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WEAK CONE-COMPLETENESS OF DIRECT SUMS IN LOCALLY CONVEX CONES

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Abstract. We consider the weak cone-completeness in locally convex cones and prove that the direct sum of a family of weakly cone-complete separated locally convex cones is weakly cone-complete. We conclude that a direct sum cone topology is barreled whenever its components are weakly cone-complete and separated with the countable bases.

1. Introduction

The notions of barreledness and weak cone completeness in locally convex cones have been defined and investigated by W. Roth in [8]. Various topics of locally convex cones have been studied from the direct sum point of view in [2-7]. In this paper, we discuss the direct sum topology of weakly cone complete locally convex cones and show that it is barreled if its components are separated and carry the countable bases.

An *ordered cone* is a set $\mathcal{P}$ endowed with an addition $(a, b) \longmapsto a + b$ and a scalar multiplication $(\alpha, a) \longmapsto \alpha a$ for real numbers $\alpha \geq 0$. The addition is supposed to be associative and commutative, there is a neutral element $0 \in \mathcal{P}$, and for the scalar multiplication the usual associative and distributive properties hold, that is, $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha + \beta)a = \alpha a + \beta a$, $\alpha(a + b) = \alpha a + \alpha b$, $1a = a$, $0a = 0$ for all $a, b \in \mathcal{P}$ and $\alpha, \beta \geq 0$. In addition, the cone $\mathcal{P}$ carries a (partial) order, i.e., a reflexive transitive relation $\leq$ that is compatible with the algebraic operations, that is $a \leq b$ implies $a + c \leq b + c$ and $\alpha a \leq \alpha b$ for all $a, b, c \in \mathcal{P}$ and $\alpha \geq 0$. For example, the extended scalar field $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ of real numbers is a preordered cone. We consider the usual order and algebraic operations in $\overline{\mathbb{R}}$; in particular, $\alpha + \infty = +\infty$ for all $\alpha \in \overline{\mathbb{R}}$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha > 0$ and $0 \cdot (+\infty) = 0$. In any cone $\mathcal{P}$, equality is obviously such an order, hence all results about ordered cones apply to cones without order structures as well.

A *full locally convex cone* $(\mathcal{P}, \mathcal{V})$ is an ordered cone $\mathcal{P}$ that contains an *abstract neighborhood system* $\mathcal{V}$, i.e., a subset of positive elements that is directed downward, closed for addition and multiplication by (strictly) positive scalars. The elements v of $\mathcal{V}$ define *upper* (*lower*) *neighborhoods* for the elements of $\mathcal{P}$ by $v(a) = \{b \in \mathcal{P} :$

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$b \leq a+v\}$ (respectively, $(a)v = \{b \in \mathcal{P} : a \leq b+v\}$), creating the *upper*, respectively *lower topologies* on $\mathcal{P}$. Their common refinement is called the *symmetric topology.* We assume all elements of $\mathcal{P}$ to be *bounded below*, i.e., for every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we have $0 \leq a+\rho v$ for some $\rho > 0$. Finally, a *locally convex cone* $(\mathcal{P}, \mathcal{V})$ is a subcone of a full locally convex cone, not necessarily containing the abstract neighborhood system $\mathcal{V}$. For a locally convex cone $(\mathcal{P}, \mathcal{V})$ the collection of all sets $\widetilde{v} \subseteq \mathcal{P}^2$, where $\widetilde{v} = \{(a, b) : a \leq b + v\}$ for all $v \in \mathcal{V}$, defines a *convex quasi-uniform structure* on $\mathcal{P}$. On the other hand, every convex quasi-uniform structure leads to a full locally convex cone, including $\mathcal{P}$ as a subcone and induces the same convex quasi-uniform structure. For details see [1, Ch I, 5.2]. For cones $\mathcal{P}$ and $\mathcal{Q}$, a map $t : \mathcal{P} \to \mathcal{Q}$ is called a *linear operator*, if $t(a + b) = t(a) + t(b)$ and $t(\alpha a) = \alpha t(a)$ for all $a, b \in \mathcal{P}$ and $\alpha \geq 0$. If $\mathcal{V}$ and $\mathcal{W}$ are abstract neighborhood systems on $\mathcal{P}$ and $\mathcal{Q}$, a linear operator $t : \mathcal{P} \to \mathcal{Q}$ is called *uniformly continuous* if for every $w \in \mathcal{W}$ there is $v \in \mathcal{V}$ such that $t(a) \leq t(b)+w$ whenever $a \leq b+v$. Uniform continuity implies continuity with respect to the upper, lower and symmetric topologies on $\mathcal{P}$ and $\mathcal{Q}$. Endowed with the neighborhood system $\varepsilon = \{\epsilon \in \mathbb{R} : \epsilon > 0\}$, $\overline{\mathbb{R}}$ is a full locally convex cone. The set of all uniformly continuous linear functionals $\mu : \mathcal{P} \to \overline{\mathbb{R}}$ is a cone called the *dual cone* of $\mathcal{P}$ and denoted by $\mathcal{P}^*$.

