (0, 1/2)$ *and* $s \in [0,1]$*, we have*

$$\min_{t\in[\mu,1-\mu]} G_1(t,s) \geq \Theta_1 G_1(s,s), \text{ for all } s \in [0,1],$$

where

$$\Theta_1 = 1 - (1-\mu)^{p_1-1}. \tag{2.6}$$

Lemma 2.3. *The function* $H_1(t,s)$ *given by* (2.2) *verifies*

(i) $H_1(t,s) \geq 0$ *for any* $t, s \in [0,1]$.
(ii) $H_1(t,s) \leq \xi_1 G_1(s,s)$ *for any* $t, s \in [0,1]$*, where*

$$\xi_1 = 1 + \frac{1}{D_1} \sum_{i=1}^{m-1} a_i \int_0^{n_i} dA(s). \tag{2.7}$$

(iii) *For any* $\mu \in (0, 1/2)$ *and* $s \in [0,1]$*, then*

$$\min_{t\in[\mu,1-\mu]} H_1(t,s) \geq \xi_1 \Theta_1 G_1(s,s), \text{ for all } s \in [0,1],$$

where ξ_1 *and* Θ_1 *are given by* (2.6) *and* (2.7) *respectively.*

In a similar way, we can obtain the above results for the BVP

$$\begin{cases} D^{p_2}v(t) = h(t), \quad t \in (0,1), \\ v'(0) = ... = v^{(r-2)}(0) = v^{(r-1)}(0) = 0, \ v(1) = \sum_{i=1}^{m-1} b_i \int_0^{n_i} v(s)dB(s). \end{cases}$$

Here, we assume that $G_2(t,s)$, $H_2(t,s)$, Θ_2, ξ_2 denote the corresponding functions and constants defined in a similar way as in Lemma 2.1-Lemma 2.3.

Let $U = \{u(t) : u \in \mathcal{C}[0,1] : D^{q_2}u \in \mathcal{C}[0,1]\}$ with the norm $\|u\|_U = \max_{t\in[0,1]} |u(t)| + \max_{t\in[0,1]} |D^{q_2}u(t)|$, and $V = \{v(t) : v \in \mathcal{C}[0,1] : D^{q_1}v \in \mathcal{C}[0,1]\}$ with the norm $\|v\|_V = \max_{t\in[0,1]} |v(t)| + \max_{t\in[0,1]} |D^{q_1}v(t)|$. Then $U \times V$ is a Banach space endowed with the norm $\|(u,v)\| = \max\{\|u\|_U, \|v\|_V\}$ for $(u,v) \in U \times V$.

Define the cone

$$P = \{(u,v) \in U \times V : u(t) \geq 0, \ v(t) \geq 0\},$$

and the operator $A : U \times V \to U \times V$ given by

$$A(u,v)(t) = (A_1v(t), A_2u(t)), \tag{2.8}$$

in which

$$\begin{aligned} A_1v(t) &= \int_0^1 H_1(t,s)f_1(s,v(s),D^{q_1}v(s))ds, \\ A_2u(t) &= \int_0^1 H_2(t,s)f_2(s,u(s),D^{q_2}u(s))ds. \end{aligned} \tag{2.9}$$

Then the fixed point of the operator A is the solution of the system (1.1).

Lemma 2.4. *$A : P \to P$ is a completely continuous operator.*

Proof. From the definition of A, it is clear that $A(P) \subset P$. Also, in view of the Arzela Ascoli theorem and the standard arguments, one can see easily that $A : P \to P$ is completely continuous. □

For the readers convenience, let us denote

$$\begin{aligned} A_1 &= \xi_1 \int_0^1 G_1(s,s)ds, \quad B_1 = \frac{1}{\Gamma(p_1 - q_2)} \int_0^1 (1-s)^{p_1-q_2-1}ds, \\ A_2 &= \xi_2 \int_0^1 G_2(s,s)ds, \quad B_2 = \frac{1}{\Gamma(p_2 - q_1)} \int_0^1 (1-s)^{p_2-q_1-1}ds, \\ M_1 &= \xi_1\Theta_1 \int_\mu^{1-\mu} G_1(s,s)ds, \quad M_2 = \xi_2\Theta_2 \int_\mu^{1-\mu} G_2(s,s)ds, \\ \Theta &= \frac{1}{\mu}. \end{aligned}$$

Theorem 2.5. *Suppose that there exist constants $R_2 > \Theta e > e > R_1 > 0$, $k_2 \geq k_1 > 0$ such that* $\dfrac{R_2}{\Theta M_i} \leq \min\left\{\dfrac{R_2}{A_i}, \dfrac{k_2}{B_i}\right\}$. *Assume*

