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Research Article

On ramification structures for finite nilpotent groups

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Abstract

We extend the characterization of abelian groups with ramification structures given by Garion and Penegini in [Beauville surfaces, moduli spaces and finite groups, Comm. Algebra, 2014] to finite nilpotent groups whose Sylow p-subgroups have a 'nice power structure', including regular p-groups, powerful p-groups and (generalized) p-central p-groups. We also correct two errors in [Beauville surfaces, moduli spaces and finite groups, Comm. Algebra, 2014] regarding abelian 2-groups with ramification structures and the relation between the sizes of ramification structures for an abelian group and those for its Sylow 2-subgroup.

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1. Introduction

An algebraic surface S is said to be *isogenous to a higher product of curves* if it is isomorphic to $(C_1 \times C_2)/G$, where C_1 and C_2 are curves of genus at least 2, and G is a finite group acting freely on $C_1 \times C_2$. Particular interesting examples of such surfaces are *Beauville surfaces.* These are algebraic surfaces isogenous to a higher product which are rigid.

Groups of surfaces isogenous to a higher product can be characterized by a purely group-theoretical condition: the existence of a 'ramification structure'.

Definition 1.1. Let G be a finite group and let $T = (g_1, g_2, \ldots, g_r)$ be a tuple of non-trivial elements of G.

(1) T is called a *spherical system of generators* of G if $\langle g_1, g_2, \ldots, g_r\rangle = G$ and $g_1 g_2 \ldots g_r = 1$.
(2) T is of *type* $\tau := (m_1, \ldots, m_r)$ if $o(g_i) = m_i$ for $g_i \in T$.
(3) $\Sigma(T)$ is the union of all conjugates of the cyclic subgroups generated by the elements of T:

$$\Sigma(T) = \bigcup_{g\in G} \bigcup_{i=1}^{r} \langle g_i\rangle^g.$$

Two tuples T_1 and T_2 are called *disjoint* if $\Sigma(T_1) \cap \Sigma(T_2) = 1$.

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Definition 1.2. An *(unmixed) ramification structure* of size (r_1, r_2) for a finite group G is a pair (T_1, T_2) of disjoint spherical systems of generators of G, where $|T_1| = r_1$ and $|T_2| = r_2$. We denote by $S(G)$ the set of all sizes (r_1, r_2) of ramification structures of G.

Observe that if d is the minimum number of generators of G, spherical systems of generators of G are of size at least $d+1$. Since clearly cyclic groups do not admit ramification structures, it follows that $r_1, r_2 \geq 3$ in Definition 1.2.

If $r_1 = r_2 = 3$, then ramification structures coincide with *Beauville structures*, which have been intensely studied in recent times; see surveys [1,2,7]. Not much is known about ramification structures that are not Beauville. In 2013, Garion and Penegini [5] proved that if $\tau_1 = (m_{1,1}, \ldots, m_{1,r_1})$ and $\tau_2 = (m_{2,1}, \ldots, m_{2,r_2})$ are tuples of natural numbers ≥ 2 and $\Sigma_{j=1}^{r_i}(1 - 1/m_{i,j}) > 2$ for $i = 1, 2$, then almost all alternating and symmetric groups admit a ramification structure of type (τ_1, τ_2), where in the case of symmetric groups there is an additional assumption that at least two components in both τ_1 and τ_2 are even. Soon afterwards, they characterized the abelian groups with ramification structures [6, Theorem 3.18].

After abelian groups, the most natural class of finite groups to consider are nilpotent groups. As we will see in Proposition 3.2, a finite nilpotent group admits a ramification structure if and only if so do its Sylow p-subgroups. The goal of this paper is to extend the characterization of abelian groups with ramification structures to finite nilpotent groups whose Sylow p-subgroups have a good behavior with respect to powers. To this purpose, we first study the existence of ramifications structures for finite p-groups with a 'nice power structure'. In particular, we generalize Theorem A in [4], which determines the conditions for such p-groups to be Beauville groups.

If G is a finite p-group, we call G *semi-p^{e-1}-abelian* if for every $x, y \in G$, we have

$$x^{p^{e-1}} = y^{p^{e-1}} \text{ if and only if } (xy^{-1})^{p^{e-1}} = 1.$$

Theorem A. Let G be a finite p-group of exponent p^e, and let $d = d(G)$. Suppose that G is semi-p^{e-1}-abelian. Then G admits a ramification structure if and only if $|\{g^{p^{e-1}} \mid g \in G\}| \geq p^s$ where $s = 2$ if $p \geq 3$ or $s = 3$ if $p = 2$. In that case, G admits a ramification structure of size (r_1, r_2) if and only if the following conditions hold:

1. $r_1, r_2 \geq d + 1$.
2. If $p = 3$ then $r_1, r_2 \geq 4$.
3. If $p = 2$ then $r_1, r_2 \geq 5$.
4. If $p = 2$ and $|\{g^{2^{e-1}} \mid g \in G\}| = 2^3$, then $(r_1, r_2) \neq (5, 5)$, and furthermore if $e = 1$, i.e. $G \cong C_2 \times C_2 \times C_2$, then r_1, r_2 are not both odd.

Note that the condition on the cardinality of the set $\{g^{p^{e-1}} \mid g \in G\}$ in Theorem A implies that if G admits a ramification structure, then $d(G) \geq 2$ if $p \geq 3$ or $d(G) \geq 3$ if $p = 2$.

According to [6, Theorem 3.18], if G is an abelian 2-group of exponent 2^e and $|G^{2^{e-1}}| = 2^3$, then G does not admit a ramification structure of size (r_1, r_2) if r_1, r_2 are both odd. However, Theorem A shows that this statement is not true, and they can be both odd provided that $G \ncong C_2 \times C_2 \times C_2$.

Theorem A applies to a wide family of p-groups, including regular p-groups (so, in particular, p-groups of exponent p or of nilpotency class less than p), powerful p-groups, and generalized p-central p-groups. A p-group is called generalized p-central if $p > 2$ and $\Omega_1(G) \leq Z_{p-2}(G)$, or $p = 2$ and $\Omega_2(G) \leq Z(G)$.