A locally convex cone $(\mathcal{P}, \mathcal{V})$ is called *weakly cone-complete* if for all $b \in \mathcal{P}$ and $v \in \mathcal{V}$, every sequence $(a_i)_{n\in\mathbb{N}}$ in $v(b) \cap (b)v$ that converges to b in the symmetric topology of $\mathcal{P}$ and $\eta_i > 0$ such that $\sum_{i=1}^{\infty} \eta_i = 1$ there is $a \in v(b) \cap (b)v$ such that $\mu(a) = \sum_{i=1}^{\infty} \eta_i \mu(a_i)$ for all $\mu \in \mathcal{P}^*$ with $\mu(b) < +\infty$. A convex subset U of $\mathcal{P}^2$ is called *barrel*, if it satisfies the following properties:

(U_1) For every $b \in \mathcal{P}$ there is a neighborhood $v \in \mathcal{V}$ such that for every $a \in v(b) \cap (b)v$ there is a $\lambda > 0$ such that $(a, b) \in \lambda \mathrm{U}$.

(U_2) If $(a, b) \notin \mathrm{U}$, then there is a $\mu \in \mathcal{P}^*$ such that $\mu(c) \leq \mu(d) + 1$ for all $(c, d) \in \mathrm{U}$ and $\mu(a) > \mu(b) + 1$.

A locally convex cone $(\mathcal{P}, \mathcal{V})$ is called *barreled* if for every barrel U and every element $b \in \mathcal{P}$ there is a neighborhood $v \in \mathcal{V}$ and a $\lambda > 0$ such that $(a, b) \in \lambda \mathrm{U}$ for all $a \in v(b) \cap (b)v$. A subset $\mathcal{V}_0$ of $\mathcal{V}$ is a *neighborhood base*, if for every $v \in \mathcal{V}$ there is $v_0 \in \mathcal{V}_0$ such that $v_0 \leq v$. Every weakly cone complete locally convex cone with a countable neighborhood base is barreled [8, Theorem 2.3].

2. Weak cone-completeness and direct sums

Let $\mathcal{P}_\gamma$, $\gamma \in \Gamma$ be cones and put $\mathcal{P} = \times_{\gamma\in\Gamma}\mathcal{P}_\gamma$. For elements $a, b \in \mathcal{P}$, $a = \times_{\gamma\in\Gamma} a_\gamma$, $b = \times_{\gamma\in\Gamma} b_\gamma$ and $\alpha \geq 0$ we set $a+b = \times_{\gamma\in\Gamma} (a_\gamma+b_\gamma)$ and $\alpha a = \times_{\gamma\in\Gamma} (\alpha a_\gamma)$. With these operations $\mathcal{P}$ is a cone which is called the *product cone* of $\mathcal{P}_\gamma$. The subcone of the product cone $\mathcal{P}$ spanned by $\cup\mathcal{P}_\gamma$ (more precisely, by $\cup j_\gamma(\mathcal{P}_\gamma)$, where $j_\gamma : \mathcal{P}_\gamma \to \mathcal{P}$ is the injection mapping) is said to be the *direct sum cone* of $\mathcal{P}_\gamma$ and denoted by $\mathcal{Q} = \sum_{\gamma\in\Gamma} \mathcal{P}_\gamma$. If $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$, $\gamma \in \Gamma$ be a family of locally convex cones, then $\mathcal{W} = \times_{\gamma\in\Gamma}\mathcal{V}_\gamma$ leads to the *finest locally convex cone topology* on $\mathcal{Q}$ such that the all injection mappings j_γ are uniformly continuous:

Definition 2.1. For elements $a, b \in \mathcal{Q}$, $a = \sum_{\gamma\in\Delta} a_\gamma$, $b = \sum_{\gamma\in\Theta} b_\gamma$ and $w \in \mathcal{W}$, $w = \times_{\gamma\in\Gamma} v_\gamma$, we set

$$a \leq_\Gamma b + w$$

if $a_\gamma \leq_\gamma b_\gamma + \alpha v_\gamma$ *for all* $\gamma \in \Delta \cup \Theta$, *where* $\sum_{\gamma\in\Delta\cup\Theta} \alpha_\gamma \leq 1$.

