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Research Article

Zariski subspace topologies on ideals

Ortaç Öneş*, Mustafa Alkan

Department of Mathematics, Akdeniz University, Antalya, Turkey

Abstract

In this paper, we show how there are tight relationships between algebraic properties of a commutative ring R and topological properties of open subsets of Zariski topology on the prime spectrum of R. We investigate some algebraic conditions for open subsets of Zariski topology to become quasi-compact, dense and irreducible. We also give a characterization for the radical of an ideal in R by using topological properties.

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1. Introduction

The notion of prime ideal has led to the development of topologies on the spectrum of the prime ideals and so many useful connections between topologies and algebraic properties have been proved. Mainly, the nilradical of a commutative ring, which has an important place in the ring theory, was characterized by using topological concepts. After that, the Zariski topology on modules has also attracted considerable attention of many authors ([1–3, 6–11, 13]). In 2011, J. Abuhlail showed that there were tight relationships between algebraic properties of modules or rings and topological properties such as ultraconnected, compact or sober.

In this paper, all rings are commutative with identity. Throughout R will denote an arbitrary ring. For an ideal I of R, we study an open subset $\mathfrak{X}_I$ of the Zariski topology on the set of all prime ideals of R denoted by $Spec(R)$. We also find the relationships between open subsets of the Zariski topology and ideals. Then we obtain some characterizations for the radical of an ideal and rings by using some topological properties. For an open subset of the Zariski topology to become quasi-compact, dense, an irreducible subset or a Noetherian spectrum, some algebraic conditions have been investigated.

In Section 2, we deal with the connections between the notions of quasi-compact space and the generating set of an ideal. Then for an ideal I of R, we prove that there are elements $r_1, r_2, ..., r_n$ of R such that $\sqrt{I} = \sqrt{Rr_1 + Rr_2 + ... + Rr_n}$ if $\mathfrak{X}_I$ is quasi-compact. In Theorem 2.2, we state a necessary and sufficient algebraic condition for $\mathfrak{X}_I$ to be a quasi-compact space. Then we define the ideal $N_I(0)$ in R, which is a generalization of a radical ideal (Definition 2.3). Here we focus on its topological properties rather than algebraic ones in order to find a characterization of irreducibility. For an ideal I of R, Theorem 2.6 points out that $N_I(0)$ is a prime ideal of R if and only if $\mathfrak{X}_I$ is irreducible.

*Corresponding Author.
Email addresses: ortacns@gmail.com (O. Öneş), alkan@akdeniz.edu.tr (M. Alkan)
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Moreover in Theorem 2.8, we find some necessary and sufficient algebraic conditions for $\mathfrak{X}_I$ to be Noetherian.

In Section 3, we concentrate on the relationships between open subsets of the Zariski topology on $Spec(R)$ and ideals of R. Then in Theorem 3.4, we focus on the irreducibility of $Spec(R)$ and verify that $Spec(R)$ is irreducible if and only if every ideal of $R/\sqrt{0}$ is essential. Thus in Corollary 3.8, we obtain a sufficient algebraic condition for an open subset to be quasi-compact. Finally, we close this paper with Theorem 3.10, which is that $Spec(R) = \bigcup_{i=1}^{n} \mathfrak{X}_{I_i}$, where $\mathfrak{X}_{I_i}$ is irreducible and I_i is an ideal of R, if and only if $R = \sum_{i=1}^{n} I_i$ and $N_{I_i}(0)$ is a prime ideal of R. As a result of Theorem 3.10, we remark on the connections between the irreducible components and ideals at the end of this paper.

Now we recall some definitions from [4, 9, 11] as follows:

- The Zariski topology is a topology on $Spec(R)$ in which closed sets are $V(I) = \{P \in Spec(R) : I \subseteq P\}$ of all ideals I of R.
- Let D be a subset of a topological space X.

(i) If every open cover of X has a finite subcover, X is said to be quasi-compact.
(ii) If $X \neq \emptyset$ and for every decomposition $X = X_1 \cup X_2$ with closed subsets X_1, $X_2 \subseteq X$, either $X = X_1$ or $X = X_2$ holds, then X is said to be irreducible.
(iii) If for every nonempty open set $U \subseteq X$, $U \cap D \neq \emptyset$, then D is said to be dense in X.
(iv) If the closed subsets of X satisfy the descending chain condition, then X is said to be Noetherian.