We want to remark that Theorem A is not valid for all finite p-groups. We will see that no condition on the cardinality of the set $\{g^{p^{e-1}} \mid g \in G\}$ can ensure the existence of ramification structures for the class of all finite p-groups.

On the other hand, if G is a finite nilpotent group and G_p is the Sylow p-subgroup of G, then we have $\bigcap_{p||G|} S(G_p) \subseteq S(G)$, and $S(G) \subseteq S(G_p)$ for odd primes p. However, it is not always true that $S(G) \subseteq S(G_2)$, even for abelian groups, contrary to what is implicit in the statement of Theorem 3.18 in [6]. We give a counterexample to that in Example 3.3. We fix this error in Theorem B.

Theorem B. Let G be a nilpotent group, and let $d = d(G)$. Let G_p denote the Sylow p-subgroup of G for every prime p dividing $|G|$. Suppose that $\exp G_p = p^{e_p}$ and all G_p are semi-p^{e_p-1}-abelian. Then G admits a ramification structure if and only if all G_p admit a ramification structure. In that case, $(r_1, r_2) \in S(G)$ if and only if the following conditions hold:

(1) $r_1, r_2 \geq d + 1$.
(2) $(r_1, r_2) \in S(G_p)$ for p odd.
(3) $(r_1, r_2) \in S(G_2)$ unless $G_2 \cong C_2 \times C_2 \times C_2$.
(4) If $G_2 \cong C_2 \times C_2 \times C_2$ then $r_1, r_2 \geq 5$ and $(r_1, r_2) \neq (5, 5)$. Furthermore, if $G \cong C_2 \times C_2 \times C_2$ then r_1, r_2 are not both odd.

Notation. If G is a finitely generated group, we write $d(G)$ for the minimum number of generators of G. If p is a prime and G is a finite p-group, then $G^{p^i} = \langle g^{p^i} \mid g \in G \rangle$ and $\Omega_i(G) = \langle g \in G \mid g^{p^i} = 1 \rangle$. The exponent of G, denoted by $\exp G$, is the maximum of the orders of all elements of G.

2. Finite p-groups

Throughout this paper all groups will be finite. In this section, we give the proof of Theorem A. Let us start with a general result related to lifting a spherical generating set of a factor group to the whole group.

Proposition 2.1. *Let G be a finite group and let $d = d(G)$. Suppose that $N \trianglelefteq G$ and $U = (\overline{x_1}, \ldots, \overline{x_r})$ is a tuple of generators of G/N. Then the following hold:*

(1) *If $r \geq d$ then there exist $n_1, \ldots, n_r \in N$ such that $T = (x_1 n_1, \ldots, x_r n_r)$ generates G.*
(2) *If $N \neq 1$, $r \geq d + 1$ and $\overline{x_1} \ldots \overline{x_r} = \overline{1}$, then we can choose T to be a spherical system of generators of G.*

Proof. (i) See Proposition 2.5.4 in [8].

(ii) Assume first that $\overline{x_i} \neq \overline{1}$ for some $i = 1, \ldots, r$. For simplicity, we suppose that $\overline{x_r} \neq \overline{1}$. The equality $\overline{x_1} \ldots \overline{x_r} = \overline{1}$ implies that $\langle \overline{x_1}, \ldots, \overline{x_{r-1}} \rangle = G/N$. Since $r - 1 \geq d$ then by (i), there is a tuple $V = (z_1, \ldots, z_{r-1})$ that generates G, where $z_i \in x_i N$ for $1 \leq i \leq r - 1$. Note that if $\overline{x_j} = \overline{1}$, then it may happen that $z_j = 1$. If this is the case, we take a nontrivial element in N as z_j. Thus, $z_i \neq 1$ for $1 \leq i \leq r - 1$.

If we call
$$T = (z_1, \ldots, z_{r-1}, (z_1 \ldots z_{r-1})^{-1}),$$
then clearly T is a spherical system of generators of G. The only thing we have to show is that $(z_1 \ldots z_{r-1})^{-1} \in x_r N$. Observe that in G/N, we have $(\overline{z_1} \ldots \overline{z_{r-1}})^{-1} = \overline{x_r}(\overline{z_1} \ldots \overline{z_{r-1}}\ \overline{x_r})^{-1} = \overline{x_r}(\overline{x_1} \ldots \overline{x_{r-1}}\ \overline{x_r})^{-1} = \overline{x_r}$. Thus, we have $(z_1 \ldots z_{r-1})^{-1} \in x_r N$. Since $\overline{x_r} \neq \overline{1}$, this implies that $z_1 \ldots z_{r-1} \neq 1$.

Now suppose that $\overline{x_i} = \overline{1}$ for all $1 \leq i \leq r$. Then $\overline{G} = \overline{1}$, and since $r \geq d + 1$, we can take any spherical system of generators T of G of size r. □

Notice that in part (ii) of Proposition 2.1, we do not require that U is a spherical system of generators of G/N. Therefore, as appears in the proof, some of $\overline{x_i} \in U$ might be the identity of G/N.

We next state a theorem characterizing the possible sizes of ramification structures of elementary abelian p-groups. Before that we need the following lemma.

Lemma 2.2. *Let G be an elementary abelian p-group of rank d with a ramification structure of size (r_1, r_2). Then the following hold:*

(1) *G admits a ramification structure of size $(r_1 + 1, r_2)$ if p is odd, and of size $(r_1 + 2, r_2)$ if $p = 2$.*
(2) *If G^* is elementary abelian of rank $d + 1$ and $r_1, r_2 \geq d + 2$, then G^* admits a ramification structure of size (r_1, r_2).*

Proof. Let (T_1, T_2) be a ramification structure of size (r_1, r_2) for G. We write $T_1 = (x_1, x_2, \ldots, x_{r_1})$.

