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Research Article

The sharper form of a Brunn-Minkowski type inequality for boxes

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Abstract

In this study, the Brunn-Minkowski inequality for boxes is studied and a sharper version of this inequality is derived by performing the results based on abstract convexity.

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1. Introduction

In recent decades, there have been many results on the extension and refinement of the famous inequalities by using different approaches and methods. Some of them are the ones based on the notion of abstract convex functions and sets. In the literature, there have been plenty of several convexity types, some of them can be seen in [5, 15, 18]. For the function classes of different convexity types, many new inequalities and its refinements such as integral, fractional integral, Hermite-Hadamard type inequalities etc. have been obtained by the different authors in [1, 2, 6–15, 17, 19–22]. Also some refined or sharper versions of the well-known inequalities have been obtained by means of the results of abstract convexity in [3, 16, 19]. In [16], the function which is abstract convex with respect to a class of certain quadratic functions defined on $\mathbb{R}^n$ is considered and the necessary condition for this function to have minimum on a set is expressed as an inequality involving global minimum point. In this study, the Brunn-Minkowski inequality for boxes is considered and sharpened by means of this condition.

The Brunn-Minkowski inequality is one of the well-known inequalities which gives the relation between Lebesgue measures of two nonempty compact sets and their Minkowski Sum in $\mathbb{R}^n$. If this measure is accepted as different geometric measures like length, area or volume, one can have interesting inequalities which can be interpreted geometrically [4]. For the volume measure V on $\mathbb{R}^n$, the Brunn-Minkowski inequality can be stated as follows:

Let A and B compact sets in $\mathbb{R}^n$. Then

$$V(A+B)^{\frac{1}{n}} \geq V(A)^{\frac{1}{n}} + V(B)^{\frac{1}{n}}$$

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where $A+B=\{a+b \mid a\in A,\ b\in B\}$. The equality occurs in three cases:
i) If these sets are compact convex sets of nonzero volume which are homothetic to each other with an arbitrary center and positive coefficient;
ii) One of the sets consists of single point;
iii) $V(A+B)=0$ [4].

If A and B are cuboids with the side lengths $(x_1,...,x_n)$ and $(y_1,...,y_n)$ in $\mathbb{R}^n$ respectively, the Brunn Minkowski Inequality for boxes is obtained as follows:

$$\left(\prod_{i=1}^{n}(x_i+y_i)\right)^{\frac{1}{n}} \geq \left(\prod_{i=1}^{n}x_i\right)^{\frac{1}{n}} + \left(\prod_{i=1}^{n}y_i\right)^{\frac{1}{n}} \tag{1.1}$$

Equality occurs when $x_i=\lambda y_i$ for $i\in\{1,2,...,n\}:=[n], \lambda>0$.

In this paper, a Brunn-Minkowski type inequality is studied on the results based on abstract convexity which is presented [3, 19]. A refinement of this inequality is derived.

The paper is arranged as follows: In the second section, some definitions and theorems related to abstract convexity are presented. In the third section, Brunn-Minkowski inequality is considered and sharper version is derived. In the forth section, the sharpness of new version of the inequality is shown on numerical experiments.

2. Preliminaries

For the easiness, let us recall some notations that are used throughout the paper. $\mathbb{R}$ is the real line; $\mathbb{R}_{-\infty}:=\mathbb{R}\cup\{-\infty\}$; $\bar{\mathbb{R}}:=\mathbb{R}\cup\{-\infty,+\infty\}$; $\mathbb{R}^n$ is an Euclidean Space; $\mathbb{R}^n_+$, $\mathbb{R}^n_{++}$ are nonnegative and positive orthants, respectively; X is a Hilbert space with the inner product $\langle .,.\rangle$ and the norm $\|x\|=\sqrt{\langle x,x\rangle}$; $B(x_0,r)=\{x\in X:\|x-x_0\|\leq r\}$ is a closed ball.

Let $g:\Omega\to\bar{\mathbb{R}}$ and $f:\Omega\to\bar{\mathbb{R}}$. If $g(x)\leq f(x)$ for all $x\in\Omega$ then $g\leq f$.