(a) $f_i(t,u,v) \leq \min\left\{\frac{R_2}{A_i}, \frac{k_2}{B_i}\right\}$ *for $t \in [0,1]$, $u \in [0,R_2]$, $v \in [-k_2,k_2]$.*

(b) $f_i(t,u,v) > \dfrac{e}{M_i}$ *for $t \in [\mu, 1-\mu]$, $u \in [e,\Theta e]$, $v \in [-k_2,k_2]$.*

(c) $f_i(t,u,v) \leq \min\left\{\frac{R_1}{A_i}, \frac{k_1}{B_i}\right\}$ *for $t \in [0,1]$, $u \in [0,R_1]$, $v \in [-k_1,k_1]$.*

(d) $f_i(t,u,v) > \dfrac{R_2}{\Theta M_i}$ *for $t \in [\mu, 1-\mu]$, $u \in [e,R_2]$, $v \in [-k_2,k_2]$.*

Then the system (1.1) *has at least three positive solutions* (u_i, v_i) $(i = 1, 2, 3)$ *with*

$$0 \le \max\left\{\max_{t\in[0,1]} u_1(t), \max_{t\in[0,1]} v_1(t)\right\} \le R_1, \quad \max\left\{\max_{t\in[0,1]} |D^{q_2}u_1(t)|, \max_{t\in[0,1]} |D^{q_1}v_1(t)|\right\} \le k_1$$

$$e < \min\left\{\min_{t\in[\mu,1-\mu]} u_2(t), \min_{t\in[\mu,1-\mu]} v_2(t)\right\} \le \max\left\{\max_{t\in[0,1]} u_2(t), \max_{t\in[0,1]} v_2(t)\right\} \le R_2,$$

$$\max\left\{\max_{t\in[0,1]} |D^{q_2}u_2(t)|, \max_{t\in[0,1]} |D^{q_1}v_2(t)|\right\} \le k_2,$$

$$k_1 \le \max\left\{\max_{t\in[0,1]} |D^{q_2}u_3(t)|, \max_{t\in[0,1]} |D^{q_1}v_3(t)|\right\} \le k_2,$$

$$R_1 \le \max\left\{\max_{t\in[0,1]} u_3(t), \max_{t\in[0,1]} v_3(t)\right\} \le R_2, \quad \min\left\{\min_{t\in[\mu,1-\mu]} u_3(t), \min_{t\in[\mu,1-\mu]} v_3(t)\right\} \le e.$$

Proof. To prove that the system has three positive solutions, we introduce the following three functionals by

$$\eta(u, v) = \max\left\{\|u\|_\infty, \|v\|_\infty\right\},$$
$$\vartheta(u, v) = \max\left\{\|D^{q_2}u\|_\infty, \|D^{q_1}v\|_\infty\right\},$$
$$\zeta(u, v) = \min\left\{\min_{t\in[\mu,1-\mu]} u(t), \min_{t\in[\mu,1-\mu]} v(t)\right\}.$$

It is apparent that ζ is a nonnegative concave functional and η, ϑ are nonnegative convex functionals satisfying (1.4) and (1.5).

Define $A : P \to P$ by (2.8) and (2.9) It follows from Lemma 2.4 that A is completely continuous. Now, we will show that all the conditions of Lemma 1.1 is verified. First, we prove that $A : \overline{P}(\eta, R_2, \vartheta, k_2) \to \overline{P}(\eta, R_2, \vartheta, k_2)$. Let $(u, v) \in \overline{P}(\eta, R_2, \vartheta, k_2)$. Then $\eta(u, v) \le R_2$, $\vartheta(u, v) \le k_2$. By virtue of assumption (a), we can get

$$\begin{aligned}\|A_1 v\|_\infty &= \max_{t\in[0,1]} \Big|\int_0^1 H_1(t, s) f_1(s, v(s), D^{q_1}v(s))ds\Big| \\ &\le \xi_1 \int_0^1 G_1(s, s) f_1(s, v(s), D^{q_1}v(s))ds \\ &\le \frac{R_2}{A_1}\xi_1 \int_0^1 G_1(s, s)ds = R_2\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}\|D^{q_2}A_1 v\|_\infty &= \max_{t\in[0,1]} \Big| -\frac{1}{\Gamma(p_1 - q_2)} \int_0^t (t - s)^{p_1 - q_2 - 1} f_1(s, v(s), D^{q_1}v(s))ds\Big| \\ &\le \frac{1}{\Gamma(p_1 - q_2)} \frac{k_2}{B_1} \int_0^1 (1 - s)^{p_1 - q_2 - 1} ds = k_2.\end{aligned} \tag{2.11}$$

