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Research Article

Qualitative analysis of solutions for a Kirchhoff-type parabolic equation with multiple nonlinearities

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Abstract

In this work, the local and global existence of weak solutions by using the Faedo-Galerkin method, the finite time blow up of the weak solution with positive initial energy and the general decay of the solution are discussed. Finally, we consider the exponential growth of the solution with sufficient conditions. This work generalizes and improves earlier results in the literature, see [L.X. Truong and N. Van Y, *On a class of nonlinear heat equations with viscoelastic term*, Comput. Math. Appl., 2016] and [L.X. Truong and N. Van Y, *Exponential growth with L^p-norm of solutions for nonlinear heat equations with viscoelastic term*, Appl. Math. Comput., 2016].

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1. Introduction

In the present work, we investigate the following model of the Kirchhoff-type nonlinear heat equation with viscoelastic term and nonlinear source term

$$(1+|u|^{q-2})u_t - M(\|\nabla u\|^2)\Delta u + \int_0^t g(t-s)\Delta u(s)ds = f(u), \quad \text{in } \Omega \times [0,\infty), \tag{1.1}$$

with boundary condition

$$u(x,t) = 0, \quad \text{on } \partial\Omega \times [0,\infty), \tag{1.2}$$

and initial condition

$$u(x,0) = u_0(x), \quad \text{in } \Omega, \tag{1.3}$$

where $\Omega \subset R^n$ $(n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, q is a positive constant, $M(s) = 1 + s^\gamma$, $(\gamma > 1$ and $s \geq 0)$. Also g and $f(u)$ will be specified later.

In the case $M \equiv 1$, Truong and Van Y [11] considered the following problem

$$(1+a|u|^{q-2})u_t - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = f(u). \tag{1.4}$$

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They showed the local and global existence of weak solutions by using the Faedo-Galerkin method. In addition, they proved the finite time blow-up and the decay of the weak solutions. Also, in [12], they investigated the result of exponentional growth of the weak solutions under the sufficient conditions of (1.4) with $f(u) = b\,|u|^{p-2}\,u$.

When $M \equiv 1$ and in the absence of $|u|^{q-2}\,u_t$ term, Eqs.(1.1) becomes the following problem

$$u_t - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = f\,(u)\,. \tag{1.5}$$

In [9], Messaoudi investigated the blow up of the weak solution for the problem (1.5) with $f(u) = b\,|u|^{p-2}\,u$. Also, the equation (1.5) without memory term ($g = 0$) has been discussed by some authors (see [1,2,5,7,10]).

In detail, the present work is organized as follows: In next section, we present some notations, assumption and lemmas. In section 3, the existence are discussed by using Faedo-Galerkin's method that is similiar to used in [11]. In section 4, we prove blow-up result for solutions with positive energy. In section 5, the exponential growth of solutions with positive initial energy is given. In last section, the decay property is studied.

2. Preliminaries

To state and prove our result, we need the following assumptions.

(**A1**) We suppose that q belongs to the interval $(2, q^*)$, where

$$q^* = \begin{cases} +\infty & \text{if } n = 1, 2, \\ \frac{2n}{n-2} & \text{if } n \geq 3. \end{cases}$$

(**A2**) The relaxation functions $g \in C^1$ is such that

$$g(0) \geq 0, \quad g'(s) \leq 0 \text{ and } 1 - \int_0^\infty g(s)ds = l > 0 \ \text{ for } s \geq 0.$$

(**A3**) The nonlinear source f is a continuous function on R. Further, there is a constant p such that

$$2 < p \leq q \quad \text{or} \quad q < p < 2 + \frac{2q}{n}, \tag{2.1}$$

and

$$|f(u)| \leq C_1 + C_2\,|u|^{p-1}\,, \quad \forall u \in R. \tag{2.2}$$

for some C_1, $C_2 > 0$ holds.

(**A4**) There is $p_1 \in (\sqrt{2p}, p)$ such that for all $m \in (0, p_1)$, there exists $\widehat{b} = \widehat{b}(m) \geq \frac{mb}{p}$ satisfying

$$f(u)u - (p-m)F(u) \geq \widehat{b}\,|u|^p\,, \ \forall u \in R.$$

(**A5**) $F(u) \leq \frac{b}{p}\,|u|^p\,, \forall u \in R$, where b is a positive constant and

$$2 < p \leq q \ \text{ or } q < p < 2 + \frac{2q}{n}.$$

(**A6**) $\int_0^\infty g(s)ds < \frac{\widehat{p}-2}{\widehat{p}-2+\widehat{p}^{-1}}$ where $\widehat{p} = \frac{p_1^2}{p}$.

Next, we introduce the energy function as follows

$$E\,(t) = \frac{1}{2}\left[(g \diamond \nabla u)(t) + \frac{1}{\gamma+1}(\|\nabla u\|^{2(\gamma+1)})\right] + \frac{1}{2}(1 - \int_0^t g(s)ds)\,\|\nabla u(t)\|^2 - \int_\Omega F\,(u)\,dx \tag{2.3}$$

where $(g \diamond w)(t) = \int_0^t g(t-s)\,\|w(s) - w(t)\|_2^2\,ds$ and $F(u) = \int_0^u f(\xi)d\xi$.

We shall give some notations and lemmas throughout this work.

$$\|.\|_2 = \|.\|_{L^2} \qquad \text{and} \qquad \|.\|_p = \|.\|_{L^p}\,.$$

Lemma 2.1 ([6]). *Suppose that* $2 \leq q < p < q^*$ *and*

$$\theta = \left(\frac{1}{q} - \frac{1}{p}\right)\left(\frac{1}{n} - \frac{1}{2} + \frac{1}{q}\right)^{-1}$$

hold. Then $\|w\|_p \leq C\|\nabla w\|_2^{\theta}\|w\|_q^{1-\theta}$ *for all* $w \in H_0^1(\Omega)$.

Lemma 2.2. *Let (A2) and (2.1) hold. Then,* $E(t)$ *is a nonincreasing function for on* $[0,\infty)$ *and*

$$E'(t) \leq 0.$$

Proof. Multiplying Eq.(1.1) by u_t, and integrating them over Ω, then summing, we have

$$E'(t) = -\left(\frac{1}{2}g(t)\|\nabla u\|^2 - \frac{1}{2}(g' \diamond \nabla u)(t) + \int_\Omega u_t^2 dx + \int_\Omega |u|^{q-2} u_t^2 dx\right) \leq 0. \tag{2.4}$$

□

3. Local existence and global existence

Firstly, we introduce the definition of weak solutions for the problem (1.1).

Definition 3.1. The function $u(x,t)$ is a weak solution of problem (1.1) on $\Omega \times [0,T)$ if

$$u \in L^\infty\left(0,T;W_0^{1,2(\gamma+1)}(\Omega)\right), \quad u_t \in L^2\left(0,T;L^2(\Omega)\right), \quad (|u|^{q/2})_t \in L^2\left(\Omega\times(0,T)\right),$$

satisfying $u(x,0) = u_0(x)$ and

$$\int_\Omega (u_t + |u|^{q-2} u_t)v dx + \int_\Omega M(\|\nabla u\|^2)\nabla u(t)\nabla v dx - \int_0^t g(t-s)\int_\Omega \nabla u(s)\nabla v dx ds$$
$$= \int_\Omega f(u)\, v dx,$$

for all $v \in L^2(0,T;W_0^{1,2(\gamma+1)}(\Omega))$ and for a.e. $t \in [0,T]$.