Let H be the class of functions on Ω. If there exists a function $h\in H$ such that $h\geq f$ then $f:\Omega\to\mathbb{R}_{-\infty}$ is majorized by H.

In [15], A. M. Rubinov gives the definition of abstract concave function as follows: Let H be the set of functions $h:\Omega\to\mathbb{R}_{+\infty}$. A function $f:\Omega\to\mathbb{R}_{-\infty}$ is called abstract concave with respect to H (or $H-concave$) if there exists a set $U\subset H$ such that $f(x)=\inf_{h\in U}h(x)$ for all $x\in\Omega$.

Let $\Omega\subset X$ and let H be the set of quadratic functions given in the following form:

$$h(x)=a\|x\|^2+\langle l,x\rangle+c,\ x\in X \tag{2.1}$$

where $a>0$, $l\in X$ and $c\in\mathbb{R}$. Then a function $f:\Omega\to\mathbb{R}_{-\infty}$ is abstract concave with respect to H if and only if f is majorized by H and f is upper semicontinuous (see [15]).

The subsequent proposition asserts that the function f satisfying certain conditions can be abstract concave with respect to the functions in the form of (2.1) (see [16]).

Proposition 2.1 ([16]). *Let $\Omega\subset X$ be a convex set and let f be a differentiable mapping defined on an open set including Ω. Suppose that ∇f is Lipschitz continuous on Ω, i.e.*

$$K:=\sup_{\substack{x,y\in\Omega\\ x\neq y}}\frac{\|\nabla f(x)-\nabla f(y)\|}{\|x-y\|}<+\infty.$$

Let $a\geq K$ and for each $t\in\Omega$

$$f_t(x)=f(t)+\langle\nabla f,x-t\rangle+a\|x-t\|^2, x\in X.$$

Then $f(x)=\min_{t\in\Omega}f_t(x),\ x\in\Omega$.

In [16], optimality conditions for the function f which can be expressed as the infimum of a family $(f_t)_{t\in T}$ of convex functions over a convex set are considered and derived. For the unconstrained minimization of a function $f : X \to \mathbb{R}$ such that $\|\nabla f(x) - \nabla f(y)\| \leq a\|x - y\|$ for all $x, y \in X$, the following result is obtained:

If x^* is a global minimum point of f over X, then

$$f(x) - f(x^*) \geq \frac{1}{4a}\|\nabla f(x)\|^2 \tag{2.2}$$

for all $x \in X$.

The following theorem which represents the general case of inequality (2.2) is proved in [16].

Theorem 2.2. *Let $\|.\|$ be the Euclidean norm and $\|.\|_\circ$ be any norm on $\mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ be a set with nonempty interior and let $f \in C^1(\Omega)$. Suppose that the mapping $x \longmapsto \nabla f(x)$ is Lipschitz continuous on Ω :*

$$K := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{\|\nabla f(x) - \nabla f(y)\|}{\|x - y\|} < \infty.$$

Let f have global minimum at $x^* \in int\,\Omega$ over Ω. Consider the ball

$$B_\circ(x^*, r) = \{x : \|x - x^*\|_\circ \leq r\} \subset int\Omega$$

and let

$$M := \max\{\|\nabla f(x)\|_\circ : x \in B_\circ(x^*, r)\}.$$

Let $q > 0$ be a number such that $B_\circ(x^*, r + q) \subset \Omega$ and let $a \geq \max\left(K, \dfrac{M}{2q}\right)$. Then

$$\frac{1}{4a}\|\nabla f(x)\|^2 \leq f(x) - f(x^*), \ x \in B_\circ(x^*, r).$$

3. Main results

If there exists $u(x) > 0$ for $x \in A$ such that $f \geq u$ on $A \subset \mathbb{R}$, then $f \geq u$ is sharper than the inequality $f \geq 0$.