Similarly, one has $\|A_2 u\|_\infty \le R_2$ and $\|D^{q_1}A_2 u\|_\infty \le k_2$. So, we have $A : \overline{P}(\eta, R_2, \vartheta, k_2) \to \overline{P}(\eta, R_2, \vartheta, k_2)$. Next, we shall show that condition (i) of Lemma 1.1 is verified. Pick $(\Theta e, \Theta e)$. Then, one can easily see that $(\Theta e, \Theta e) \in \overline{P}(\eta, \Theta e, \vartheta, k_2, \zeta, e)$, where $\zeta(u, v) > e$. Thus, $\Big\{(u, v) \in \overline{P}(\eta, \Theta e, \vartheta, k_2, \zeta, e) : \zeta(u, v) > e\Big\} \ne \emptyset$. If we choose $(u, v) \in \overline{P}(\eta, \Theta e, \vartheta, k_2, e)$, then $u(t), v(t) \in [e, \Theta e]$ for any $t \in [\mu, 1-\mu]$. Thus, from assumption (b) we get

$$\begin{aligned}\min_{t\in[\mu,1-\mu]} A_1 v(t) &\geq \Theta_1 \xi_1 \int_\mu^{1-\mu} G_1(s,s) f_1(s, v(s), D^{q_1} v(s)) ds \\ &> \frac{e}{M_1} \Theta_1 \xi_1 \int_\mu^{1-\mu} G_1(s,s) ds \\ &= e.\end{aligned}$$

Also, it is evident that $\min_{t\in[\mu,1-\mu]} A_2 u(t) > e$. So, we obtain that $\zeta(A(u,v)) > e$. Thus (i) of Lemma 1.1 is verified.

In a similar way as in (2.10) and (2.11), we can prove that the operator A holds the condition (ii) of Lemma 1.1.

Finally, we show that the last condition of Lemma 1.1 is satisfied. For $u \in \overline{P}(\eta, R_2, \vartheta, k_2, \zeta, e)$ with $\eta(A(u,v)) > \Theta e$, we know that $\eta(A(u,v)) \leq R_2$. Then (d) implies that

$$\begin{aligned}\min_{t\in[\mu,1-\mu]} A_1 v(t) &\geq \Theta_1 \xi_1 \int_\mu^{1-\mu} G_1(s,s) f_1(s, v(s), D^{q_1} v(s)) ds \\ &> \frac{R_2}{\Theta M_1} \Theta_1 \xi_1 \int_\mu^{1-\mu} G_1(s,s) ds \\ &= \frac{R_2}{\Theta} \geq \frac{\eta(A(u,v))}{\Theta} > e.\end{aligned}$$

Similarly, one has $\min_{t\in[\mu,1-\mu]} A_2 u(t) > e$. Hence, $\zeta(A(u,v)) > e$, (iii) of Lemma 1.1 is verified.

Consequently, it follows from Lemma 1.1 that the operator A has at least three positive solutions (u_i, v_i)
$(i = 1, 2, 3)$ with

$$\begin{aligned}&(u_1, v_1) \in P(\eta, R_1, \vartheta, k_1), \quad (u_2, v_2) \in \left\{\overline{P}(\eta, R_2; \vartheta, k_2; \zeta, e) : \zeta(u) > e\right\}, \\ &(u_3, v_3) \in \overline{P}(\eta, R_2, \vartheta, k_2) \setminus (\overline{P}(\eta, R_2; \vartheta, k_2; \zeta, e) \cup \overline{P}(\eta, R_1; \vartheta, k_1)).\end{aligned}$$

The proof is complete. □

Example 2.6. Consider

$$\begin{cases} D^{7/2} u(t) + f_1(t, v(t), D^{3/2} v(t)) = 0, \quad t \in (0,1), \\ D^{7/2} v(t) + f_2(t, u(t), D^{3/2} u(t)) = 0, \quad t \in (0,1), \\ u'(0) = u''(0) = 0, \ u(1) = \frac{1}{2}\int_0^{1/3} u(s) dA(s) + \frac{1}{2}\int_0^{1/2} u(s) dA(s) + \frac{1}{2}\int_0^1 u(s) dA(s), \\ v'(0) = v''(0) = 0, \ v(1) = \frac{1}{2}\int_0^{1/3} v(s) dB(s) + \frac{1}{2}\int_0^{1/2} v(s) dB(s) + \frac{1}{2}\int_0^1 v(s) dB(s), \end{cases} \tag{2.12}$$

where $p_1 = p_2 = \frac{7}{2}$, $q_1 = q_2 = \frac{3}{2}$, $m = 4$, $A(s) = B(s) = \frac{s^4}{4}$, $a_1 = a_2 = a_3 = \frac{1}{2}$, $n_1 = \frac{1}{3}$, $n_2 = \frac{1}{2}$, $n_3 = 1$,