We first prove (i). If

$$T_1' = \begin{cases} (x_1^2, x_2, \ldots, x_{r_1}, x_1^{-1}) & \text{if } p \text{ is odd,} \\ (T_1, x_1, x_1) & \text{if } p = 2, \end{cases}$$

then (T_1', T_2) is a ramification structure as desired.

We next prove (ii). Let $G^* = G \times \langle y \rangle$ be an elementary abelian p-group of rank $d + 1$. Since G is of rank d and $r_1, r_2 \geq d + 2$, both T_1 and T_2 have at least two elements, say $a_1, b_1 \in T_1$ and $a_2, b_2 \in T_2$, that belong to the subgroup generated by the rest of the elements in T_1 and T_2, respectively. We modify T_1, T_2 to T_1^* and T_2^*, by multiplying a_1, a_2 with y and b_1, b_2 with y^{-1}. Then (T_1^*, T_2^*) is a ramification structure of size (r_1, r_2) for G^*. □

Note that the roles of r_1 and r_2 are symmetric. Thus in Lemma 2.2, G also admits a ramification structure of size $(r_1, r_2 + 1)$ if p is odd and of size $(r_1, r_2 + 2)$ if $p = 2$.

Theorem 2.3. *Let G be an elementary abelian p-group of rank d and let $r_1, r_2 \geq d + 1$. Then G admits a ramification structure of size (r_1, r_2) if and only if the following conditions hold:*

(1) *$d \geq 2$ if $p \geq 3$ or $d \geq 3$ if $p = 2$.*
(2) *If $p = 3$ then $r_1, r_2 \geq 4$.*
(3) *If $p = 2$ then $r_1, r_2 \geq 5$, and furthermore if $d = 3$ then r_1, r_2 are not both odd.*

Proof. We first assume that G admits a ramification structure (T_1, T_2) of size (r_1, r_2). We already know that $d \geq 2$. If $p = 2$ and $G \cong C_2 \times C_2$, then clearly $\Sigma(T_1) \cap \Sigma(T_2) \neq 1$, a contradiction. Thus, if $p = 2$ then $d \geq 3$.

We next assume that $p = 3$. We will show that $r_1, r_2 \geq 4$. Suppose, on the contrary, that $r_1 = 3$. Then $G \cong C_3 \times C_3$. If we write $T_1 = (x_1, x_2, (x_1x_2)^{-1})$, then $\Sigma(T_1)$ contains 6 distinct nontrivial elements of G. The other two nontrivial elements of G are $x_1x_2^2$ and $x_1^2x_2^4$. Since they do not generate G, there is no ramification structure for G, which is a contradiction.

We now assume that $p = 2$. We show that $r_1, r_2 \geq 5$. Suppose that $r_1 = 4$. Then $G \cong C_2 \times C_2 \times C_2$. We write $T_1 = (x_1, x_2, x_3, (x_1x_2x_3)^{-1})$. Then T_2 can only contain x_1x_2, x_1x_3 and x_2x_3. However, $\langle x_1x_2, x_1x_3, x_2x_3 \rangle \neq G$, again a contradiction.

Finally, we show that if $G \cong C_2 \times C_2 \times C_2$ then r_1, r_2 are not both odd. Suppose that r_1 is odd. Then observe that T_1 contains at least 4 distinct nontrivial elements. Otherwise, if T_1 has 3 distinct nontrivial elements, say u, v, t, then (u, v, t) is a minimal system of generators of G. Since the product of the elements of T_1 is equal to 1, each of u, v, t appears an even number of times in T_1, which is not possible since r_1 is odd.

We now prove the converse. To this purpose, it is enough to find ramification structures of sizes $(3, 3)$ or $(4, 4)$ according as $p \geq 5$ or $p = 3$ if $d = 2$, of sizes $(5, 6)$ or $(6, 6)$ if $d = 3$ and $p = 2$, and finally of size $(5, 5)$ if $d = 4$ and $p = 2$. Then by applying (i) and (ii) in Lemma 2.2 repeatedly, we get the result.

Let $G = \langle x_1 \rangle \times \langle x_2 \rangle \cong C_p \times C_p$ where $p \geq 3$. If we take

$$T_1 = \begin{cases} (x_1, x_2, (x_1x_2)^{-1}) & \text{if } p \geq 5, \\ (x_1, x_1^{-1}, x_2, x_2^{-1}) & \text{if } p = 3, \end{cases}$$

and

$$T_2 = \begin{cases} (x_1x_2^2, x_1x_2^4, (x_1^2x_2^6)^{-1}) & \text{if } p \geq 5, \\ (x_1x_2, (x_1x_2)^{-1}, x_1x_2^2, (x_1x_2^2)^{-1}) & \text{if } p = 3, \end{cases}$$

then (T_1, T_2) is a ramification structure for G of size $(3, 3)$ if $p \geq 5$, or of size $(4, 4)$ if $p = 3$.

Now assume that $G = \langle x_1 \rangle \times \langle x_2 \rangle \times \langle x_3 \rangle \cong C_2 \times C_2 \times C_2$. If we take

$$T_1 = \begin{cases} (x_1x_2, x_1x_3, x_2x_3, x_1x_2x_3, x_1x_2x_3) & \text{if } r_1 = 5, \\ (x_1x_2, x_1x_3, x_1x_2x_3, x_1x_2, x_1x_3, x_1x_2x_3) & \text{if } r_1 = 6, \end{cases}$$

and $T_2 = (x_1, x_2, x_3, x_1, x_2, x_3)$, then (T_1, T_2) is a ramification structure for G of size $(5, 6)$ or $(6, 6)$.