2. The subspace on an ideal

Let I be an ideal of a ring R, $\mathfrak{X}_I = Spec(R) \backslash V(I)$ and $\tilde{V}(B) = V(B) \backslash V(I)$, where B is an ideal of R. Then it is clear that

$$\Gamma_I = \left\{ \tilde{V}(B) : B \text{ is an ideal of } R \right\}$$

satisfies the axioms for closed sets of a topological space on $\mathfrak{X}_I$, called the complement Zariski topology of I in R.

It is clear that the open subset $\mathfrak{X}_I$ is equal to $Spec(R)$ when $I = R$ and also $\mathfrak{X}_I = \emptyset$ when $I = 0$.

We start with the following proposition revealing some connections between $\mathfrak{X}_I$ and the ideal I.

Proposition 2.1. *Let I be an ideal of a ring R. The following hold:*

(i) *For $r \in R$, $(\mathfrak{X}_I)^r = \mathfrak{X}_I \backslash \tilde{V}(r) = Spec(R) \backslash (V(rI))$ forms a base for $\mathfrak{X}_I$.*
(ii) *$(\mathfrak{X}_I)^r \cap (\mathfrak{X}_I)^s = (\mathfrak{X}_I)^{rs}$ for every element $r,s \in R$.*
(iii) *$(\mathfrak{X}_I)^r = \emptyset$ if and only if $rI \subseteq \sqrt{0}$.*
(iv) *If r is unit, then $(\mathfrak{X}_I)^r = \mathfrak{X}_I$.*
(v) *$(\mathfrak{X}_I)^r = (\mathfrak{X}_I)^s$ if and only if $\sqrt{rI} = \sqrt{sI}$.*
(vi) *If $(\mathfrak{X}_I)^r = \mathfrak{X}_I$, then we have $\sqrt{rI} = \sqrt{I} \subseteq \sqrt{\langle r \rangle}$.*

By the standard argument, Proposition 2.1 can be proved.

Let I be a proper ideal of a ring R. I is said to satisfy the condition $(*)$ if there is a finite subset Δ of an index set Λ such that

$$\sqrt{\langle \{r_i \in R : i \in \Lambda\} \rangle} = \sqrt{\langle \{r_j \in R : j \in \Delta\} \rangle}$$

whenever $\sqrt{I} \subseteq \sqrt{\langle \{r_i \in R : i \in \Lambda\} \rangle}$.

If R is a Noetherian ring, then every ideal of R satisfies the condition $(*)$. More specifically, if $R/\sqrt{I}$ is a Noetherian ring, then I satisfies the condition $(*)$ but it is clear that the converse is not true.

In this paper, let us denote the finite set $\Delta = \{1, 2, ..., n\}$ for a positive integer n.

The following theorem gives some connections between algebraic properties and topological properties.

Theorem 2.2. *Let I be a proper ideal of a ring R. Then the following hold:*

(i) $(\mathfrak{X}_I)^r$ *is quasi-compact for every* $r \in R$.

(ii) *If $\mathfrak{X}_I$ is quasi-compact, then there is a finite subset $\{r_1, r_2, ..., r_n\}$ of R such that $\sqrt{I} = \sqrt{Rr_1 + ... + Rr_n}$.*

(iii) *If I satisfies the condition $(*)$, then $\mathfrak{X}_I$ is quasi-compact.*

Proof. (i) By the standard argument, it can easily be proved.

(ii) Let $\mathfrak{X}_I$ be quasi-compact.