Theorem 2.2 has been used to derive sharper versions of some inequalities in [3, 19]. In a similar way, by means of this theorem, the inequality in following theorem is derived as the sharper version of the Brunn-Minkowski inequality for boxes in certain conditions.

Theorem 3.1. *Let $\lambda > 0$, $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n_{++}$, $\lambda' = \min\limits_{i\in\{1,2,\ldots,n\}}\{\lambda y_i\}$, $0 < r < \lambda'$ and*

$$a_{\lambda',r} = \min_{r<d<\lambda'} \max\left\{a_1(\lambda', d), \frac{M_0}{2(d - r)}\right\}$$

where

$$a_1(\lambda, d) = \frac{\sqrt{2(n-1)}}{n\sqrt{n}} \times \left(\sum_{\delta\in\{\lambda,\lambda+1\}} \left[\prod_{i\in[n]} A_i(y, \delta, d)^{\frac{1}{n}} \sum_{k\in[n]} A_k(y, \delta, -d)^{-2}\right]\right)^{\frac{1}{2}},$$

$A_i(y, \alpha, \beta) = \alpha y_i + \beta$ and

$$M_0 = \frac{1}{n}\max_{i\in[n]}\left\{\sum_{\delta\in\{\lambda,\lambda+1\}} \frac{\prod_{i\in[n]} A_i(y, \delta, r)^{\frac{1}{n}}}{A_i(y, \delta, -r)}\right\}. \tag{3.1}$$

Then the following inequality holds for all $x \in \mathbb{R}^n_{++}$ such that $\|x - \lambda y\|_\infty \leq r$:

$$\prod_{i=1}^n (x_i + y_i)^{\frac{1}{n}} \geq \left(\prod_{i=1}^n x_i\right)^{\frac{1}{n}} + \left(\prod_{i=1}^n y_i\right)^{\frac{1}{n}} + \frac{1}{4n^2 a_{\lambda',r}} \sum_{k=1}^n \left[\frac{\prod_{i=1}^n (x_i + y_i)^{\frac{1}{n}}}{(x_k + y_k)} - \frac{\prod_{i=1}^n x_i^{\frac{1}{n}}}{x_k}\right]^2.$$

Proof. Let $y = (y_1, y_2, \cdots, y_n)$ be arbitrary point in $\mathbb{R}^n_{++}$ and let us define $f_y(x)$ on $\mathbb{R}^n_{++}$ as follows

$$f_y(x) = \prod_{i=1}^{n}(x_i + y_i)^{\frac{1}{n}} - \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} - \left(\prod_{i=1}^{n} y_i\right)^{\frac{1}{n}}.$$

For $x \in \mathbb{R}^n_{++}$, $f_y(x) \geq 0$ and $f_y(x) = 0$ is valid if and only if $x = \lambda y$, $\lambda > 0$. It is seen that f has global minimum on the points λy over $\mathbb{R}^n_{++}$. Now considering the function in Theorem 2.2 as $f_y(x)$, we can sharpen the Brunn-Minkowski inequality, namely, $f_y(x) \geq 0$. Some easy calculations imply that

$$\nabla f_y(x) = \frac{1}{n}\left[\frac{\prod\limits_{i=1}^{n}(x_i + y_i)^{\frac{1}{n}}}{(x_1 + y_1)} - \frac{\prod\limits_{i=1}^{n} x_i^{\frac{1}{n}}}{x_1}, \cdots, \frac{\prod\limits_{i=1}^{n}(x_i + y_i)^{\frac{1}{n}}}{(x_n + y_n)} - \frac{\prod\limits_{i=1}^{n} x_i^{\frac{1}{n}}}{x_n}\right].$$