Finally if $p = 2$ and $G = \langle x_1 \rangle \times \langle x_2 \rangle \times \langle x_3 \rangle \times \langle x_4 \rangle$, then we take $T_1 = (x_1, x_2, x_3, x_4, (x_1x_2x_3x_4)^{-1})$ and $T_2 = (x_1x_2, x_2x_3, x_3x_4, x_1x_2x_3, x_2x_3x_4)$. Then clearly (T_1, T_2) is a ramification for G of size $(5, 5)$. This completes the proof. □

Theorem 2.3 can also be deduced from Theorem 3.18 in [6] that characterizes abelian groups with ramification structures. However, note that the statement of that theorem corresponding to abelian 2-groups is not true in general. According to Theorem 3.18 in [6], if G is an abelian 2-group of exponent 2^e with $|G^{2^{e-1}}| = 2^3$ and G admits a ramification structure of size (r_1, r_2), then r_1, r_2 cannot be both odd. However, the next example shows that this is not necessarily the case. We fix this mistake in Theorem 2.8.

Example 2.4. Let $G = \langle a \rangle \times \langle x \rangle \times \langle y \rangle \times \langle z \rangle \cong C_2 \times C_4 \times C_4 \times C_4$. Now $\exp G = 4$ and $|G^2| = 2^3$. If we take

$$T_1 = (x, y, z, x^{-1}, y^{-1}, z^{-1}a, a),$$

and

$$T_2 = (xya, xz, yz, xyz, xyza),$$

then clearly (T_1, T_2) is a ramification structure for G of size $(7, 5)$.

We next see that the existence of ramification structures for a group of exponent p can be deduced from Theorem 2.5.

Theorem 2.5. *Let G be a p-group of exponent p. Then G admits a ramification structure of size (r_1, r_2) if and only if $G/\Phi(G)$ admits a ramification structure of size (r_1, r_2).*

Proof. Note that if $p = 2$ then G is an elementary abelian 2-group, and hence G coincides with $G/\Phi(G)$. Thus we assume that $p \geq 3$. We first show that if $G/\Phi(G)$ admits a ramification structure (U_1, U_2) of size (r_1, r_2), then so does G.

Consider a lift of (U_1, U_2) to G, say (T_1, T_2), such that T_1 and T_2 are spherical systems of generators of G. Since $\exp G = p$, all elements in T_1 and T_2 are of order p. We claim that (T_1, T_2) is a ramification structure of size (r_1, r_2) for G. Suppose, on the contrary, that there are $a \in T_1$ and $b \in T_2$ such that $\langle a \rangle^g = \langle b \rangle$ for some $g \in G$. Since $G/\Phi(G)$ is abelian, we get $\langle \overline{a} \rangle = \langle \overline{b} \rangle$, which is a contradiction.

Let us now prove the converse. Assume that G admits a ramification structure of size (r_1, r_2). Note that $G/\Phi(G)$ has rank at least 2. Then by Theorem 2.3, any elementary abelian p-group of rank ≥ 2 for $p \geq 5$ admits a ramification structure of size (r_1, r_2) if $r_1, r_2 \geq 3$.

Finally we assume that $p = 3$. According to Theorem 2.3, we only need to prove that G does not admit a ramification structure with $r_1 = 3$. By way of contradiction, suppose that $r_1 = 3$. It then follows that G is a 2-generator group with $\exp G = 3$. Then [9, 14.2.3] implies that G is of order 3^3. Observe that each element in T_1 falls into a distinct maximal subgroup of G. Since G has 4 maximal subgroups and not all elements in T_2 fall into the same maximal subgroup, it then follows that there are elements in T_1 and T_2, say $a \in T_1$ and $b \in T_2$, which are in the same maximal subgroup. Then we have

$$b = a^i c,$$

for some $c \in \Phi(G) = G'$ and for $i \in \{1, 2\}$. Since $|G| = 3^3$ and a^i is a generator of G, we can write $c = [a^i, g]$ for some $g \in G$. It then follows that $b = (a^i)^g$, a contradiction. □

We now introduce a property which is essential to our result, and then we describe some families of finite p-groups satisfying this property.

Let G be a finite p-group, and let $i \geq 1$ be an integer. Following Xu [11], we say that G is *semi-p^i-abelian* if the following condition holds for every $x, y \in G$:

$$x^{p^i} = y^{p^i} \quad \text{if and only if} \quad (xy^{-1})^{p^i} = 1. \tag{2.1}$$

If G is semi-p^i-abelian, then we have [11, Lemma 1]:

(SA1) $\Omega_i(G) = \{x \in G \mid x^{p^i} = 1\}$.

(SA2) $|G : \Omega_i(G)| = |\{x^{p^i} \mid x \in G\}|$.

If G is semi-p^i-abelian for every $i \geq 1$, then G is called *strongly semi-p-abelian*.

By [10, Theorem 3.14], regular p-groups are strongly semi-p-abelian. On the other hand, by Lemma 3 in [3], a powerful p-group of exponent p^e is semi-p^{e-1}-abelian. Furthermore, by Theorem 2.2 in [4], generalized p-central p-groups, i.e. groups in which $\Omega_1(G) \leq Z_{p-2}(G)$ for odd p, or $\Omega_2(G) \leq Z(G)$ for $p = 2$, are strongly semi-p-abelian.

Before we proceed to prove Theorem A, we need the following lemma.

Lemma 2.6. *Let G be a p-group of exponent p^e and let $d = d(G)$. Suppose that G is semi-p^{e-1}-abelian. Then the following hold:*

(1) *If (T_1, T_2) is a ramification structure for G, then $(\overline{T}_1 \smallsetminus \{\overline{1}\}, \overline{T}_2 \smallsetminus \{\overline{1}\})$ is a ramification structure for $G/\Omega_{e-1}(G)$.*

(2) *If (U_1, U_2) is a ramification structure of size (r_1, r_2) for $G/\Omega_{e-1}(G)$ and $r_1, r_2 \geq d + 1$, then there is a lift of (U_1, U_2) to G which is a ramification structure of size (r_1, r_2) for G.*

Proof. We first prove (i) by way of contradiction. Note that $G/\Omega_{e-1}(G)$ is of exponent p. Suppose that there are $\overline{a} \in \overline{T}_1 \smallsetminus \{\overline{1}\}$ and $\overline{b} \in \overline{T}_2 \smallsetminus \{\overline{1}\}$ such that $\langle \overline{a} \rangle = \langle \overline{b} \rangle^{\overline{g}}$ for some $\overline{g} \in G/\Omega_{e-1}(G)$, i.e. $\overline{b}^{\overline{g}} = \overline{a}^i$ for some i not divisible by p. Then we have $b^g a^{-i} \in \Omega_{e-1}(G)$, and consequently $(b^g a^{-i})^{p^{e-1}} = 1$, by (SA1). Since G is semi-p^{e-1}-abelian, we get $(b^g)^{p^{e-1}} = a^{ip^{e-1}}$. This is a contradiction, since both a and b are of order p^e and $\langle a \rangle \cap \langle b \rangle^g = 1$.