Let $I = \langle\{r_i \in R : i \in \Lambda\}\rangle$. Since $V(\langle\{r_i : i \in \Lambda\}\rangle) = V(I)$, it follows that $\tilde{V}(\langle\{r_i : i \in \Lambda\}\rangle) = \emptyset$. Thus

$$\begin{aligned} \mathfrak{X}_I &= \mathfrak{X}_I \backslash \emptyset = \mathfrak{X} \backslash V\left\{\bigcup_{i \in \Lambda} r_i I\right\} = \mathfrak{X} \backslash \left(\bigcap_{i \in \Lambda} V(r_i I)\right) \\ &= \bigcup_{i \in \Lambda} (\mathfrak{X} \backslash V(r_i I)) = \bigcup_{i \in \Lambda} (\mathfrak{X}_I)^{r_i}. \end{aligned}$$

Thus $\mathfrak{X}_I$ has an open cover and since $\mathfrak{X}_I$ is quasi-compact, there is a finite set $\Delta = \{1, 2, ..., n\} \subseteq \Lambda$ such that $\mathfrak{X}_I = \bigcup_{i \in \Delta} (\mathfrak{X}_I)^{r_i} = \mathfrak{X}_I \backslash \tilde{V}(\langle r_1, r_2, ..., r_n\rangle)$. Thus $\tilde{V}(\langle r_1, r_2, ..., r_n\rangle) = \emptyset$, which means that $V(\langle r_1, r_2, ..., r_n\rangle) \subseteq V(I)$. Hence it follows that $\sqrt{I} \subseteq \sqrt{Rr_1 + Rr_2 + ... + Rr_n}$ and we have the equality since $r_1, r_2, ..., r_n \in I$.

(iii) Let I satisfy the condition $(*)$.

Let $\{A_i : i \in \Lambda\}$ be an open cover of $\mathfrak{X}_I$. Since A_i can be expressed as a union of the sets of $(\mathfrak{X}_I)^r$, we may assume that $A_i = (\mathfrak{X}_I)^{r_i}$ for every $i \in \Lambda$. Then

$$\begin{aligned} \mathfrak{X}_I &= \bigcup_{i \in \Lambda} (\mathfrak{X}_I)^{r_i} = \bigcup_{i \in \Lambda} \left(\mathfrak{X}_I \backslash \tilde{V}(r_i)\right) \\ &= \mathfrak{X}_I \backslash \bigcap_{i \in \Lambda} \tilde{V}(r_i) \\ &= \mathfrak{X}_I \backslash \tilde{V}(\{r_i : i \in \Lambda\}). \end{aligned}$$

Thus $\tilde{V}(\langle\{r_i : i \in \Lambda\}\rangle) = \emptyset$ and so $V(\langle\{r_i : i \in \Lambda\}\rangle) \subseteq V(I)$.

In this case, $\sqrt{I} \subseteq \sqrt{\langle\{r_i : i \in \Lambda\}\rangle}$. By the condition $(*)$, there is a finite subset $\Delta \subseteq \Lambda$ such that $\sqrt{I} = \sqrt{\langle\{r_j : j \in \Delta\}\rangle}$. Then $V(I) = V(\langle\{r_j : j \in \Delta\}\rangle)$ and so $\tilde{V}(\langle\{r_j : j \in \Delta\}\rangle) = \emptyset$. Then

$$\begin{aligned} \mathfrak{X}_I &= \mathfrak{X}_I \backslash \tilde{V}(\langle\{r_j : j \in \Delta\}\rangle)) = \mathfrak{X}_I \backslash \bigcap_{j \in \Delta} \tilde{V}(r_j) \\ &= \bigcup_{j \in \Delta} \left(\mathfrak{X}_I \backslash \tilde{V}(r_j)\right) = \bigcup_{j \in \Delta} (\mathfrak{X}_I)^{r_j}. \end{aligned}$$

Since $\mathfrak{X}_I$ is covered by a finite number of open subsets $(\mathfrak{X}_I)^{r_j}$, $\mathfrak{X}_I$ is quasi-compact. □

For an ideal I of R, we recall the definition of a radical ideal from [12] as follows:

$$\sqrt{I} = \{r \in R : r^n \in I \text{ for a positive integer } n\}.$$

If $I = 0$, then $\sqrt{0}$ is called the nilradical. It is well known that the nilradical ideal is prime if and only if $Spec(R)$ is irreducible. To obtain similar results for open subsets, we define a new class of ideals as follows:

Definition 2.3. Let I be an ideal of a ring R. The set $N_I(T)$ is defined as the intersection of all prime ideals containing T which does not contain I. In other words, $N_I(T) = \cap \{P \in Spec(R) : T \subseteq P \text{ and } I \nsubseteq P\}$.