Lemma 3.2 ([11]). *Let* $2 < q < p < q^*$. *For each* $\varepsilon > 0$, *there exists a positive constant* C_ε *such that*

$$\|w\|_p^p \leq \varepsilon\|\nabla w\|_2^2 + C_\varepsilon\|w\|_q^{\alpha q},$$

for all $w \in H_0^1(\Omega) \cap L^q(\Omega)$, *where*

$$\alpha = \frac{2p(1-\theta)}{q(2-p\theta)}, \quad \theta = \left(\frac{1}{q} - \frac{1}{p}\right)\left(\frac{1}{n} - \frac{1}{2} + \frac{1}{q}\right)^{-1}.$$

Now, we shall find global existence and local existence in the following theorem that.

Theorem 3.3. *Let (A1), (A2) and (A3) hold. If*

$$2 < p \leq q,$$

then for each $u_0 \in W_0^{1,2(\gamma+1)}(\Omega)$, *the problem (1.1) has a global weak solution of class*

$$u \in L^\infty(0,T;W_0^{1,2(\gamma+1)}(\Omega)), \quad u_t \in L^2(0,T;L^2(\Omega)), \quad (|u|^{q/2})_t \in L^2(\Omega\times[0,T)),$$

for all $T > 0$. *If*

$$q < p < 2 + \frac{2q}{n},$$

then, for any $u_0 \in W_0^{1,2(\gamma+1)}(\Omega)$, *there is* $T_* > 0$ *such that the problem (1.1) has a weak solution of class*

$$u \in L^\infty(0,T;W_0^{1,2(\gamma+1)}(\Omega)), \quad u_t \in L^2(0,T;L^2(\Omega)), \quad (|u|^{q/2})_t \in L^2(\Omega\times[0,T)),$$

for all $T \in (0,T_*)$.

Proof. **Approximate solutions**

We establish the existence of weak solution of the problem (1.1) via the Faedo-Galerkin's method. Let $\{v_j\}_{j=1}^{\infty}$ forms be an orthonormal basis for $L^2(\Omega)$ as well as $W_0^{1,2(\gamma+1)}(\Omega)$. Let $W_k = span\{v_1, v_2, ..., v_k\}$ and u_{0k} be an element of W_k given by

$$u_{0k} = \sum_{j=1}^{k} \alpha_{kj} v_j \longrightarrow u_0 \text{ strongly in } W_0^{1,2(\gamma+1)}(\Omega), \text{ as } k \mapsto \infty. \tag{3.1}$$

We search for an approximate solution of problem (1.1) such that

$$u_k(t) = \sum_{j=1}^{k} c_{kj}(t) v_j, \tag{3.2}$$

where c_{kj} $(1 \leq j \leq k)$ satisfy the system of integro-differential equations

$$\begin{aligned} & \int_\Omega (1 + |u_k(t)|^{q-2}) u_k'(t) v_i dx + \int_\Omega M(\|\nabla u_k(t)\|^2) \nabla u_k(t) \nabla v_i dx \\ & - \int_\Omega \int_0^t g(t-s) \nabla u_k(s) \nabla v_i ds dx \\ = & \int_\Omega f(u_k(t)) v_i dx, \quad i = 1, 2, ..., k, \end{aligned} \tag{3.3}$$

where the initial conditions

$$c_{kj}(0) = \alpha_{kj}, \quad j \in \{1, 2, ..., k\}. \tag{3.4}$$

A priori estimate I

Multiplying (3.3) by $c_{ki}(t)$ and take the sum respect to i from 1 to k and then integrating on $[0, t]$. Afterwards we get

$$A_k(t) = A_k(0) + \int_0^t ds \int_0^s g(s-\tau) \int_\Omega \nabla u_k(\tau) \nabla u_k(s) dx d\tau + \int_0^t ds \int_\Omega f(u_k(s)) u_k(s) dx, \tag{3.5}$$

where

$$A_k(t) = \frac{1}{2} \|u_k(t)\|^2 + \frac{1}{q} \|u_k(t)\|_q^q + \int_0^t \left(\|\nabla u(s)\|^{2(\gamma+1)} + \|\nabla u(s)\|^2 \right) ds.$$

As applying the embedding $W^{1,2(\gamma+1)}(\Omega) \hookrightarrow L^p(\Omega)$, $p \geq 1$, it follows from (3.1) that

$$A_k(0) = \frac{1}{2} \|u_{0k}\|^2 + \frac{1}{q} \|u_{0k}\|_q^q \leq C_3, \tag{3.6}$$

for positive constant C_3 not dependent on k. To estimate the second term on right-hand side of (3.5), we use Hölder and Young inequality,

$$\begin{aligned} & \int_0^t ds \int_0^s g(s-\tau) \int_\Omega \nabla u_k(\tau) \nabla u_k(s) dx d\tau \\ \leq & \frac{1}{2} \int_0^t \|\nabla u_k(s)\|^2 ds + \frac{1}{2} \int_0^t \left(\int_0^s g(s-\tau) \|\nabla u_k(\tau)\|_2 d\tau \right)^2 ds \\ \leq & \frac{1}{2}(2-l) \int_0^t \|\nabla u_k(s)\|_2^2 ds. \end{aligned} \tag{3.7}$$

Firstly, we take into account the case

$$2 < p \leq q.$$

Then, by $(A3)$ and using the embedding $L^q(\Omega) \hookrightarrow L^p(\Omega)$, $p \le q$, we obtain

$$\begin{aligned}\int_0^t ds \int_\Omega f(u_k(s))u_k(s)dx &\le \int_0^t \int_\Omega \left(C_1 + C_2 |u_k(s)|^{p-1}\right)|u_k(s)|\, dxds \\ &\le C_4 \int_0^t A_k(s)ds + C_5, \end{aligned} \tag{3.8}$$

where C_4, $C_5 > 0$, not dependent of k.

Combining (3.5)-(3.6), (3.7)-(3.8) and using the Gronwall-Bellman-Bihari type inequality (see [3], Theorem 21), we obtain

$$A_k(t) \le C_T, \quad \text{for any } t \in [0, T], \text{ for all } k \in N,$$

for all $T > 0$.

Now, we consider the case

$$q < p < 2 + \frac{2q}{n}.$$

In the same way, from the assumption $(A3)$ and Lemma 3.2, we get

$$\int_0^t ds \int_\Omega f(u_k(s))u_k(s)dx \le \varepsilon \int_0^t \|\nabla u_k(s)\|^2 ds + C_6 \int_0^t A_k^\alpha(s)ds + C_7, \tag{3.9}$$

where $\alpha = \frac{2p(1-\theta)}{q(2-p\theta)} > 1$, and C_i $(i = 6, 7)$ are positive constants that independent on k. Thus, if we pick ε to be sufficiently small, it follows from (3.5), (3.6), (3.7), (3.9) that

$$A_k(t) \le C_8 + C_9 \int_0^t A_k^\alpha(s)ds,$$

where C_8 and C_9 are positive constants that independent on k. Therefore, applying the Gronwall-Bellman-Bihari type inequality, there exists a positive constant $T_* > 0$ such that, for all $T \in (0, T_*)$,

$$A_k(t) \le C_{T_*}, \ \forall t \in [0, T], \ \forall k \in N.$$

So, in all the cases considered above, we obtained estimates of the form

$$A_k(t) \le \text{constant independent of } k, \ \forall t \in [0, T]; \tag{3.10}$$

furthermore, if $2 < p \le q$ then the quantity $T > 0$ is arbitrarily large and fixed, whereas if

$$q < p < 2 + \frac{2q}{n},$$

the quantity $T > 0$ is rather small ($T \in (0, T_*)$, for some $T_* > 0$).