We next prove (ii). By part (ii) of Proposition 2.1, we can take a lift of (U_1, U_2) to G, say (T_1, T_2), such that T_1 and T_2 are spherical systems of generators of G. Observe that all elements in T_1 and T_2 are of order p^e. We next show that T_1 and T_2 are disjoint. Suppose, on the contrary, that there are $a \in T_1$ and $b \in T_2$ such that

$$\langle a^{p^{e-1}} \rangle^g = \langle b^{p^{e-1}} \rangle,$$

for some $g \in G$, i.e $(a^g)^{p^{e-1}} = b^{ip^{e-1}}$ for some integer i not divisible by p. Since G is semi-p^{e-1}-abelian, then $a^g b^{-i} \in \Omega_{e-1}(G)$, and consequently, $\langle \overline{a} \rangle^{\overline{g}} = \langle \overline{b} \rangle$ in $G/\Omega_{e-1}(G)$, which is a contradiction since (U_1, U_2) is a ramification structure for $G/\Omega_{e-1}(G)$. □

We are now ready to prove Theorem A. We deal separately with the cases $p \geq 3$ and $p = 2$.

Theorem 2.7. *Let G be a p-group of exponent p^e with $p \geq 3$, and let $d = d(G)$. Suppose that G is semi-p^{e-1}-abelian. Then G admits a ramification structure if and only if $|\{g^{p^{e-1}} \mid g \in G\}| \geq p^2$. In that case, G admits a ramification structure of size (r_1, r_2) if and only if $r_1, r_2 \geq d + 1$, and also $r_1, r_2 \geq 4$ provided that $p = 3$.*

Proof. We first assume that G admits a ramification structure (T_1, T_2). By (SA2), the cardinality of the set $X = \{g^{p^{e-1}} \mid g \in G\}$ is a power of p. Suppose that $|X| = p$. It then follows that the subgroup $G^{p^{e-1}}$ is cyclic of order p. Note that by (SA1), we have $\exp \Omega_{e-1}(G) = p^{e-1}$. Then there are elements $a \in T_1$ and $b \in T_2$ such that $o(a) = o(b) = p^e$. Thus,

$$G^{p^{e-1}} = \langle a^{p^{e-1}} \rangle = \langle b^{p^{e-1}} \rangle,$$

which is a contradiction.

We next prove that if $p = 3$ and G admits a ramification structure of size (r_1, r_2), then $r_1, r_2 \geq 4$. Suppose, by way of contradiction, that $r_1 = 3$. Then since $|X| \geq 3^2$, we have $|G/\Omega_{e-1}(G)| \geq 3^2$, by (SA2). Part (i) of Lemma 2.6 implies that $G/\Omega_{e-1}(G)$ admits a ramification structure of size (r, s) where $r \leq r_1 \leq 3$. However, according to Theorems 2.3 and 2.5 this is not possible.

Now assume that $|X| \geq p^2$. Let us use the bar notation $\overline{G}$ for the factor group $G/\Omega_{e-1}(G)$. Then $|\overline{G}| \geq p^2$ and $d(\overline{G}) \geq 2$. It follows from Theorems 2.3 and 2.5 that $\overline{G}$ admits a ramification structure of size (r, s) for all $r, s \geq d(\overline{G}) + 1$, and $r, s \geq 4$ provided that $p = 3$. If we take $r_1, r_2 \geq d + 1 \geq d(\overline{G}) + 1$, and $r_1, r_2 \geq 4$ provided that $p = 3$, then part (ii) of Lemma 2.6 implies that G admits a ramification structure of size (r_1, r_2). This completes the proof. ☐

We next deal with the prime 2.

Theorem 2.8. *Let G be a 2-group of exponent 2^e, and let $d = d(G)$. Suppose that G is semi-2^{e-1}-abelian. Then G admits a ramification structure if and only if $|\{g^{2^{e-1}} \mid g \in G\}| \geq 2^3$. In that case, G admits a ramification structure of size (r_1, r_2) if and only if the following conditions hold:*

(1) *$r_1, r_2 \geq d + 1$.*
(2) *$r_1, r_2 \geq 5$.*
(3) *If $|\{g^{2^{e-1}} \mid g \in G\}| = 2^3$, then $(r_1, r_2) \neq (5, 5)$, and furthermore if $e = 1$, i.e. $G \cong C_2 \times C_2 \times C_2$, then r_1, r_2 are not both odd.*

Proof. We first assume that G admits a ramification structure. Suppose that $X = \{g^{2^{e-1}} \mid g \in G\}$ is of cardinality at most 2^2, so that $|G : \Omega_{e-1}(G)| \leq 2^2$. Then according to Theorem 2.3, $G/\Omega_{e-1}(G)$ does not admit a ramification structure. Thus, G has no ramification structure, as follows from Lemma 2.6(i). This is a contradiction. So we have $|X| \geq 2^3$.