Hence $\|\nabla f(x)\|^2 = \frac{1}{n^2}\sum\limits_{k=1}^{n}\left[\frac{\prod_{i=1}^{n}(x_i+y_i)^{\frac{1}{n}}}{(x_k+y_k)} - \frac{\prod_{i=1}^{n} x_i^{\frac{1}{n}}}{x_k}\right]^2$. Later we will use not only the norm $\|.\| = \|.\|_2$ but also the norm $\|.\|_\infty$. Let us consider the ball

$$\begin{aligned} V_{\lambda',d} = B_\infty(\lambda y, d) &= \{x \in \mathbb{R}^n : \|x - \lambda y\|_\infty \leq d\} \\ &= \{x \in \mathbb{R}^n : \lambda y_i - d \leq x_i \leq \lambda y_i + d, i = 1, ..., n\} \end{aligned}$$

where $\lambda' = \min\limits_i \{\lambda y_i\} > d > 0$. Since $d < \lambda y_i$, it follows that $V_{\lambda',d} \subset \mathbb{R}^n_{++}$. Let $\rho_i(x) = \frac{\prod_{i=1}^{n}(x_i+y_i)^{\frac{1}{n}}}{(x_k+y_k)} - \frac{\prod_{i=1}^{n} x_i^{\frac{1}{n}}}{x_k}$. Since it will be used to show that ∇f meets the condition of Lipschitz continuity, we need to estimate $\|\nabla \rho_i(x)\|$ for $x \in V_{\lambda',d}$. We have

$$\begin{aligned} \left|\frac{\partial \rho_i}{\partial x_i}(x)\right| &= \frac{n-1}{n}\left|-\frac{\prod\limits_{i=1}^{n}(x_i + y_i)^{\frac{1}{n}}}{(x_i + y_i)^2} + \frac{\prod\limits_{i=1}^{n} x_i^{\frac{1}{n}}}{x_i^2}\right| \\ &\leq \frac{n-1}{n}\left[\frac{\prod\limits_{i=1}^{n}(d + \lambda y_i)^{\frac{1}{n}}}{(\lambda y_i - d)^2} + \frac{\prod\limits_{i=1}^{n}(d + (1+\lambda)y_i)^{\frac{1}{n}}}{((\lambda+1)y_i - d)^2}\right] \\ \left|\frac{\partial \rho_i}{\partial x_j}(x)\right| &= \frac{1}{n}\left|\frac{\prod\limits_{i=1}^{n}(x_i + y_i)^{\frac{1}{n}}}{(x_i + y_i)(x_j + y_j)} - \frac{\prod\limits_{i=1}^{n} x_i^{\frac{1}{n}}}{x_i x_j}\right| \\ &\leq \frac{1}{n}\left[\frac{\prod\limits_{i=1}^{n}(d + \lambda y_i)^{\frac{1}{n}}}{(\lambda y_i - d)^2} + \frac{\prod\limits_{i=1}^{n}(d + (1+\lambda)y_i)^{\frac{1}{n}}}{((\lambda+1)y_i - d)^2}\right] \end{aligned}$$

so

$$\|\nabla \rho_i(x)\| \leq \sqrt{\frac{2(n-1)}{n}}\left[\frac{\prod\limits_{i=1}^{n}(d + \lambda y_i)^{\frac{1}{n}}}{(\lambda y_i - d)^2} + \frac{\prod\limits_{i=1}^{n}(d + (1+\lambda)y_i)^{\frac{1}{n}}}{((\lambda+1)y_i - d)^2}\right]^{\frac{1}{2}} = s_i. \tag{3.2}$$

Let $x,z \in V_{\lambda',d}$. With the help of the Mean Value Theorem and the Cauchy-Schwarz Inequality, it can be deduced that there exist numbers $\theta_i \in (0,1)$, $i=1,...,n$ such that

$$\begin{aligned}
\|\nabla f_y(x)-\nabla f_y(z)\| &= \frac{1}{n}\left\|\left[\rho_1(x)-\rho_1(z)\right],\left[\rho_2(x)-\rho_2(z)\right],...,\left[\rho_n(z)-\rho_n(z)\right]\right\| \\
&= \frac{1}{n}\left(\sum_{k=1}^{n}\left[\rho_k(x)-\rho_k(z)\right]^2\right)^{\frac{1}{2}} \\
&= \frac{1}{n}\left(\sum_{k=1}^{n}\left[\nabla\rho_k(x+\theta_k(z-x))(x-z)\right]^2\right)^{\frac{1}{2}} \\
&\leq \frac{1}{n}\left(\sum_{k=1}^{n}\|\nabla\rho_k(x+\theta_k(z-x))\|^2\right)^{\frac{1}{2}}\|x-z\| \\
&\leq \frac{\sqrt{2(n-1)}}{n\sqrt{n}}\times\|x-z\|\times A,
\end{aligned}$$