A priori estimate II

Now, multiplying the equation of (3.3) by $c'_{ki}(t)$, summing with respect to i and integrating over $(0, t)$, we get

$$\begin{aligned} &\int_0^t \|u_{ks}\|^2 ds + \int_0^t \int_\Omega |u_k|^{q-2} u_{ks}^2 dxds + \psi_k(t) \\ = \ & \psi_k(0) - \frac{1}{2}\int_0^t g(s) \|\nabla u_k(s)\|^2 ds \\ &+ \frac{1}{2}\int_0^t (g' \diamond \nabla u_k)(s)ds - \int_\Omega F(u_{0k})dx + \int_\Omega F(u_k)dx, \end{aligned} \tag{3.11}$$

where

$$\begin{aligned}\psi_k(t) = \ & \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u_k(t)\|^2 + \frac{1}{2}(g \diamond \nabla u_k)(t) \\ &+ \frac{1}{2(\gamma+1)} \|\nabla u_k(t)\|^{2(\gamma+1)}.\end{aligned}$$

By using assumptions, Lemma 3.2 and the estimates (3.6), (3.10) we arrive at

$$\begin{aligned} &\int_0^t \|u_{ks}\|^2 ds + \int_0^t \int_\Omega |u_k|^{q-2} u_{ks}^2 dxds + \frac{l}{2} \|\nabla u_k(t)\|^2 \\ \leq\ &\text{constant independent of } k,\ \forall t \in [0,T]; \end{aligned} \quad (3.12)$$

where $T > 0$ is arbitrary large if $2 < p \leq q$ and $T \in (0, T_*)$ if

$$q < p < 2 + \frac{2q}{n}.$$

Passing to the limit

Let $T > 0$ be an arbitrarily number if $2 < p \leq q$; or $T \in (0, T_*)$ if

$$q < p < 2 + \frac{2q}{n}.$$

From priori estimate (I) - (II) (see (3.10) and (3.12)), there exists function u and a subsequences of $\{u_k\}_{k=1}^\infty$ that we still denote by $\{u_k\}_{k=1}^\infty$ such that, as $k \to \infty$,

$$\begin{aligned} u_k &\to u \quad \text{weakly star in} \quad L^\infty(0,T; W_0^{1,2(\gamma+1)}(\Omega)), && (3.13)\\ u_{kt} &\to u_t \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)), && (3.14)\\ (|u_k|^{q-2})_t &\to \chi \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)). && (3.15) \end{aligned}$$

Since the sequence $\{u_k\}$ is bounded in

$$W = \left\{ v \in L^\infty(0,T; W_0^{1,2(\gamma+1)}(\Omega)) : v' \in L^2(0,T;L^2(\Omega)) \right\}$$

uniformly in k, it follows from Aubin-Lions compactness theorem that

$$u_k \to u \text{ strongly in } C(0,T;L^r(\Omega)),\ \forall r \in [1, q^*).$$

Therefore, we have u_k strongly converges to u in $L^q(\Omega \times (0,T))$ and so, by Krasnoselskii's theorem for the Nemytskii operator we have

$$|u_k|^{q/2-2} u_k \longrightarrow |u|^{q/2-2} u \text{ strongly in } L^{2q/(q-2)}(\Omega \times (0,T)). \quad (3.16)$$

Similarly, we also have $|u_k|^{q/2}$ strongly converges to $|u|^{q/2}$ in $L^2(\Omega \times (0,T))$. Therefore,

$$(|u_k|^{q/2})_t \longrightarrow (|u|^{q/2})_t \text{ weakly}^* \text{ in } D'(\Omega \times (0,T)) \quad (3.17)$$

where $D'(\Omega \times (0,T))$ is space of distributions on $\Omega \times (0,T)$.

By (3.15) and (3.17) we deduce that

$$\chi = (|u|^{q/2})_t \ \text{ and } \ (|u_k|^{q/2})_t \longrightarrow (|u|^{q/2})_t \ \text{ weakly in } L^2(\Omega \times (0,T)). \quad (3.18)$$

Combining (3.16) and (3.18) we get

$$|u_k|^{q-2} u_{kt} = \frac{2}{q} |u_k|^{q/2-2} u_k (|u_k|^{q/2})_t \longrightarrow |u|^{q-2} u_t \quad \text{weakly in } L^2(\Omega \times (0,T)). \quad (3.19)$$

On the other hand, by the continuity of function f, it is obvious that

$$f(u_k) \longrightarrow f(u) \text{ for all } (x,t) \in (\Omega \times (0,T)) \quad (3.20)$$

Conversely, we have

$$\int_0^T \int_\Omega |f(u_k)|^{\frac{p}{p-1}} dxds \leq \int_0^T \int_\Omega (C_1 + C_2 |u_k|^{p-1})^{p/(p-1)} dxds \leq C_{10}, \quad (3.21)$$

for all k. By Lions's lemma (see [4]), it follows from (3.20) and (3.21) that

$$f(u_k) \longrightarrow f(u) \quad \text{weakly in } L^{\frac{p}{p-1}}(\Omega \times (0,T)). \quad (3.22)$$

By (3.13)-(3.15), (3.18), (3.19) and (3.22), passing to the limit as $k \longrightarrow \infty$ in (3.3) and (3.4), we obtain that $u(x,0)=u_0$,

$$\begin{aligned}&\int_\Omega (1+|u(t)|^{q-2})u'(t)vdx+\int_\Omega M(\|\nabla u(t)\|^2)\nabla u(t)\nabla vdx\\&-\int_0^t g(t-s)\int_\Omega \nabla u(s)\nabla vdxds\\=&\int_\Omega f\left(u(t)\right)vdx \end{aligned}\tag{3.23}$$

for all $v\in W_0^{1,2(\gamma+1)}(\Omega)$ and for any $t\in[0,T]$.

This completes the proof. □

4. Blow up of solutions

In this section, we deal with the blow up results of the solution for the problem (1.1). In order to obtain the result of blow up of solutions, we assume that following assumption:

$$p\in(q,2+\frac{2q}{n}).$$

We denote by C_ρ the best embedding constant for $W_0^{1,2(\gamma+1)}(\Omega)\hookrightarrow L^\rho(\Omega)$ and $C_{\rho,1}=\frac{C_\rho}{\sqrt{l}}$. We put

$$\alpha_1=C_{p,1}^{-p/(p-2)}b^{-1/(p-2)},\qquad E_1=(\frac{1}{2}-\frac{1}{p})\alpha_1^2,\quad E_2=(\frac{1}{p_1}-\frac{1}{p})\alpha_1^2.$$

The following lemma will play an essential role in the proof of our main result in this section. It is similar to a lemma used firstly by Vitillaro ([13]). Also, the following lemma's proof can be easily established by the Lemma 3.6 in [11]. Therefore, we omit it.

Lemma 4.1. *Suppose that assumptions (A1), (A2) and (A5) hold. Moreover, assume that $E(0)<E_1$ and u is a weak solution of problem (1.1)*

(i) If $\|\nabla u_0\|^2<\alpha_1$ then there exists $\tilde{\alpha}_1\in[0,\alpha_1]$ such that

$$\left((1-\int_0^t g(s)ds)\,\|\nabla u(t)\|^2+(g\diamond\nabla u)(t)+\frac{1}{\gamma+1}\,\|\nabla u(t)\|^{2(\gamma+1)}\right)^{\frac{1}{2}}<\tilde{\alpha}_1,\ \forall t\in[0,T).\tag{4.1}$$

(ii) If $\|\nabla u_0\|^2>\alpha_1$ then there exists a constant $\widehat{\alpha}_1>\alpha_1$ such that

$$\left((1-\int_0^t g(s)ds)\,\|\nabla u(t)\|^2+(g\diamond\nabla u)(t)+\frac{1}{\gamma+1}\,\|\nabla u(t)\|^{2(\gamma+1)}\right)^{\frac{1}{2}}\geq\widehat{\alpha}_1,\tag{4.2}$$

and

$$\|u\|_p\geq C_{p,1}\widehat{\alpha}_1,\tag{4.3}$$

for all $t\in[0,T)$.