We show that this generalization is different from the radical of an ideal with the following example.

Example 2.4. Let $I = 6\mathbb{Z}$ and $T = 10\mathbb{Z}$ be ideals of the ring of integers $R = \mathbb{Z}$. Then $N_I(T) = 5\mathbb{Z}$ but $\sqrt{T} = 10\mathbb{Z} = T$.

We also give some algebraic properties of the ideal $N_I(T)$ without proof as follows:

Lemma 2.5. *Let I be a proper ideal of a ring R. The following statements hold:*

(i) *For the ideal T of R, $N_I(T) = \sqrt{T}$ when $I = R$.*
(ii) *$N_I(T)$ is an ideal of R.*
(iii) *$N_{I/K}(T/K) = N_I(T)/K$, where $K \subseteq T$ is an ideal of R.*
(iv) *$N_I(0) = N_{\sqrt{I}}(0)$.*

We obtain a connection between topological property of $\mathfrak{X}_I$ and algebraic property of $N_I(0)$ as follows:

Theorem 2.6. *Let I be a proper ideal of a ring R and $\sqrt{I} \neq \sqrt{0}$. Then $\mathfrak{X}_I$ is irreducible if and only if $N_I(0)$ is a prime ideal of R.*

Proof. Let $N_I(0)$ be a prime ideal of R and K be a nonempty open subset of $\mathfrak{X}_I$. Then $K = \mathfrak{X}_I \backslash \tilde{V}(E) = Spec(R) \backslash (V(I) \cup V(E))$, where E is an ideal of R. Take $P \in K$. Then we have $P \notin V(E) \cup V(I)$, which means that $I \nsubseteq P$ and $E \nsubseteq P$. Thus $N_I(0) \subseteq P$, so $E \nsubseteq N_I(0) \subseteq P$. This implies that $N_I(0) \notin V(E)$ and by the definition of $N_I(0)$, we get $N_I(0) \notin V(I)$. Thus $N_I(0) \in K$. Therefore any nonempty open subset of $\mathfrak{X}_I$ contains $N_I(0)$. This means that $\mathfrak{X}_I$ is irreducible.

Let $\mathfrak{X}_I$ be irreducible. Suppose that $N_I(0)$ is not a prime ideal of R. Then there exist elements $a, b \in R$ such that $ab \in N_I(0)$ and $a, b \in R \backslash N_I(0)$.

Since $\sqrt{I} \neq \sqrt{0}$ and $a \in R \backslash N_I(0)$, it follows that $\tilde{V}(a) \neq \emptyset$ and $\tilde{V}(a) \neq \mathfrak{X}_I$, which implies $(\mathfrak{X}_I)^a \neq \emptyset$. By the same argument, $(\mathfrak{X}_I)^b$ is a nonempty open subset. Therefore, we get

$$\begin{aligned} (\mathfrak{X}_I)^a \cap (\mathfrak{X}_I)^b &= (\mathfrak{X}_I)^{ab} = \mathfrak{X}_I \backslash \tilde{V}(ab) \\ &\subseteq \mathfrak{X}_I \backslash \tilde{V}(N_I(0)) \\ &= Spec(R) \backslash (V(N_I(0)) \cup V(I)) = \emptyset. \end{aligned}$$

This contradicts with the hypothesis. Thus $N_I(0)$ is a prime ideal of R. □

As an application of Theorem 2.6, we can deduce that open subsets $\mathfrak{X}_{a\mathbb{Z}}$ of the Zariski topology on $Spec(\mathbb{Z})$ are not irreducible for any positive integer a since $N_{a\mathbb{Z}}(0)$ is not prime.

We find more connections between a topological space and a ring under the following condition.

A ring R is said to satisfy TN-condition for an ideal I, if for any chain $N_I(U_1) \subseteq N_I(U_2) \subseteq N_I(U_3) \subseteq ...$, there is a positive integer m such that $N_I(U_m) = N_I(U_{m+i})$ for all positive integers i.