If the ramification structure for G is of size (r_1, r_2), then we have $r_1, r_2 \geq d + 1$. By Theorem 2.3, ramification structures of $G/\Omega_{e-1}(G)$ have size (r, s) where $r, s \geq 5$, and furthermore r, s are not both odd if $|G/\Omega_{e-1}(G)| = 2^3$. Hence, by part (i) of Lemma 2.6, we have $r_1, r_2 \geq 5$ and furthermore, if $|G/\Omega_{e-1}(G)| = 2^3$ then $(r_1, r_2) \neq (5, 5)$. Finally if $G \cong C_2 \times C_2 \times C_2$ then r_1, r_2 are not both odd, by Theorem 2.3.

We now work under the assumption $|X| \geq 2^3$. Suppose that $r_1, r_2 \geq d+1$, $r_1, r_2 \geq 5$ and furthermore that r_1, r_2 are not both odd if $|X| = 2^3$. Then by Theorem 2.3, $G/\Omega_{e-1}(G)$ admits a ramification structure of size (r_1, r_2). Lemma 2.6(ii) implies that G admits a ramification structure of size (r_1, r_2).

It remains to prove that if $r_1, r_2 \geq 5$, $(r_1, r_2) \neq (5, 5)$ and both r_1, r_2 are odd, then G admits a ramification structure of size (r_1, r_2) under the assumptions $|X| = 2^3$ and $e \geq 2$. We may assume that $r_2 \geq 7$. Then $G/\Omega_{e-1}(G)$ admits a ramification structure of size $(r_1, r_2 - 1)$.

Since G/G^2 is elementary abelian of rank d and $G/\Omega_{e-1}(G)$ is of rank 3, we have $\Omega_{e-1}(G)/G^2$ is of rank $d-3$. We take a generating set $\{n_1,\dots,n_{d-3}\}$ of $\Omega_{e-1}(G)$ modulo G^2. Call $n = n_1 \dots n_{d-3}$ and let $o(n) = 2^k < 2^e$. If $1 \neq n^{2^{k-1}} = x^{2^{e-1}}$ for some $x \in G$, then since $x \notin \Omega_{e-1}(G)$ we take a generating set of $G/\Omega_{e-1}(G)$ containing $\overline{x}$, say $G/\Omega_{e-1}(G) = \langle \overline{x} \rangle \times \langle \overline{y} \rangle \times \langle \overline{z} \rangle$. Otherwise, if $n^{2^{k-1}} \neq g^{2^{e-1}}$ for any $g \in G$, then we take any generating set of $G/\Omega_{e-1}(G)$.

Now consider the following ramification structure of $G/\Omega_{e-1}(G)$:

$$U_1 = (\overline{xy}, \overline{yz}, \overline{xz}, \overline{xyz}, \overline{xyz}, \overline{xy}, \dots, \overline{xy}) \quad \text{and}$$

$$U_2 = (\overline{x}, \overline{y}, \overline{z}, \overline{x}, \overline{y}, \overline{z}, \overline{x}, \dots, \overline{x}),$$

where $|U_1| = r_1$ and $|U_2| = r_2 - 1$. Since $r_1 \geq d+1$, by part (ii) of Proposition 2.1, we take a lift T_1 of U_1 so that T_1 is a spherical system of generators of G. Then consider the following lift of U_2 to G:

$$T_2 = (x, y, z, xn_1, yn_2, zn_3, xn_4, \dots, xn_{d-3}, x, \dots, x),$$

where $|T_2| = r_2 - 1$. Clearly, T_2 generates G. Observe that the product of all components of T_2 is n modulo G^2, i.e. the product is equal to wn for some $w \in G^2$. Now consider the following tuple:

$$T_2^* = (w^{-1}x, y, z, xn_1, yn_2, zn_3, xn_4, \dots, xn_{d-3}, x, \dots, x, n^{-1}),$$

where $|T_2| = r_2$. Since $w \in G^2 = \Phi(G)$, it follows that T_2^* generates G and furthermore, it is spherical. Our claim is that (T_1, T_2^*) is a ramification structure of size (r_1, r_2) for G.

Notice that all elements in $T_1 \cup T_2^*$ are of order 2^e except n^{-1}. Then by using the same argument in the proof of part (ii) of Lemma 2.6, we conclude that $\langle a \rangle^g \cap \langle b \rangle = 1$ for any $g \in G$, $a \in T_1$ and $b \in T_2^* \smallsetminus \{n^{-1}\}$. On the other hand, if $n^{2^{k-1}} = x^{2^{e-1}}$ then since $\langle x^{2^{e-1}} \rangle \neq \langle a^{2^{e-1}} \rangle^g$ for any $g \in G$ and $a \in T_1$, we have $\langle n \rangle \cap \Sigma(T_1) = 1$. Otherwise, if $n^{2^{k-1}} \neq g^{2^{e-1}}$ for any $g \in G$, then clearly $\langle n \rangle \cap \Sigma(T_1) = 1$. This completes the proof. □

We close this section by showing that the assumption of being semi-p^{e-1}-abelian is essential in Theorem A. As we next see, for a general finite p-group G, the cardinality of the set $\{g^{p^{e-1}} \mid g \in G\}$ does not control the existence of ramification structures for G. To this purpose, we will work with 2-generator p-groups constructed in [4]. For more details, we suggest readers to see pages 11-13 of [4].

Lemma 2.9. *Let G be a Beauville group. Then G admits a ramification structure of size (r_1, r_2) for any $r_1, r_2 \geq 3$.*

Proof. Assume that G is a Beauville group, that is it admits a ramification structure (U_1, U_2) of size $(3, 3)$. Let $U_1 = (x_1, y_1, (x_1y_1)^{-1})$, $U_2 = (x_2, y_2, (x_2y_2)^{-1})$. Consider the following tuples:

$$T_1 = (x_1, y_1, y_1^{-1}, x_1^{-1}) \quad \text{or} \quad T_1 = U_1,$$

and

$$T_2 = (x_2, y_2, y_2^{-1}, x_2^{-1}) \quad \text{or} \quad T_2 = U_2.$$

By adding x_1, x_1^{-1} to T_1 and x_2, x_2^{-1} to T_2 repeatedly, we obtain a pair of spherical systems of generators (T_1^*, T_2^*) for G of size (r_1, r_2) for any $r_1, r_2 \geq 3$. Then since (U_1, U_2) is a ramification structure for G, so does (T_1^*, T_2^*). □

The following result shows that the 'only if' part of Theorem A fails for a general finite p-group.