Theorem 4.2. *Suppose that (A1), (A2), (A4), (A5) and (A6) hold. Let $u_0\in W_0^{1,2(\gamma+1)}(\Omega)$ with*

$$0\leq E(0)<E_2,\qquad\qquad \|\nabla u_0\|>\alpha_1.$$

Then the solution u of the problem (1.1) blows up in finite time.

Proof. For this purpose, we set

$$H(t)=E_2-E(t).\tag{4.4}$$

By the definition of $H\left(t\right)$ and since $E'(t)\leq 0$, we get

$$H'\left(t\right)=-E'\left(t\right)\geq 0.$$

Conversely, by using (2.3) and Lemma 4.1 follows that

$$\begin{aligned} 0 &< H(0) \le H(t) = E_2 - \frac{1}{2}\left[(1 - \int_0^t g(s)ds)\,\|\nabla u(t)\|^2 + (g \diamond \nabla u)(t)\right] \\ &\quad -\frac{1}{2(\gamma+1)}(\|\nabla u\|^{2(\gamma+1)}) + \int_\Omega F(u)\,dx \\ &\le E_2 - \frac{1}{2}\widehat{\alpha}_1^2 + \int_\Omega F(u)\,dx \\ &\le E_1 - \frac{1}{2}\alpha_1^2 + \int_\Omega F(u)\,dx \\ &= -\frac{1}{p}\alpha_1^2 + \int_\Omega F(u)\,dx. \end{aligned} \tag{4.5}$$

Therefore, combining $(A5)$ and (4.5), we have

$$0 < H(0) \le H(t) \le \int_\Omega F(u)\,dx \le \frac{b}{p}\|u(t)\|_p^p. \tag{4.6}$$

Let's define the functional

$$\Psi(t) = H^{1-\sigma}(t) + \frac{\varepsilon}{2}\int_\Omega |u|^2\,dx, \tag{4.7}$$

where ε small to be chosen later and

$$0 < \sigma \le 1 - \frac{q}{p}.$$

By taking a derivative of (4.7) with respect to t and using Eq.(1.1), we have

$$\begin{aligned} \Psi'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\int_\Omega u_t u\,dx \\ &= (1-\sigma)H^{-\sigma}(t)H'(t) - \varepsilon\int_\Omega |u|^{q-2}uu_t\,dx - \varepsilon\|\nabla u\|^2 - \varepsilon\|\nabla u\|^{2(\gamma+1)} \\ &\quad +\varepsilon\int_0^t g(t-s)\int_\Omega \nabla u(s)\nabla u(t)\,dx\,ds + \varepsilon\int_\Omega uf(u)\,dx. \end{aligned} \tag{4.8}$$

By the assumption $(A4)$, we get

$$\begin{aligned} \Psi'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) - \varepsilon\int_\Omega |u|^{q-2}uu_t\,dx - \varepsilon\|\nabla u\|^2 - \varepsilon\|\nabla u\|^{2(\gamma+1)} \\ &\quad +\varepsilon(p-m)\int_\Omega F(u)\,dx + \varepsilon\widehat{b}\,\|u(t)\|_p^p + \varepsilon\left(\int_0^t g(s)ds\right)\|\nabla u\|^2 \\ &\quad -\varepsilon\int_0^t g(t-s)\int_\Omega |\nabla u(s) - \nabla u(t)|\,|\nabla u(t)|\,dx\,ds. \end{aligned} \tag{4.9}$$

Furthermore, we use Young's inequality to estimate last term of right-hand of (4.9), we obtain

$$\begin{aligned} \Psi'(t) &\ge (1-\sigma)H^{-\sigma}(t)H'(t) - \varepsilon\int_\Omega |u|^{q-2}uu_t\,dx - \varepsilon\|\nabla u\|^{2(\gamma+1)} \\ &\quad +\varepsilon(p-m)\int_\Omega F(u)\,dx + \varepsilon\widehat{b}\,\|u(t)\|_p^p - \varepsilon\delta(g \diamond \nabla u)(t) \\ &\quad -\varepsilon\left[1 - \left(1 - \frac{1}{4\delta}\right)\int_0^t g(s)ds\right]\|\nabla u\|^2, \end{aligned} \tag{4.10}$$

where $\delta > 0$. Then, using definition $H'(t)$ and Young's inequality to estimate the second terms in (4.10), we get

$$\int_\Omega |u|^{q-2} uu_t dx \leq \delta_1 \int_\Omega |u|^q dx + \frac{1}{\delta_1}\int_\Omega |u|^{q-2} u_t^2 dx \leq \delta_1 \|u(t)\|_q^q + \frac{1}{\delta_1} H'(t), \tag{4.11}$$

for all $\delta_1 > 0$. Moreover, by using (4.6) and definition of $H(t)$, we get

$$\|u(t)\|_p^p \geq \frac{p}{b}\int_\Omega F(u)dx = \frac{p}{b}\left\{H(t) + \frac{1}{2}\left[(g \diamond \nabla u)(t) + \frac{1}{\gamma+1}\|\nabla u\|^{2(\gamma+1)}\right]\right. \tag{4.12}$$

$$= \left. +\frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|^2 - E_2\right\}. \tag{4.13}$$

On the other hand,

$$\varepsilon\widehat{b}\|u(t)\|_p^p = \varepsilon\widehat{b}\left[\frac{p-p_1}{p}\|u(t)\|_p^p + \frac{p_1}{p}\|u(t)\|_p^p\right]$$
$$\geq \varepsilon\widehat{b}\left\{\frac{p-p_1}{p}\|u(t)\|_p^p - \frac{p_1}{p}\frac{p}{b}E_2 + \frac{p_1}{p}\left[\frac{p}{b}H(t) + \frac{p}{2b}(g\diamond\nabla u)(t)\right.\right.$$
$$\left.\left. + \frac{p}{2b}(1 - \int_0^t g(s)ds)\|\nabla u(t)\|^2 + \frac{p}{2b(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}\right]\right\}. \tag{4.14}$$

and from (4.3), we have

$$\frac{p-p_1}{p}\|u(t)\|_p^p - \frac{p_1}{p}\frac{p}{b}E_2 \geq \left[\frac{p-p_1}{p} - p_1\frac{1}{b}E_2(C_{p,1}\widehat{\alpha}_1)^{-p}\right]\|u(t)\|_p^p. \tag{4.15}$$

Combining (4.14) and (4.15), we have

$$\varepsilon\widehat{b}\|u(t)\|_p^p \geq \varepsilon\widehat{b}\left\{K_0\|u(t)\|_p^p + \frac{p_1}{p}\left[\frac{p}{b}H(t) + \frac{p}{2b}(g\diamond\nabla u)(t)\right.\right.$$
$$\left.\left. + \frac{p}{2b}(1 - \int_0^t g(s)ds)\|\nabla u(t)\|^2 + \frac{p}{2b(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}\right]\right\} \tag{4.16}$$

since $\frac{1}{b}E_2(C_{p,1}\widehat{\alpha}_1)^{-p} < \frac{1}{b}E_2(C_{p,1}\alpha_1)^{-p} = \frac{E_2}{\alpha_1^2} = \frac{p-p_1}{p_1 p}$, $K_0 = \frac{p-p_1}{p} - p_1\frac{1}{b}E_2(C_{p,1}\widehat{\alpha}_1)^{-p} > 0$.