Theorem 2.7. *Let I be a proper ideal of a ring R. Then the following statements are equivalent:*

(i) *R satisfies the TN-condition.*
(ii) *$\mathcal{X}_I$ is a Noetherian topological space.*

Proof. (i) $\Rightarrow$ (ii) Assume that R satisfies the TN-condition. Take the sequence $\tilde{V}(U_1) \supseteq \tilde{V}(U_2) \supseteq \tilde{V}(U_3) \supseteq ...$, where U_i is an ideal of R. Then we have the sequence $N_I(U_1) \subseteq N_I(U_2) \subseteq N_I(U_3) \subseteq ...$ and there exists an integer m such that $N_I(U_m) = N_I(U_{m+i})$ for all positive integers i since R satisfies the TN-condition. Therefore we have $\tilde{V}(U_m) = \tilde{V}(U_{m+i})$ for all positive integers i. Thus $\mathcal{X}_I$ is Noetherian.

(ii) $\Rightarrow$ (i) Let $\mathcal{X}_I$ be a Noetherian topological space. Take the sequence $N_I(U_1) \subseteq N_I(U_2) \subseteq N_I(U_3)...$, where U_i is an ideal of R. Then this yields the sequence $\tilde{V}(U_1) \supseteq \tilde{V}(U_2) \supseteq \tilde{V}(U_3) \supseteq ....$ Since $\mathcal{X}_I$ is Noetherian, there exits an integer m such that $\tilde{V}(U_m) = \tilde{V}(U_{m+i})$ for all positive integers i. This implies $N_I(U_m) = N_I(U_{m+i})$ for all positive integers i. Therefore R satisfies TN-condition. □

We close this section with the following theorem, which shows tight relationships between algebraic and topological properties.

Theorem 2.8. *Let R be a ring. Then the following are equivalent:*

(i) *$\mathcal{X}$ is a Noetherian topological space.*
(ii) *$\mathcal{X}_I$ is a Noetherian topological space for every ideal I of R.*
(iii) *R satisfies the TN-condition.*
(iv) *R satisfies ascending chain condition on the radical ideals of R.*

Proof. (i) $\Rightarrow$ (ii), (iv) $\Leftrightarrow$ (i) It is clear.

(ii) $\Rightarrow$ (i) Take the sequence $V(U_1) \supseteq V(U_2) \supseteq V(U_3) \supseteq ...$, where U_i is an ideal of R. Let $I = \cap U_i$ be an ideal of R. Consider the Zariski topology on $\mathcal{X}_I$. Then we have the sequence $\tilde{V}(U_1) \supseteq \tilde{V}(U_2) \supseteq \tilde{V}(U_3) \supseteq ....$ Since $\mathcal{X}_I$ is Noetherian, there exists an integer m such that $\tilde{V}(U_m) = \tilde{V}(U_{m+i})$ for all positive integers i. Thus we have $V(U_m) = V(U_{m+i})$ for all positive integers i. Thus $\mathcal{X}$ is Noetherian.

(ii) $\Leftrightarrow$ (iii) It is Theorem 2.7. □

3. The relationships between ideals and subspaces

In this section we find some algebraic and topological tools for ideals and some characterizations for rings.

Theorem 3.1. *Let I, J and K be proper ideals of a ring R. Then we have the following properties:*

(i) *Any open set of $\mathcal{X}$ is of the form $\mathcal{X}_I$.*
(ii) *$\mathcal{X}_I = \mathcal{X}_J$ if and only if $\sqrt{I} = \sqrt{J}$.*
(iii) *$\mathcal{X}_I \cap \mathcal{X}_J = \mathcal{X}_K$ if and only if $\sqrt{I} \cap \sqrt{J} = \sqrt{IJ} = \sqrt{K}$.*
(iv) *$\mathcal{X}_I \subseteq \mathcal{X}_J$ if and only if $\sqrt{I} \subseteq \sqrt{J}$.*

The proof of Theorem 3.1 is straightforward, hence we omit it. The following corollary is an immediate consequence of Theorem 3.1.