Proposition 2.10. *Let $p \geq 5$ be a prime. Then there exists a p-group G such that:*

(1) $|\{g^{p^{e-1}} \mid g \in G\}| = p$, *where* $p^e = \exp G$.
(2) *G admits a ramification structure of size (r_1, r_2) for any $r_1, r_2 \geq 3$.*

Proof. In the proof of Corollary 2.12 in [4], it was shown that there exists a Beauville p-group G with $\exp G = p^e$ such that $|G^{p^{e-1}}| = p$. It then follows that $|\{g^{p^{e-1}} \mid g \in G\}| = p$ and hence (i) holds. Since G is a Beauville group, (ii) readily follows from Lemma 2.9. $\square$

Finally, the following result shows that for every power of p, there is a p-group G such that the cardinality of the set $\{g^{p^{e-1}} \mid g \in G\}$ is exactly that power and G does not admit a ramification structure.

Proposition 2.11. *For every prime $p \geq 5$, and positive integer m, there exists a p-group G such that:*

(1) $|\{g^{p^{e-1}} \mid g \in G\}| = p^m$, *where* $p^e = \exp G$.
(2) *G does not admit a ramification structure.*

Proof. Consider the group G in the second part of the proof of Corollary 2.12 in [4]. Then G is a 2-generator p-group G with $\exp G = p^e$ such that $|G^{p^{e-1}}| = p^m$ for some m. One can also observe from the proof that the subgroup $G^{p^{e-1}}$ coincides with the set $\{g^{p^{e-1}} \mid g \in G\}$. Furthermore, it was shown that for every pair of generating sets (x_1, y_1) and (x_2, y_2), there are elements, say x_1 and x_2, such that $\langle x_1^i \rangle = \langle x_2^j \rangle \neq 1$ for some integers i, j. Thus, G does not admit a ramification structure. Furthermore, Corollary 2.13 in [4] implies that m can be any positive integer. $\square$

3. Finite nilpotent groups

In this section, we prove Theorem B. We give the possible sizes of ramification structures for nilpotent groups whose Sylow p-subgroups are semi-p^{e-1}-abelian if the exponent is p^e. To this purpose, we need the following result regarding a direct product of groups of coprime order.

Proposition 3.1. *Let G and G^* be groups of coprime order. Then the following hold:*

(1) *If G and G^* admit ramification structures of size (r_1, r_2) and (r_1^*, r_2^*), respectively, then $G \times G^*$ admits a ramification structure of size (r, s) where $r = \max\{r_1, r_1^*\}$ and $s = \max\{r_2, r_2^*\}$.*
(2) *If $G \times G^*$ admits a ramification structure of size (r, s), then G and G^* admit ramification structures of size (r_1, r_2) and (r_1^*, r_2^*), respectively, for some $r_1, r_1^* \leq r$ and $r_2, r_2^* \leq s$. Furthermore, if G is of odd order, we also have $r_1 = r$ and $r_2 = s$.*

Proof. We first prove (i). Assume that (T_1, T_2) and (T_1^*, T_2^*) are ramification structures of size (r_1, r_2) and (r_1^*, r_2^*) for G and G^*, respectively. Let $r = \max\{r_1, r_1^*\}$ and $s = \max\{r_2, r_2^*\}$. Then by adding as many times the identity as needed to T_1, T_2, T_1^* and T_2^*, we obtain U_1, U_2, U_1^* and U_2^* where $|U_1| = |U_1^*| = r$ and $|U_2| = |U_2^*| = s$. Let

$$U_1 = (x_1, \ldots, x_r) \quad \text{and} \quad U_2 = (y_1, \ldots, y_s),$$
$$U_1^* = (x_1^*, \ldots, x_r^*) \quad \text{and} \quad U_2^* = (y_1^*, \ldots, y_s^*).$$

Then let

$$A_1 = ((x_1, x_1^*), \ldots, (x_r, x_r^*)) \quad \text{and}$$
$$A_2 = ((y_1, y_1^*), \ldots, (y_s, y_s^*)).$$

Observe that since G and G^* have coprime order, both A_1 and A_2 generate $G \times G^*$. We will see that (A_1, A_2) is a ramification structure for $G \times G^*$. Otherwise, there exist $(a, a^*) \in A_1$ and $(b, b^*) \in A_2$ such that

$$\langle (a, a^*) \rangle^{(g, g^*)} \cap \langle (b, b^*) \rangle \neq \{(1, 1)\},$$

for some $(g, g^*) \in G \times G^*$. It then follows that either $\langle a \rangle^g \cap \langle b \rangle \neq 1$ or $\langle a^* \rangle^{g^*} \cap \langle b^* \rangle \neq 1$, which is a contradiction.

Let us now prove (ii). Assume that

$$A_1 = ((x_1, x_1^*), \ldots, (x_r, x_r^*)) \quad \text{and} \quad A_2 = ((y_1, y_1^*), \ldots, (y_s, y_s^*))$$

form a ramification structure of size (r, s) for $G \times G^*$. Assume that after deleting the identity element in $(x_1, \ldots, x_r)$ and $(y_1, \ldots, y_s)$ we get $T_1 = (z_1, \ldots, z_{r_1})$ and $T_2 = (t_1, \ldots, t_{r_2})$ for some $r_1 \leq r$ and $r_2 \leq s$. We claim that (T_1, T_2) is a ramification structure of size (r_1, r_2) for G. The same arguments apply to G^*. For every $(a, a^*) \in A_1$ and $(b, b^*) \in A_2$ we have

$$\langle (a, a^*) \rangle^{(g, g^*)} \cap \langle (b, b^*) \rangle = \{(1, 1)\}, \tag{3.1}$$

for all $(g, g^*) \in G \times G^*$. Let $|G| = l$ and $|G^*| = m$, where $\gcd(l, m) = 1$. Then by equation (3.1), we get

$$\langle ((a^m)^g, 1) \rangle \cap \langle (b^m, 1) \rangle = \{(1, 1)\},$$

and hence $\langle a^m \rangle^g \cap \langle b^m \rangle = 1$. Since $\gcd(l, m) = 1$, it then follows that $\langle a \rangle^g \cap \langle b \rangle = 1$.