Now combining (4.10), (4.11), (4.16) and let $\delta_1 = K_1^{-1}H^\sigma(t)$, we get

$$\Psi'(t) \geq (1-\sigma-K_1\varepsilon)H^{-\sigma}(t)H'(t) - \varepsilon K_1^{-1}H^\sigma(t)\|u(t)\|_q^q$$
$$+\varepsilon\left(\lambda_m\frac{p_1}{2(\gamma+1)} - 1\right)(\|\nabla u\|^{2(\gamma+1)})$$
$$+\varepsilon(p - m + \lambda_m p_1)H(t) + \varepsilon\widehat{b}K_0\|u(t)\|_p^p + \varepsilon\left(\lambda_m\frac{p_1}{2} - \delta\right)(g\diamond\nabla u)(t)$$
$$+\varepsilon\left\{\lambda_m\frac{p_1}{2}\left(1 - \int_0^t g(s)ds\right) - \left[1 - \left(1 - \frac{1}{4\delta}\right)\int_0^t g(s)ds\right]\right\}\|\nabla u(t)\|^2, \tag{4.17}$$

where $\lambda_m = \frac{\widehat{b(m)}}{b}$ and $K_1 > 0$.

Thanks to (4.6) and embedding $L^p(\Omega) \hookrightarrow L^q(\Omega)$, $1 \leq q < p$, we obtain

$$H^\sigma(t)\|u(t)\|_q^q \leq K_2\|u(t)\|_p^{\sigma p+q}, \tag{4.18}$$

where $K_2 > 0$ is a constant. Since $q < p$ and $0 < \sigma < 1 - q/p$, now applying the following algebraic inequality

$$x^l \leq (x+1) \leq (1+\frac{1}{z})(x+z), \quad \forall x \geq 0,\ 0 < l < 1,\ z > 0,$$

especially, taking $x = \|u(t)\|_p^p$, $l = (p\sigma + q)/p$, $z = H(0)$, we get

$$\begin{aligned} \|u(t)\|_p^{p\sigma+q} &\leq \left(1+\frac{1}{H(0)}\right)(\|u(t)\|_p^p + H(0)) \\ &\leq \left(1+\frac{1}{H(0)}\right)(\|u(t)\|_p^p + H(t)). \end{aligned} \tag{4.19}$$

Inserting (4.18) and (4.19) into (4.17), we conclude that

$$\begin{aligned} \Psi'(t) \geq\ & (1-\sigma-K_1\varepsilon)H^{-\sigma}(t)\,H'(t) + \varepsilon\left[p-m+\lambda_m p_1 - K_1^{-1}K_2\left(1+\frac{1}{H(0)}\right)\right]H(t) \\ &+\varepsilon\left(\lambda_m\frac{p_1}{2(\gamma+1)}-1\right)(\|\nabla u\|^{2(\gamma+1)}) + \varepsilon\left(\lambda_m\frac{p_1}{2}-\delta\right)(g\diamond\nabla u)(t) \\ &+\varepsilon\left[\widehat{b}K_0 - K_1^{-1}K_2\left(1+\frac{1}{H(0)}\right)\right]\|u(t)\|_p^p \\ &+\varepsilon\left\{\lambda_m\frac{p_1}{2}\left(1-\int_0^t g(s)ds\right) - \left[1-\left(1-\frac{1}{4\delta}\right)\int_0^t g(s)ds\right]\right\}\|\nabla u(t)\|^2. \end{aligned} \tag{4.20}$$

At this point, we point out that the estimate (4.20) holds for all $m \in (0, p_1]$, K_1 is large number and for all δ, $\varepsilon > 0$. By using the assumption $(A6)$, we can choose m such that

$$\int_0^t g(s)ds < \frac{\lambda_m p_1 - 2}{\lambda_m p_1 - 2 + \lambda_m^{-1}p_1^{-1}}.$$

Then, we pick

$$\lambda_m\frac{p_1}{2} > \max\{\delta,\ \gamma+1\}$$

such that

$$\lambda_m - \frac{2}{p_1}\frac{1-\left(1-\frac{1}{4\delta}\right)\int_0^\infty g(s)ds}{1-\int_0^\infty g(s)ds)} > 0.$$

Then, we pick K_1 such that

$$K_1 > \max\left\{\frac{K_2(1+\frac{1}{H(0)})}{p-m+\lambda_m p_1},\ \frac{K_2(1+\frac{1}{H(0)})}{\widehat{b}K_0}\right\},$$

and then pick $\varepsilon \in \left(0, \frac{1-\sigma}{K_1}\right)$.

So (4.20) become

$$\Psi'(t) \geq C(\|u(t)\|_p^p + H(t)), \tag{4.21}$$

for

$$C = \varepsilon\min\left\{p-m+\lambda_m p_1 - K_1^{-1}K_2\left(1+\frac{1}{H(0)}\right),\ \ \widehat{b}K_0 - K_1^{-1}K_2\left(1+\frac{1}{H(0)}\right)\right\}.$$

Consequently, we arrive at

$$\Psi(t) \geq \Psi(0) > 0,\ \forall t \geq 0.$$

Now, we estimate $\Psi(t)^{\frac{1}{1-\sigma}}$

$$\begin{aligned} \Psi^{\frac{1}{1-\sigma}}(t) &= \left(H^{1-\sigma}(t) + \frac{\varepsilon}{2}\|u(t)\|_2^2\right)^{\frac{1}{1-\sigma}} \\ &\leq C(H(t) + \|u(t)\|_p^{\frac{2}{1-\sigma}}). \end{aligned} \tag{4.22}$$

We again apply above algebraic inequality, we get

$$\|u(t)\|_p^{\frac{2}{1-\sigma}} \leq \left(1+\frac{1}{H(0)}\right)(\|u(t)\|_p^p + H(t)). \tag{4.23}$$

By combining of (4.21), (4.22) and (4.23) we arrive at

$$\Psi'(t) \geq \xi \Psi^{\frac{1}{1-\sigma}}(t), \tag{4.24}$$

where ξ is a positive constant.

A simple integration of (4.24) over $(0,t)$ yields $\Psi^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{\Psi^{-\frac{\sigma}{1-\sigma}}(0) - \frac{\xi\sigma t}{1-\sigma}}$, which implies that the solution blows up in a finite time T^*, with

$$T^* \leq \frac{1-\sigma}{\xi\sigma\Psi^{\frac{\sigma}{1-\sigma}}(0)}.$$

□

5. Exponential growth of solutions

In this section, we estimate that the energy will grow up as an exponential function as time as goes to infinity.

Theorem 5.1. *Suppose that $f(u) = b|u|^{p-2}u$, (A2), (A6) hold. Let $u_0 \in W_0^{1,2(\gamma+1)}(\Omega)$ with*

$$E(0) < E_2, \qquad \|\nabla u_0\| > \alpha_1.$$

Then there exist positive constant C and ϱ such that

$$\|u(t)\|_p \geq C\exp(\varrho t),$$

for all $t \in [0,T)$.