$$f_1(t,u,v) = \begin{cases} \left(\frac{1}{8}\right)^{t+1} + \frac{100u}{8} + \frac{v^2}{10^4}, & u \in [0,8], \\ \left(\frac{1}{8}\right)^{t+1} + 950u - 7500 + \frac{v^2}{10^4}, & u \in [8,10], \\ \left(\frac{1}{8}\right)^{t+1} + \frac{100u + 17000}{9} + \frac{v^2}{10^4}, & u \in [10,\infty], \end{cases}$$

and

$$f_2(t,u,v) = \begin{cases} \left(\frac{1}{6}\right)^t + \frac{100u}{8} + \frac{|v|}{10^5}, & u \in [0,8], \\ \left(\frac{1}{6}\right)^t + 950u - 7500 + \frac{|v|}{10^5}, & u \in [8,10], \\ \left(\frac{1}{6}\right)^t + \frac{100u+17000}{9} + \frac{|v|}{10^5}, & u \in [10,\infty]. \end{cases}$$

By easy calculation, we obtain $D = \frac{81}{432}$, $\sum_{i=1}^{3} a_i \int_0^{n_i} dA(s)$, $\sum_{i=1}^{3} a_i \int_0^{n_i} dB(s) \in (0,1)$, $\Theta_1 = \Theta_2 = \frac{8}{15\sqrt{\pi}}$, $\xi_1 = \xi_2 = \frac{332}{81}$. Let $\mu = \frac{1}{4}$, then $A_1 = A_2 = \frac{108}{2905\sqrt{\pi}}$, $B_1 = B_2 = \frac{1}{2}$, and $M_1 = M_2 \approx 0.005593$. Choosing $R_1 = 8$, $e = 10$, $R_2 = 100$, $k_1 = 100$, $k_2 = 2000$, one gets

$$f_1(t,u) \leq \min\left\{\tfrac{R_2}{A_1}, \tfrac{k_2}{B_1}\right\} = 4000, \text{ for } t \in [0,1],\ u \in [0,100],\ v \in [-2000,2000],$$

$$f_1(t,u,v) > \frac{e}{M_1} \approx 1787.949, \text{ for } t \in [\tfrac{1}{4}, \tfrac{3}{4}],\ u \in [10, 10\Theta],\ v \in [-2000,2000],$$

$$f_1(t,u,v) \leq \min\left\{\tfrac{R_1}{A_i}, \tfrac{k_1}{B_i}\right\} = 200 \text{ for } t \in [0,1],\ u \in [0,8],\ v \in [-100,100],$$

$$f_1(t,u,v) > \frac{R_2}{\Theta M_i} \approx 2760 \text{ for } t \in [\tfrac{1}{4}, \tfrac{3}{4}],\ u \in [10,100],\ v \in [-2000,2000];$$

i.e., f_1 holds the conditions of Theorem 2.1. Similarly, f_2 verifies the conditions of Theorem 2.1. Consequently, Theorem 2.1 implies that the system (2.12) has at least three positive solutions (u_i, v_i) for $i \in \{1,2,3\}$ with

$$0 \leq \max\left\{\max_{t\in[0,1]} u_1(t), \max_{t\in[0,1]} v_1(t)\right\} \leq 8, \quad \max\left\{\max_{t\in[0,1]} |D^{q_2} u_1(t)|, \max_{t\in[0,1]} |D^{q_1} v_1(t)|\right\} \leq 100,$$

$$10 < \min\left\{\min_{t\in[\mu,1-\mu]} u_2(t), \min_{t\in[\mu,1-\mu]} v_2(t)\right\} \leq \max\left\{\max_{t\in[0,1]} u_2(t), \max_{t\in[0,1]} v_2(t)\right\} \leq 100,$$

$$\max\left\{\max_{t\in[0,1]} |D^{q_2} u_2(t)|, \max_{t\in[0,1]} |D^{q_1} v_2(t)|\right\} \leq 1000,$$

$$100 \leq \max\left\{\max_{t\in[0,1]} |D^{q_2} u_3(t)|, \max_{t\in[0,1]} |D^{q_1} v_3(t)|\right\} \leq 1000,$$

$$8 \leq \max\left\{\max_{t\in[0,1]} u_3(t), \max_{t\in[0,1]} v_3(t)\right\} \leq 100, \quad \min\left\{\min_{t\in[\mu,1-\mu]} u_3(t), \min_{t\in[\mu,1-\mu]} v_3(t)\right\} \leq 10.$$

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