Finally we assume that G is of odd order. If $r - r_1$ is even, then we take $T_1 = (z_1, \ldots, z_{r_1}, z_1, z_1^{-1}, \ldots, z_1, z_1^{-1})$. Now suppose that $r - r_1$ is odd. Since G is of odd order, we have $o(z_1) \neq 2$. Then in this case we take

$$T_1 = (z_1^2, z_1^{-1}, z_2, \ldots, z_{r_1}, z_1, z_1^{-1}, \ldots, z_1, z_1^{-1}).$$

In both cases, T_1 is a spherical system of generators of G of size r. By using the same arguments, we can make $|T_2| = s$. Then by the previous paragraph, (T_1, T_2) is a ramification structure of size (r, s) for G, as desired. This completes the proof. □

The following proposition is easily deduced from Proposition 3.1.

Proposition 3.2. *Let G be a nilpotent group. Then*

(1) *G admits a ramification structure if and only if all Sylow p-subgroups of G admit a ramification structure.*
(2) *If G is of odd order, then G admits a ramification structure of size (r_1, r_2) if and only if all Sylow p-subgroups of G admit a ramification structure of size (r_1, r_2).*

In order to characterize abelian groups with ramification structures, Garion and Penegini [6] reduced the study to their Sylow p-subgroups. However, as far as the sizes of ramification structures are concerned, this reducing argument is not correct in general. More precisely, if G is an abelian group of even order, then the size of a ramification structure of G need not be inherited by the Sylow 2-subgroup of G, as we see in the next example. We fix this mistake in Theorem 3.4.

Example 3.3. Let $G = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \cong C_6 \times C_6 \times C_2$. If we take

$$T_1 = (a, b, c, b^{-1}, (ac)^{-1}),$$

and

$$T_2 = (ab, ab, (ab)^{-2}, abc, (abc)^{-1}, a^2bc, (a^2bc)^{-1}),$$

then (T_1, T_2) is a ramification structure of size $(5, 7)$ for G . However, the Sylow 2-subgroup of G, which is $C_2 \times C_2 \times C_2$, does not admit a ramification structure of size $(5, 7)$.

We close the paper by proving Theorem B.

Theorem 3.4. *Let G be a nilpotent group, and let $d = d(G)$. Let G_p denote the Sylow p-subgroup of G for every prime p dividing $|G|$. Suppose that $\exp G_p = p^{e_p}$ and all G_p are semi-$p^{e_p - 1}$-abelian. Then G admits a ramification structure of size (r_1, r_2) if and only if the following conditions hold:*

(1) *$r_1, r_2 \geq d + 1$.*
(2) *$(r_1, r_2) \in S(G_p)$ for p odd.*
(3) *$(r_1, r_2) \in S(G_2)$ unless $G_2 \cong C_2 \times C_2 \times C_2$.*

(4) *If* $G_2 \cong C_2 \times C_2 \times C_2$ *then* $r_1, r_2 \geq 5$ *and* $(r_1, r_2) \neq (5, 5)$*. Furthermore, if* $G \cong C_2 \times C_2 \times C_2$ *then* r_1, r_2 *are not both odd.*

Proof. We first assume that $(r_1, r_2) \in S(G)$. We know that (i) holds, and by Proposition 3.1(ii), we have (ii). We next assume that $G_2 \neq 1$. Then again by Proposition 3.1(ii), G_2 admits a ramification structure of size (r, s) for some $r \leq r_1$ and $s \leq r_2$. Then by Theorem 2.8, $r, s \geq 5$, and furthermore $(r, s) \neq (5, 5)$ if $|\{g^{e_2-1} \mid g \in G_2\}| = 2^3$. This implies that $r_1, r_2 \geq 5$, and furthermore $(r_1, r_2) \neq (5, 5)$ if $|\{g^{e_2-1} \mid g \in G_2\}| = 2^3$. Then the first part of (iv) follows, and (iii) follows from Theorem 2.8. Finally if $G \cong C_2 \times C_2 \times C_2$ then r_1, r_2 are not both odd, by Theorem 2.3.

Conversely, assume that conditions (i)-(iv) hold. Then all G_p admit a ramification structure of size (r_1, r_2) unless $G_2 \cong C_2 \times C_2 \times C_2$. Thus, if $G_2 \not\cong C_2 \times C_2 \times C_2$, by Proposition 3.1(i), we conclude that G admits a ramification structure of size (r_1, r_2).

Finally we assume that conditions (i)-(iv) hold and $G_2 = \langle x \rangle \times \langle y \rangle \times \langle z \rangle \cong C_2 \times C_2 \times C_2$. If $G = G_2$ then we already know the result, by Theorem 2.8. Thus, we assume that $G \neq G_2$. Let R be the direct product of the Sylow p-subgroups of G for all odd primes p dividing $|G|$. Then Proposition 3.2(ii), together with condition (ii), implies that R admits a ramification structure of size (r_1, r_2).

If r_1, r_2 are not both odd, then G_2 also admits a ramification structure of size (r_1, r_2). Otherwise, if both r_1, r_2 are odd, then we may assume that $r_2 \geq 7$, and thus G_2 admits a ramification structure of size $(r_1, r_2 - 1)$, by Theorem 2.3. Then in both cases, Proposition 3.1(i) implies that $G = R \times G_2$ admits a ramification structure of size (r_1, r_2). This completes the proof. □

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