where $A = \left(\prod\limits_{i=1}^{n}(d+\lambda y_i)^{\frac{1}{n}}\sum\limits_{k=1}^{n}(\lambda y_k-d)^{-2}+\prod\limits_{i=1}^{n}(d+(1+\lambda)y_i)^{\frac{1}{n}}\sum\limits_{k=1}^{n}((\lambda+1)y_i-d)^{-2}\right)^{\frac{1}{2}}$. Because $x,z \in V_{\lambda',d}$, $x+\theta_i(z-x) \in V_{\lambda',d}$ for all i.

Using (3.2), we derive that $\|\nabla f_y(x)-\nabla f_y(z)\| \leq a_1(\lambda,d)\|x-z\|$, $x,z \in V_{\lambda',d}$, where

$$a_1(\lambda,d) = \frac{\sqrt{2(n-1)}}{n\sqrt{n}}\times$$
$$\left(\prod_{i=1}^{n}(d+\lambda y_i)^{\frac{1}{n}}\sum_{k=1}^{n}(\lambda y_k-d)^{-2}+\prod_{i=1}^{n}(d+(1+\lambda)y_i)^{\frac{1}{n}}\sum_{k=1}^{n}((\lambda+1)y_i-d)^{-2}\right)^{\frac{1}{2}}.$$

As a consequence, the mapping $x \to \nabla f(x)$ is Lipschitz continuous on $V_{\lambda',d}$ with the Lipschitz constant $K \leq a_1(\lambda',d)$. Let us apply Theorem 2.2 to a set $\Omega = V_{\lambda',d}$ where $d < \lambda' = \min\limits_i\{\lambda y_i\}$ and the global minimum points $x^* = \lambda y$ of the function f. Suppose that $\|.\|_\circ$ is used in Theorem 2.2 as $\|.\|_\infty$. Let $r \in (0,d)$ and $q = d-r$. Let us estimate $M = \max\left\{\|\nabla f_y(x)\|_\infty : x \in V_{\lambda',r}\right\}$ as follows:

$$\begin{aligned}
M = \max_{x\in V_{\lambda',r}}\left\{\|\nabla f(x)\|_\infty\right\} &= \frac{1}{n}\max_{x\in V_{\lambda',r}}\left\{\max_{1\leq i\leq n}\left|\frac{\prod\limits_{i=1}^{n}(x_i+y_i)^{\frac{1}{n}}}{(x_k+y_k)}-\frac{\prod\limits_{i=1}^{n}x_i^{\frac{1}{n}}}{x_k}\right|\right\} \\
&\leq \frac{1}{n}\max_{1\leq i\leq n}\left\{\frac{\prod\limits_{i=1}^{n}(r+\lambda y_i)^{\frac{1}{n}}}{(\lambda y_i-r)}+\frac{\prod\limits_{i=1}^{n}(r+(1+\lambda)y_i)^{\frac{1}{n}}}{((\lambda+1)y_i-r)}\right\} \equiv M_0.
\end{aligned}$$