Corollary 3.2. *Let I and J be proper ideals of a ring R. Then $\mathcal{X}_I \cap \mathcal{X}_J = \emptyset$ if and only if $\sqrt{I} \cap \sqrt{J} = \sqrt{IJ} = \sqrt{0}$.*

Theorem 3.3. *Let I be a proper ideal of a ring R. Then $\mathcal{X}_I$ is dense in $\mathcal{X}$ if and only if $\sqrt{IJ} \neq \sqrt{0}$ for every proper ideal J not in $\sqrt{0}$.*

Proof. Let $\mathcal{X}_I$ be dense in $\mathcal{X}$ and let J be a proper ideal of R not in $\sqrt{0}$. Then $\mathcal{X}_J = Spec(R)\backslash V(J)$ is a nonempty open set in the Zariski topology and by the hypothesis, the intersection of $\mathcal{X}_I$ and $\mathcal{X}_J$ is nonempty. Then by Corollary 3.2, $\sqrt{IJ} \neq \sqrt{0}$.

Let $\sqrt{IJ} \neq \sqrt{0}$ for every proper ideal J of R not in $\sqrt{0}$. By Corollary 3.2, since $\mathcal{X}_I \cap \mathcal{X}_J \neq \emptyset$, it follows that $\mathcal{X}_I$ is dense in $\mathcal{X}$. □

The following theorem gives a characterization for the ring $R/\sqrt{0}$ by using topological properties.

Theorem 3.4. *Let R be a ring. The following statements are equivalent:*

(i) *$\sqrt{0}$ is a prime ideal of R.*
(ii) *$Spec(R)$ is irreducible.*
(iii) *Every ideal of $R/\sqrt{0}$ is essential.*
(iv) *Every open subset of $Spec(R)$ is dense.*

Proof. (i) $\Leftrightarrow$ (ii) This part is already a well known fact.

(iii) $\Rightarrow$ (iv) Let $\mathfrak{X}_I$ and $\mathfrak{X}_J$ be open subsets for ideals I, J. Then $\left(J+\sqrt{0}\right)/\sqrt{0}$ and $\left(I+\sqrt{0}\right)/\sqrt{0}$ are ideals of $R/\sqrt{0}$. Then

$$\begin{aligned}\sqrt{0} &\neq \sqrt{\left(J+\sqrt{0}\right)\cap\left(I+\sqrt{0}\right)} \\ &= \sqrt{\left(J+\sqrt{0}\right)\left(I+\sqrt{0}\right)} = \sqrt{\left(JI+\sqrt{0}\right)}\end{aligned}$$

and so $\sqrt{IJ}\neq\sqrt{0}$. This means that $\mathfrak{X}_I$ is dense.

(iv) $\Rightarrow$ (ii) $\Rightarrow$ (iii) By using the same argument, it is proved. □

Theorem 3.5. *Let I_i be a proper ideal of a ring R for all $i\in\Lambda$. Then $\bigcup\limits_{i\in\Lambda}\mathfrak{X}_{I_i}=\mathfrak{X}_D$ for any ideal D of R if and only if $\sqrt{D}=\sqrt{\sum\limits_{i\in\Lambda}I_i}$.*

Proof. Indeed, $\bigcup\limits_{i\in\Lambda}\mathfrak{X}_{I_i}=\mathfrak{X}_D\Leftrightarrow\bigcup\limits_{i\in\Lambda}(\mathfrak{X}\backslash V(I_i))=\mathfrak{X}_D\Leftrightarrow$

$\mathfrak{X}\backslash\left(\bigcap\limits_{i\in\Lambda}V(I_i)\right)=\mathfrak{X}\backslash V(D)\Leftrightarrow\bigcap\limits_{i\in\Lambda}V(I_i)=V(D)\Leftrightarrow$

$V\left(\bigcup\limits_{i\in\Lambda}I_i\right)=V(D)\Leftrightarrow V\left(\sum\limits_{i\in\Lambda}I_i\right)=V(D)\Leftrightarrow\sqrt{\sum\limits_{i\in\Lambda}I_i}=\sqrt{D}$. □

Theorem 3.6. *Let I_i be a proper ideal of a ring R for all $i\in\Lambda$ and let D be a finitely generated ideal of R. Then the following statements are equivalent:*