Proof. Let

$$L(t) = H(t) + \frac{\varepsilon}{2}\int_\Omega u^2(x,t)dx, \tag{5.1}$$

where $H(t) = E_2 - E(t)$ and ε small to be chosen later. By taking the time derivative of (5.1) and taking into account $E'(t)$, we obtain

$$\begin{aligned} L'(t) &= H'(t) + \varepsilon\int_\Omega uu_t dx \\ &= -E'(t) + \varepsilon\int_\Omega uu_t dx \\ &= \frac{1}{2}g(t)\|\nabla u\|^2 - \frac{1}{2}(g' \diamond \nabla u)(t) + \int_\Omega u_t^2 dx + \int_\Omega |u|^{q-2}u_t^2 dx + \varepsilon\int_\Omega uu_t dx. \end{aligned} \tag{5.2}$$

By using Eq.(1.1), we get

$$\begin{aligned} \int_\Omega uu_t dx &= \int_\Omega uf(u)dx - \int_\Omega |u|^{q-2}uu_t dx - \|\nabla u(t)\|^{2(\gamma+1)} \\ &\quad - \|\nabla u(t)\|^2 + \int_0^t g(t-s)\int_\Omega \nabla u(s)\nabla u(t)dxds \\ &\geq b\|u\|_p^p - \int_\Omega |u|^{q-2}uu_t dx - \|\nabla u(t)\|^{2(\gamma+1)} \\ &\quad - \|\nabla u(t)\|^2 + \int_0^t g(t-s)\|\nabla u(t)\|^2 ds \\ &\quad - \int_0^t g(t-s)\int_\Omega |\nabla u(s) - \nabla u(t)||\nabla u(t)|dxds. \end{aligned} \tag{5.3}$$

In order to estimate $\int_0^t g(t-s)\int_\Omega |\nabla u(s)-\nabla u(t)|\,|\nabla u(t)|\,dxds$ term, using the Schwarz inequality, we obtain

$$\begin{aligned}\int_\Omega uu_t dx &\geq b\,\|u(t)\|_p^p-\left(1-\int_0^t g(t-s)ds\right)\|\nabla u\,(t)\|^2-\|\nabla u(t)\|^{2(\gamma+1)}\\ &\quad-\int_\Omega |u|^{q-2}\,uu_t dx-\int_0^t g(t-s)\,\|\nabla u(t)\|_2\,\|\nabla u(s)-\nabla u(t)\|_2\,ds. \quad (5.4)\end{aligned}$$

Then, using Young's inequality to the last term of (5.4), we deduce that

$$\begin{aligned}\int_\Omega uu_t dx &\geq b\,\|u(t)\|_p^p-\left[1-\left(1-\frac{1}{4\delta}\right)\int_0^t g(t-s)ds\right]\|\nabla u\,(t)\|^2-\|\nabla u(t)\|^{2(\gamma+1)}\\ &\quad-\delta(g\diamond\nabla u)(t)-\int_\Omega |u|^{q-2}\,uu_t dx, \quad (5.5)\end{aligned}$$

for all $\delta>0$. Insert (5.5) into (5.2), we arrive at

$$\begin{aligned}L'(t) &\geq \int_\Omega u_t^2 dx+\int_\Omega |u|^{q-2}\,u_t^2 dx\\ &\quad+\varepsilon\left[b\,\|u(t)\|_p^p-\left[1-\left(1-\frac{1}{4\delta}\right)\int_0^t g(t-s)ds\right]\|\nabla u\,(t)\|^2\right.\\ &\quad\left.-\|\nabla u(t)\|^{2(\gamma+1)}-\delta(g\diamond\nabla u)(t)-\int_\Omega |u|^{q-2}\,uu_t dx\right]. \quad (5.6)\end{aligned}$$

In order to estimate $\int_\Omega |u|^{q-2}\,uu_t dx$ term, applying Young's inequality and the embedding $L^p(\Omega)\hookrightarrow L^q(\Omega)$, we get

$$\begin{aligned}\int_\Omega |u|^{q-2}\,uu_t dx &\leq \alpha\int_\Omega |u(t)|^q\,dx+\frac{1}{\alpha}\int_\Omega |u|^{q-2}\,u_t^2 dx\\ &\leq \alpha C\left(\|u(t)\|_p^p\right)^{q/p}+\frac{1}{\alpha}\int_\Omega |u|^{q-2}\,u_t^2 dx, \quad (5.7)\end{aligned}$$

for positive embedding constant C. Now, again apply the above algebraic inequality and we have used the fact that $0<q/p<1$, we obtain

$$\left(\|u(t)\|_p^p\right)^{q/p}\leq\left(1+\frac{1}{H(0)}\right)(\|u(t)\|_p^p+H(0)). \quad (5.8)$$

Therefore, by (5.7) and (5.8), we get

$$\int_\Omega |u|^{q-2}\,uu_t dx\leq\alpha C\left(1+\frac{1}{H(0)}\right)\left(1+\frac{b}{p}\right)\|u(t)\|_p^p+\frac{1}{\alpha}\int_\Omega |u|^{q-2}\,u_t^2 dx. \quad (5.9)$$

Substituting (5.9) into (5.6), we have

$$\begin{aligned}L'(t) &\geq \int_\Omega u_t^2 dx+\left(1-\frac{\varepsilon}{\alpha}\right)\int_\Omega |u|^{q-2}\,u_t^2 dx\\ &\quad+\varepsilon(b-\alpha c')\,\|u(t)\|_p^p-\varepsilon\delta(g\diamond\nabla u)(t)\\ &\quad-\varepsilon\left[1-\left(1-\frac{1}{4\delta}\right)\int_0^t g(t-s)ds\right]\|\nabla u\|^2-\varepsilon\,\|\nabla u(t)\|^{2(\gamma+1)}, \quad (5.10)\end{aligned}$$

where

$$c'=C\left(1+\frac{1}{H(0)}\right)\left(1+\frac{b}{p}\right).$$

By using the relation

$$\|u(t)\|_p^p=\frac{p}{b}\left[\begin{array}{c} H(t)+\frac{1}{2}(g\diamond\nabla u)(t)+\frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}\\ +\frac{1}{2}(1-\int_0^t g(s)ds)\,\|\nabla u(t)\|^2-E_2\end{array}\right],$$

we obtain

$$\begin{aligned}
& \varepsilon(b-\alpha c')\left\|u(t)\right\|_p^p \\
= \; & \varepsilon(b-\alpha c')\left(\frac{p-p_1}{p}\left\|u(t)\right\|_p^p+\frac{p_1}{p}\left\|u(t)\right\|_p^p\right) \\
= \; & \varepsilon(b-\alpha c')\left\{\frac{p-p_1}{p}\left\|u(t)\right\|_p^p-\frac{p_1}{p}.\frac{p}{b}E_2+\frac{p_1}{p}\left[\frac{p}{b}H(t)+\frac{p}{2b}(g\diamond\nabla u)(t)\right.\right. \\
& \left.\left.+\frac{p}{2b}\left(1-\int_0^t g(s)ds\right)\left\|\nabla u(t)\right\|^2+\frac{p}{2b(\gamma+1)}\left\|\nabla u\right\|^{2(\gamma+1)}\right]\right\}. \qquad (5.11)
\end{aligned}$$

Further, it follows from (4.3) that

$$\frac{p-p_1}{p}\left\|u(t)\right\|_p^p-\frac{p_1}{p}.\frac{p}{b}E_2 \geq \left[\frac{p-p_1}{p}-p_1\frac{1}{b}E_2(C_{p,1}\widehat{\alpha}_1)^{-p}\right]\left\|u(t)\right\|_p^p \geq 0, \qquad (5.12)$$

since

$$\frac{1}{b}E_2(C_{p,1}\widehat{\alpha}_1)^{-p}<\frac{1}{b}E_2(C_{p,1}\alpha_1)^{-p}=\frac{E_2}{\alpha_1^2}=\frac{p-p_1}{p_1p}.$$