Let

$$a_2(\lambda',d,r) = \frac{M_0}{2(d-r)}$$

and

$$a(\lambda',d,r) = \max\left\{a_1(\lambda',d),a_2(\lambda',d,r)\right\}.$$

Note that $\lim\limits_{d\to\lambda'-0} a(\lambda',d,r) = \lim\limits_{d\to r+0} a(\lambda',d,r) = +\infty$ so the function $d \longmapsto a(\lambda',d,r)$ takes its minimum on the interval (r,λ'). Let $a_{\lambda',r} = \min\limits_{r<d<\lambda'} a(\lambda',d,r)$. Applying Theorem 2.2

we conclude that

$$\prod_{i=1}^{n}(x_i+y_i)^{\frac{1}{n}} \geq \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} + \left(\prod_{i=1}^{n} y_i\right)^{\frac{1}{n}} + \frac{1}{4n^2 a_{\lambda',r}} \sum_{k=1}^{n} \left[\frac{\prod_{i=1}^{n}(x_i+y_i)^{\frac{1}{n}}}{(x_k+y_k)} - \frac{\prod_{i=1}^{n} x_i^{\frac{1}{n}}}{x_k}\right]^2$$

for $x \in V_{\lambda',r}$. □

4. Some numerical experiments on the sharper inequality

To illustrate the numerical efficacy of the refined inequality in comparison to usual inequality, some numerical experiments have been carried out. First, for some x_i and y_i values for certain parameters n, λ and r numerical results are given as the tables. Second, the effect of the variation of parameters n, λ, r is considered empirically.

The value of $a_{\lambda',r}$ is determined by searching of 10^4 uniformly separated points in the interval (r, λ')

Throughout the numerical experiments, results are shown by at least six significant figures in scientific notation.

In the tables, for given values of n, λ, x_i, y_i, r the values of $f_y(x)$, $u(x)$ and $\frac{u(x)}{f_y(x)}$ are computed where $f_y(x)$ represents the difference between the right side and left side in (1.1),

$$u(x) = \frac{1}{4n^2 a_{\lambda',r}} \sum_{k=1}^{n} \left[\frac{\prod_{i=1}^{n}(x_i+y_i)^{\frac{1}{n}}}{(x_k+y_k)} - \frac{\prod_{i=1}^{n} x_i^{\frac{1}{n}}}{x_k}\right]^2$$

denotes the sharpness amount and $\frac{u(x)}{f_y(x)}$ can be called relative sharpness ratio.

To illustrate the numerical efficacy of sharper version, we chose y_i in two ways. First, we have taken $y_i = 10i\cos^2 i$ which gives the fluctuating values and $x_i = \lambda y_i + r\cos i$ which does not violate the condition $\lambda y_i - r < x_i < \lambda y_i + r$ in Theorem 2.2. For $n = 5$, 10, 20 and various λ, r such that $r < \lambda'$, results are shown in Table 1, Table 2 and Table 3 respectively.

Table 1. Results for $n = 5$, $y_i = 10i\cos^2 i$, $x_i = \lambda y_i + r\cos i$

λ	r	λ'	$f_y(x)$	$u(x)$	$\frac{u(x)}{f_y(x)}$
1	2	2.91927	7.40765×10^{-2}	8.85441×10^{-4}	1.19531×10^{-2}
5	10	14.59632	1.22432×10^{-1}	2.95644×10^{-4}	2.41475×10^{-3}
10	20	29.19276	1.33422×10^{-1}	1.32525×10^{-4}	9.93275×10^{-4}
20	30	58.38532	7.93102×10^{-2}	4.47886×10^{-5}	5.64727×10^{-4}
40	100	116.77063	2.23874×10^{-1}	1.54104×10^{-5}	6.88354×10^{-5}

It is seen that for each experiment the amount of sharpness $u(x)$ changes. In Table 3, it is too small because λ' is relatively too small hence r in comparison to experiments in Table 1 and Table 2. Although n is larger here, it stems from the choice of y_i affecting λ' since y_i is too small for some $i > 10$.