(i) $\bigcup\limits_{i\in\Lambda}\mathfrak{X}_{I_i}=\mathfrak{X}_D$.
(ii) *There is a finite subset Δ of Λ such that $\bigcup\limits_{i\in\Delta}\mathfrak{X}_{I_i}=\mathfrak{X}_D$.*
(iii) *There is a finite subset Δ of Λ such that $\sqrt{\sum\limits_{i\in\Delta}I_i}=\sqrt{D}$.*

Proof. (i) $\Rightarrow$ (iii) Let $\sqrt{\sum\limits_{i\in\Lambda}I_i}=\sqrt{D}$ and let D be an ideal generated by the set $\{d_1,...,d_t\}$. For each d_i, there is a positive number n_i such that $d_i^{n_i}\in\sum\limits_{i\in\Lambda}I_i$ and so there is a finite subset Δ_i of Λ such that $d_i^{n_i}\in\sum\limits_{i\in\Delta_i}I_i$. If $n=\max\{n_1,...n_t\}$ and $\Delta=\bigcup\limits_{i=1}^{n}\Delta_i$ then $\sqrt{\sum\limits_{i\in\Delta}I_i}=\sqrt{D}$.

(iii) $\Rightarrow$ (ii) Theorem 3.5.

(ii) $\Rightarrow$ (i) It is clear. □

The following corollary is a special case of Theorem 3.6.

Corollary 3.7. *Let I_i be a proper ideal of a ring R for all $i\in\Lambda$. Then the following statements are equivalent:*

(i) $\bigcup_{i\in\Lambda} \mathcal{X}_{I_i} = Spec(R)$.

(ii) *There is a finite subset* Δ *of* Λ *such that* $\bigcup_{i\in\Delta} \mathcal{X}_{I_i} = Spec(R)$.

(iii) *There is a finite subset* Δ *of* Λ *such that* $\sum_{i\in\Delta} I_i = R$.

We also state a generalization of a well known result as follows:

Corollary 3.8. *Let* D *be a finitely generated ideal of a ring* R. *Then* $\mathcal{X}_D$ *is quasi-compact.*

Using topological properties, we are now ready to prove a characterization of the nil-radical ideal $\sqrt{0}$.

Theorem 3.9. *Let* R *be a ring satisfying the* TN*-condition for every ideal. Then there are proper ideals* $I_1, ..., I_n$ *of* R *such that* $\sqrt{0} = \sqrt{I_1...I_n}$.

Proof. Let $\mathcal{X} = Spec(R)$ be a Noetherian topological space. By [5], $\mathcal{X}$ has only a finite number of distinct irreducible components U_i such that $\bigcup_{i=1}^{n} U_i = \mathcal{X}$. It is well known that any irreducible component in a topology space is closed and so for each i, there is an ideal I_i such that $U_i = V(I_i)$. Then

$$\emptyset = \mathcal{X}\backslash \bigcup_{i=1}^{n} V(I_i) = \bigcap_{i=1}^{n} (\mathcal{X}\backslash V(I_i)) = \bigcap_{i=1}^{n} \mathcal{X}_{I_i}.$$

Thus, by Theorem 3.1, $\sqrt{0} = \bigcap_{i=1}^{n} \sqrt{I_i} = \sqrt{I_1...I_n}$. □

By combining Theorem 2.6 and Theorem 3.5, we close this paper with the following theorem.

Theorem 3.10. *Let* I_i *be an ideal of a ring* R *for all* $i \in \{1, ..., n\}$. *Then* $\mathcal{X} = \bigcup_{i=1}^{n} \mathcal{X}_{I_i}$, *where* $\mathcal{X}_{I_i}$ *is irreducible if and only if* $R = \sum_{i=1}^{n} I_i$ *and* $N_{I_i}(0)$ *is a prime ideal of* R.

We think that Theorem 2.6 gives a lead to characterize the irreducible components of the Zariski topology on $Spec(R)$. We also wonder if the irreducible components of the Zariski topology on $Spec(R)$ are characterized by the ideal $N_I(0)$.

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