From (5.11) and (5.12), we deduce that

$$\begin{aligned}
\varepsilon(b-\alpha c')\left\|u(t)\right\|_p^p \geq \; & \varepsilon\left(1-\frac{\alpha c'}{b}\right).\frac{p_1}{p}\left[pH(t)+\frac{p}{2}(g\diamond\nabla u)(t)\right. \\
& \left.+\frac{p}{2}(1-\int_0^t g(s)ds)\left\|\nabla u(t)\right\|^2+\frac{p}{2(\gamma+1)}\left\|\nabla u\right\|^{2(\gamma+1)}\right]. \qquad (5.13)
\end{aligned}$$

Combining (5.10) and (5.13), we obtain

$$\begin{aligned}
L'(t) \geq \; & \int_\Omega u_t^2dx+\left(1-\frac{\varepsilon}{\alpha}\right)\int_\Omega |u|^{q-2}u_t^2dx+\varepsilon p_1\left(1-\frac{\alpha c'}{b}\right)H(t) \\
& +\varepsilon\left\{\frac{p_1}{2}\left(1-\frac{\alpha c'}{b}\right)\left(1-\int_0^t g(s)ds\right)-\left[1-\left(1-\frac{1}{4\delta}\right)\int_0^t g(t-s)ds\right]\right\}\left\|\nabla u(t)\right\|^2 \\
& +\varepsilon\left[\frac{p_1}{2}\left(1-\frac{\alpha c'}{b}\right)-\delta\right](g\diamond\nabla u)(t) \\
& +\varepsilon\left[\frac{p_1}{2(\gamma+1)}\left(1-\frac{\alpha c'}{b}\right)-1\right]\left\|\nabla u(t)\right\|^{2(\gamma+1)}. \qquad (5.14)
\end{aligned}$$

By using the assumption $(A6)$, we can choose $\delta<p_1/2$ such that

$$1-\frac{2}{p_1}.\frac{1-\left(1-\frac{1}{4\delta}\right)\int_0^\infty g(t-s)ds}{1-\int_0^\infty g(s)ds}>0.$$

Then, we continue choosing α and ε, respectively, such that

$$0<\alpha<\frac{b}{c'}\min\left\{\frac{p_1}{2}-\delta,\frac{p_1}{2(\gamma+1)}-1,1-\frac{2}{p_1}.\frac{1-\left(1-\frac{1}{4\delta}\right)\int_0^\infty g(s)ds}{1-\int_0^\infty g(s)ds}\right\},$$

and $\varepsilon\in(0,\alpha)$. Hence, we deduce from (5.14) that

$$L'(t)\geq C_1\left(H(t)+\left\|\nabla u(t)\right\|^2\right) \qquad (5.15)$$

with C_1 is a positive constant define by

$$\begin{aligned}
C_1 = \; & \varepsilon\min\left\{p_1\left(1-\frac{\alpha c'}{b}\right);\frac{p_1}{2}\left(1-\frac{\alpha c'}{b}\right)-\delta;\frac{p_1}{2(\gamma+1)}\left(1-\frac{\alpha c'}{b}\right)-1;\right. \\
& \left.\frac{p_1}{2}\left(1-\frac{\alpha c'}{b}\right)\left(1-\int_0^t g(s)ds\right)-\left[1-\left(1-\frac{1}{4\delta}\right)\int_0^t g(t-s)ds\right]\right\}.
\end{aligned}$$

Moreover, by definition of $L(t)$ and Poincare's inequality, we get

$$\begin{aligned} L(t) &= H(t)+\frac{\varepsilon}{2}\int_{\Omega}u^2dx \\ &\leq H(t)+\frac{\varepsilon C_\alpha^2}{2}\left\|\nabla u(t)\right\|^2 \\ &\leq C_2\left(H(t)+\left\|\nabla u(t)\right\|^2\right) \end{aligned} \tag{5.16}$$

where C_α is the Poincare constant and $C_2 = 1+\frac{\varepsilon C_\alpha^2}{2}$. Hence, by (5.15) and (5.16), we arrive at

$$L'(t) \geq rL(t), \quad \forall t \in [0, T). \tag{5.17}$$

where $r = C_1/C_2$. Integration of (5.17) between 0 and t gives us the desired results. □

6. Decay of solution

In this part, we prove the general decay of solutions of the problem (1.1). Along this part we consider both cases of p, that is

$$2 < p \leq q, \quad \text{or} \quad q < p < 2+\frac{2q}{n}.$$

Lemma 6.1 ([8]). *Let $E : R^+ \to R^+$ be a noncreasing function; $\Theta : R^+ \to R^+$ be a strictly increasing function of class C^1 such that*

$$\Theta(0) = 0 \text{ and } \lim_{t\to+\infty}\Theta(t) = +\infty.$$

Assume that there exist $\eta \geq 0$ and $\lambda > 0$ such that

$$\forall S \geq 0, \quad \int_S^{+\infty} E^{1+\eta}(t)\Theta'(t) \leq \frac{1}{\lambda}E^\eta(0)E(S).$$

Then E has the following decay properties:

$$\begin{aligned} \text{If } \eta &= 0 \text{ then } E(t) \leq E(0)\exp(1-\lambda\Theta(t)), \ \forall t \geq 0, \\ \text{If } \eta &> 0 \text{ then } E(t) \leq E(0)\exp\left(\frac{1+\eta}{1+\lambda\eta\Theta(t)}\right)^{1/\eta}, \ \forall t \geq 0. \end{aligned}$$

Theorem 6.2. *Suppose that (A1), (A2), (A5),*

$$uf(u) - pF(u) \leq \sum_{i=1}^M M_i\left|u\right|^{p_i}, \ \forall u \in \mathbb{R},$$

and $\forall t \geq 0$, let $\xi : R^+ \to R^+$ is a nonincreasing function

$$g'(t) \leq -\xi(t)g(t), \qquad \int_0^{+\infty}\xi(t)dt = +\infty,$$

where $M_i \geq 0$ are constants, $p_i \in [2, p)$ $(i = 1, 2, ..., M)$ hold.

Assume further that the initial data $u_0 \in W_0^{1,2(\gamma+1)}(\Omega)$ satisfies conditions $\left\|\nabla u_0\right\|_2 < \alpha_1$, $0 \leq E(0) < E_1$, and

$$C_{p,1}^p\left(\frac{2p}{p-2}E(0)\right)^{(p-2)/2} b + \frac{p}{p-2}\sum_{i=1}^M C_{p_i,1}^{p_i}\left(\frac{2p}{p-2}E(0)\right)^{(p_i-2)/2} M_i < 1. \tag{6.1}$$

Then, solution u of problem (1.1) satisfies the following energy decay estimate for some $\lambda > 0$,

$$E(t) \leq E(0)\exp(1-\lambda\int_0^t \xi(\tau)d\tau), \ \forall t \geq 0.$$

Proof. Firstly, we show that, for $t \geq 0$,

$$\begin{aligned}\ell^{1/2}\|\nabla u(t)\|_2 &\leq \left[\left(1-\int_0^t g(s)ds\right)\|\nabla u(t)\|^2+(g\diamond\nabla u)(t)+\frac{1}{\gamma+1}\|\nabla u(t)\|^{2(\gamma+1)}\right]^{1/2}\\ &< \alpha_1=b^{-1/(p-2)}C_{p,1}^{-p/(p-2)}.\end{aligned}$$