Table 2. Results for $n = 10,\ \ y_i = 10i\cos^2 i,\ x_i = \lambda y_i + r\cos i$

λ	r	λ'	$f_y(x)$	$u(x)$	$\frac{u(x)}{f_y(x)}$
1	1	1.69362	2.21459×10^{-2}	2.76298×10^{-4}	1.24763×10^{-2}
5	5	8.46810	3.65856×10^{-2}	1.03438×10^{-4}	2.82728×10^{-3}
10	10	16.93621	3.98410×10^{-2}	4.90442×10^{-5}	1.23100×10^{-3}
20	25	33.87242	6.45404×10^{-2}	1.81733×10^{-5}	2.81581×10^{-4}
40	50	67.74448	6.60825×10^{-2}	7.58624×10^{-6}	1.14800×10^{-4}

Table 3. Results for $n = 20,\ y_i = 10i\cos^2 i,\ x_i = \lambda y_i + r\cos i$

λ	r	λ'	$f_y(x)$	$u(x)$	$\frac{u(x)}{f_y(x)}$
1	10^{-3}	2.15455×10^{-3}	8.24187×10^{-7}	6.38044×10^{-8}	7.74150×10^{-2}
5	10^{-2}	1.07727×10^{-2}	5.47901×10^{-6}	3.19564×10^{-9}	5.83252×10^{-4}
10	10^{-2}	2.15455×10^{-2}	1.49772×10^{-6}	1.84856×10^{-8}	1.23425×10^{-2}
20	10^{-2}	4.30910×10^{-2}	3.92740×10^{-7}	5.10792×10^{-9}	1.30059×10^{-2}
40	10^{-2}	8.61820×10^{-2}	1.00643×10^{-7}	8.87448×10^{-10}	8.81775×10^{-3}

In the second way, we have chosen y_i as arithmetically increasing numbers and x_i as alternating numbers in $[\lambda y_i - r, \lambda y_i + r]$. For $n = 5$, 10, 20 and various λ, r such that $r < \lambda'$ results are shown in Table 4, Table 5 and Table 6 respectively.

Table 4. Results for $n = 5,\ y_i = 20i,\ x_i = \lambda y_i + \frac{(-1)^i r}{2}$

λ	r	λ'	$f_y(x)$	$u(x)$	$\frac{u(x)}{f_y(x)}$
1	2	20	8.92929×10^{-3}	2.23510×10^{-4}	2.50311×10^{-2}
5	50	100	4.30586×10^{-1}	1.48812×10^{-3}	3.45604×10^{-2}
10	100	200	4.73842×10^{-1}	6.47896×10^{-4}	1.36732×10^{-3}
20	200	400	4.98942×10^{-1}	2.57650×10^{-4}	5.16393×10^{-4}
40	400	800	5.12529×10^{-1}	9.74210×10^{-5}	1.90079×10^{-4}

Table 5. Results for $n = 10,\ \ y_i = 20i,\ x_i = \lambda y_i + \frac{(-1)^i r}{2}$

λ	r	λ'	$f_y(x)$	$u(x)$	$\frac{u(x)}{f_y(x)}$
1	10	20	2.48602×10^{-1}	6.64682×10^{-2}	2.67568×10^{-2}
5	50	100	4.31267×10^{-1}	2.08523×10^{-3}	4.83513×10^{-3}
10	100	200	4.75102×10^{-1}	9.17230×10^{-4}	1.93060×10^{-3}
20	200	400	5.00580×10^{-1}	3.67309×10^{-4}	7.33767×10^{-4}
40	400	800	5.14385×10^{-1}	1.39541×10^{-4}	2.71278×10^{-4}

Table 6. Results for $n = 20$, $y_i = 20i$, $x_i = \lambda y_i + \frac{(-1)^i r}{2}$

λ	r	λ'	$f_y(x)$	$u(x)$	$\frac{u(x)}{f_y(x)}$
1	10	20	2.39523×10^{-1}	8.23519×10^{-3}	3.43816×10^{-2}
5	50	100	4.16243×10^{-1}	2.71825×10^{-3}	6.53044×10^{-3}
10	100	200	4.58746×10^{-1}	1.21076×10^{-3}	2.63927×10^{-3}
20	200	400	4.83347×10^{-1}	4.89261×10^{-4}	1.01198×10^{-3}
40	400	800	4.96867×10^{-1}	1.87122×10^{-4}	3.76604×10^{-4}

In order to see how the alteration in variables x_i, y_i and parameters n, λ, r affects the sharpness of inequality numerically, all variables and parameter n, λ and r except one has been kept constant. So it can be shown the effect of variation of this variable into the inequality. Since the variation of n requires adding or removing some x_i or y_i values and variation of λ requires the changing x_i values which should obey the condition $\lambda y_i - r < x_i < \lambda y_i + r$, only the effect of variation of r can be observed provided that x_i values are constant.

Numerical results are given in Table 7, Table 8, Table 9 and Table 10 in order to show the impact of the change in parameter r. According to conditions imposed by Theorem 2.2, r can be selected any positive number smaller than λ'.

Table 7. Results for $n = 10$, $\lambda = 10$, $\lambda' = 500$, $y_i = 50i$, $x_i = 500i + 50 \cos i$

r	$f_y(x)$	$u(x)$	$\frac{u(x)}{f_y(x)}$
50	5.30226×10^{-2}	4.62241×10^{-5}	8.71781×10^{-4}
100	5.30226×10^{-2}	4.14820×10^{-5}	7.82345×10^{-4}
200	5.30226×10^{-2}	3.17775×10^{-5}	5.99320×10^{-4}
400	5.30226×10^{-2}	1.06071×10^{-4}	2.00049×10^{-4}
499	5.30226×10^{-2}	7.52190×10^{-8}	1.41862×10^{-7}

Table 8. Results for $n = 20$, $\lambda = 10$, $\lambda' = 500$, $y_i = 50i$, $x_i = 500i + 50 \cos i$

r	$f_y(x)$	$u(x)$	$\frac{u(x)}{f_y(x)}$
50	5.06348×10^{-2}	5.34698×10^{-5}	1.05599×10^{-3}
100	5.06348×10^{-2}	4.82349×10^{-5}	9.52603×10^{-4}
200	5.06348×10^{-2}	3.72388×10^{-5}	7.35437×10^{-4}
400	5.06348×10^{-2}	1.22733×10^{-5}	2.42388×10^{-4}
499	5.06348×10^{-2}	7.39036×10^{-9}	1.45954×10^{-7}

Table 9. Results for $n = 10$, $\lambda = 100$, $\lambda' = 1.69362 \times 10^2$, $y_i = 10i\cos^2 i$, $x_i = \lambda y_i + 50\cos i$

r	$f_y(x)$	$u(x)$	$\frac{u(x)}{f_y(x)}$
10	4.58435×10^{-4}	6.24138×10^{-8}	1.36146×10^{-4}
50	4.58435×10^{-4}	4.66779×10^{-8}	1.01820×10^{-4}
100	4.58435×10^{-4}	2.61021×10^{-8}	5.69375×10^{-5}
150	4.58435×10^{-4}	4.84804×10^{-9}	1.05752×10^{-5}
168	4.58435×10^{-4}	5.46284×10^{-12}	1.19163×10^{-7}

Table 10. Results for $n = 20$, $\lambda = 100$, $\lambda' = 2.15455 \times 10^{-1}$, $y_i = 10i\cos^2 i$, $x_i = \lambda y_i + 50\cos i$

r	$f_y(x)$	$u(x)$	$\frac{u(x)}{f_y(x)}$
10^{-4}	1.63523×10^{-12}	7.49248×10^{-15}	4.58192×10^{-3}
10^{-3}	1.63523×10^{-12}	7.42960×10^{-15}	4.54285×10^{-3}
10^{-2}	1.63523×10^{-12}	6.80921×10^{-15}	4.16407×10^{-3}
10^{-1}	1.63523×10^{-12}	2.13159×10^{-15}	1.30293×10^{-3}
2.14×10^{-1}	1.63523×10^{-12}	4.73303×10^{-19}	2.89441×10^{-7}

It is observed empirically from Table 7, Table 8, Table 9 and Table 10 that when r is chosen close to λ', the sharpening amount $u(x)$ and relative sharpening ratio $\frac{u(x)}{f_y(x)}$ decreases. This shows that the increase in r values might have little effect on sharpening the inequality.

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