Herefrom, we obtain

$$\begin{aligned}\mathcal{F}(t) &:= \left(1-\int_0^t g(s)ds\right)\|\nabla u(t)\|^2+(g\diamond\nabla u)(t)\\ &\quad+\frac{1}{\gamma+1}\|\nabla u(t)\|^{2(\gamma+1)}-p\int_\Omega F(u)dx\\ &\geq \ell\|\nabla u(t)\|^2-b\|u(t)\|_p^p\\ &\geq \ell\|\nabla u(t)\|^2-bC_{p,1}^p\ell^{p/2}\|\nabla u(t)\|^p\\ &\geq \ell\|\nabla u(t)\|^2\left(1-bC_{p,1}^p\ell^{(p-2)/2}\|\nabla u(t)\|^{p-2}\right)>0.\end{aligned}$$

Conversely, using Eq.(2.3), we get

$$E(t)=\frac{p-2}{2p}\left[\left(1-\int_0^t g(s)ds\right)\|\nabla u(t)\|^2+(g\diamond\nabla u)(t)+\frac{1}{\gamma+1}\|\nabla u(t)\|^{2(\gamma+1)}\right]+\frac{1}{p}\mathcal{F}(t). \tag{6.2}$$

Furthermore, from (6.2) and (A5), we easily see that

$$\ell\|\nabla u(t)\|^2\leq\left(1-\int_0^t g(s)ds\right)\|\nabla u(t)\|^2\leq\frac{2p}{p-2}E(t)\leq\frac{2p}{p-2}E(0). \tag{6.3}$$

By using embedding theorem, we get

$$\|u(t)\|_p^p\leq C_p^p\|\nabla u(t)\|^p\leq\frac{2p\vartheta_p}{p-2}E(t),\ \forall t\in[0,T), \tag{6.4}$$

where

$$\vartheta_p=C_{p,1}^p\left(\frac{2p}{p-2}E(0)\right)^{(p-2)/2}.$$

In the same way, we obtain

$$\|u(t)\|_{p_i}^{p_i}\leq\frac{2p\vartheta_{p_i}}{p-2}E(t),\ \forall t\in[0,T), \tag{6.5}$$

with

$$\vartheta_{p_i}=C_{p_i,1}^{p_i}\left(\frac{2p}{p-2}E(0)\right)^{(p_i-2)/2},\quad\forall i\quad(i=1,2,...,M).$$

Let's consider the function

$$\Theta(t)=\int_0^t\xi(s)ds,\ t\geq 0,$$

which gives, as Θ is non-decreasing function of class C^1 on $\mathbb{R}^+$ and

$$\Theta(t)\to+\infty\ \text{ as }\ t\to+\infty.$$

Multiplying Eq.(1.1) by $\xi(t)u(x,t)$ and then integrating them over $\Omega\times[S,T]$, we get

$$\begin{aligned}&\int_S^T\xi(t)\left\{\int_\Omega(1+|u(t)|^{q-2})uu_tdx+\left(1-\int_0^t g(s)ds\right)\|\nabla u\|^2+\|\nabla u\|^{2(\gamma+1)}\right.\\ &\left.-\int_0^t\int_\Omega g(t-s)(\nabla u(s)-\nabla u(t))\nabla u(t)dxds-\int_\Omega f(u)udx\right\}dt=0,\end{aligned} \tag{6.6}$$

where $0\leq S<T<+\infty$.

Taking into account the definition of $E(t)$ and (6.6), we arrive at

$$\begin{aligned} 2\int_S^T E(t)\xi(t)dt &= \int_S^T \xi(t)\Big\{-\int_\Omega (1+|u(t)|^{q-2})uu_t dx \\ &\quad + \int_0^t\int_\Omega g(t-s)(\nabla u(s)-\nabla u(t))\nabla u(t)dxds \\ &\quad + \int_\Omega [f(u)u - 2F(u)]\,dx - \left(1-\frac{1}{\gamma+1}\right)\|\nabla u\|^{2(\gamma+1)} + (g\diamond\nabla u)(t)\Big\}\,dt \\ &= \int_S^T \xi(t)\left\{I_1+I_2+I_3-\left(1-\frac{1}{\gamma+1}\right)\|\nabla u\|^{2(\gamma+1)}+(g\diamond\nabla u)(t)\right\}dt, \end{aligned} \tag{6.7}$$

in what follows we will estimate $I_1+I_2+I_3$ in (6.7). By using Sobolev's embedding theorem, Young's inequality and (2.4), (6.3), we deduce that

$$\begin{aligned} I_1 &\le \varepsilon\int_\Omega (u^2+|u|^q)dx + c(\varepsilon)\int_\Omega (u_t^2+|u|^{q-2}\,u_t^2)dx \\ &\le \varepsilon cE(t) - c(\varepsilon)E'(t). \end{aligned} \tag{6.8}$$

$$\begin{aligned} I_2 &\le \varepsilon\|\nabla u(t)\|^2 + c(\varepsilon)\left(\int_0^t g(t-s)\|\nabla u(s)-\nabla u(t)\|\,ds\right)^2 \\ &\le \varepsilon cE(t) + c(\varepsilon)\left(\int_0^t g(s)ds\right)\left(\int_0^t g(t-s)\|\nabla u(s)-\nabla u(t)\|^2\,ds\right) \\ &\le \varepsilon cE(t) + c(\varepsilon)(g\diamond\nabla u)(t). \end{aligned} \tag{6.9}$$

For I_3 in (6.7), applying the assumptions (A5), since $uf(u)-pF(u)\le\sum_{i=1}^M M_i|u|^{p_i}$ and (6.4)-(6.5), we get

$$\begin{aligned} I_3 &= \int_\Omega [f(u)u - pF(u)]\,dx + (p-2)\int_\Omega F(u)dx \\ &\le b.\frac{p-2}{p}\int_\Omega |u|^p\,dx + \sum_{i=1}^M M_i\|u(t)\|_{p_i}^{p_i} \\ &\le 2\left(\vartheta_p b + \frac{p}{p-2}\sum_{i=1}^M M_i\vartheta_{p_i}\right)E(t). \end{aligned} \tag{6.10}$$

Then, combining these estimates (6.8)-(6.10), we arrive at

$$\begin{aligned} 2\int_S^T E(t)\xi(t)dt &\le \left[\varepsilon c + 2\left(\vartheta_p b + \frac{p}{p-2}\sum_{i=1}^M M_i\vartheta_{p_i}\right)\right]\int_S^T E(t)\xi(t)dt \\ &\quad + c(\varepsilon)\int_S^T \xi(t)(g\diamond\nabla u)(t)dt - c(\varepsilon)\int_S^T E'(t)\xi(t)dt. \end{aligned} \tag{6.11}$$

Moreover, since $g'(t)\le-\xi(t)g(t)$, we have

$$\int_S^T \xi(t)(g\diamond\nabla u)(t)dt \le -\int_S^T (g'\diamond\nabla u)(t)dt \le -c\int_S^T E'(t)dt \le cE(S). \tag{6.12}$$

It is obvious that

$$-\int_S^T E'(t)\xi(t)dt \le \xi(0)\int_S^T -E'(t)dt \le cE(S). \tag{6.13}$$

Since $\vartheta_p b + \frac{p}{p-2}\sum_{i=1}^M M_i\vartheta_{p_i} < 1$ we can choose ε to be a sufficent small number so that it follows from (6.13) that

$$\int_S^T E(t)\xi(t)dt \le cE(S), \tag{6.14}$$

which implies

$$\int_S^{+\infty} E(t)\xi(t)dt \leq cE(S), \quad \forall S \geq 0.$$

We end the proof by applying Lemma 6.1. □

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