The subsets $\{(a,b) \in \mathcal{Q}^2 : a \leq_\Gamma b + w\}$ for all $w \in \mathcal{W}$ describe the finest convex quasi-uniform structure on $\mathcal{Q}$ which makes every injection mapping uniformly continuous. According to [1, Ch I, 5.4], there exists a full cone $\mathcal{Q} \oplus \mathcal{W}_0$ with abstract neighborhood system $\mathrm{W} = \{0\} \oplus \mathcal{W}$, whose neighborhoods yield the same quasi-uniform structure on $\mathcal{Q}$. The elements $w \in \mathcal{W}, w = \times_{\gamma\in\Gamma} v_\gamma$ form a basis for W in the following sense: For every $\mathrm{w} \in \mathrm{W}$ there is $w \in \mathcal{W}$ such that $a \leq_\Gamma b + w$ for $a, b \in \mathcal{Q}$ implies that $a \leq_\Gamma b \oplus \mathrm{w}$. The locally convex cone topology on $\mathcal{Q}$ induced by $\mathcal{W}$ is called the *locally convex direct sum cone of* $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ and denoted by $(\mathcal{Q}, \mathcal{W})$. For details see [3].

Proposition 2.1. *If $\mathcal{Q} = \sum_{\gamma\in\Gamma} \mathcal{P}_\gamma$ is the locally convex direct sum cone, then*

(a) *if $b \in \mathcal{Q}$ and $(a_i)_{i\in\mathbb{N}} \subset \mathcal{Q}$ converges to b in the symmetric topology of $\mathcal{Q}$, then for each $\gamma \in \Gamma$, $(\varphi_\gamma(a_i))_{i\in\mathbb{N}}$ converges to $\varphi_\gamma(b)$ in the symmetric topology of $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$,*

(b) *$\mathcal{Q}^* = \times_{\gamma\in\Gamma} \mathcal{P}^*_\gamma$, where $\mathcal{Q}^*$ is the dual cone of $(\mathcal{Q}, \mathcal{W})$.*

Proof. (a) Fix $\gamma \in \Gamma$ and let $v_\gamma \in \mathcal{V}_\gamma$. If we set $w = \times_{\xi\in\Gamma} v_\xi$, where $v_\xi = v_\gamma$ for $\xi = \gamma$ and $v_\xi \in \mathcal{V}_\gamma$ otherwise, then $a \leq_\Gamma b + w$ for $a, b \in \mathcal{Q}$ implies that $\varphi_\gamma(a) \leq \varphi_\gamma(b) + v_\gamma$, i.e., φ_γ is uniformly continuous. For (b), see Theorem 3.10 in [7]. □

Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. A subset A of $\mathcal{P}$ is *bounded* in the weak topology $\sigma(\mathcal{P}, \mathcal{P}^*)$ if, $-\infty < \inf_{\mu\in F, x\in A} \mu(x) \leq \sup_{\mu\in F, x\in A} \mu(x) < +\infty$ for all finite sets $F \subset \mathcal{P}^*$ [5, 6]. The cone $\mathcal{P}$ is *separated* if $\bar{a} = \bar{b}$ for $a, b \in \mathcal{P}$ implies $a = b$, where $\overline{a}$ is the closure of a with respect to the lower topology of $\mathcal{P}$; for example $\overline{\mathbb{R}}$ with the neighborhoods system $\varepsilon = \{\epsilon \in \mathbb{R} : \epsilon > 0\}$ is separated [1, Ch I, 3.12]. A locally convex cone direct sum cone is separated if and only if its components are separated [7, Corollary 3,3].

Lemma 2.2. *Suppose $(\mathcal{Q}, \mathcal{W})$ is the locally convex direct sum cone of $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$, $b \in \mathcal{Q}$ and let $\mathcal{Q}^*_b$ be the subcone of all $\mu \in \mathcal{Q}^*$ with $\mu(b) < +\infty$. If $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ is separated for all $\gamma \in \Gamma$ and $(a_i)_{i\in\mathbb{N}} \subset \mathcal{Q}$ converges to b in the symmetric topology of $\mathcal{Q}$, then there is a finite subset Δ of Γ such that for each $i \in \mathbb{N}$, $\varphi_\gamma(a_i) = 0$ for all $\gamma \in \Gamma \setminus \Delta$.*

Proof. Let $\mu_1, ..., \mu_n \in \mathcal{Q}^*_b$ and let $w' \in \mathcal{W}$ such that $w' \leq w$ and $\mu_i \in w'^{\circ}$ for all $i = 1, 2, ..., n$. By the assumption, there is $i_0 \in \mathbb{N}$ such that $a_i \in w'(b) \cap (b)w'$ for all $i \geq i_0$ which yields

$$-\infty < \inf_{1\leq j\leq n, i\geq i_0} \mu_j(a_i) \leq \sup_{1\leq j\leq n, i\geq i_0} \mu_j(a_i) < +\infty,$$

i.e., $\{a_i : i \geq i_0\}$ is $\sigma(\mathcal{Q}, \mathcal{Q}^*_b)$-bounded so, by [6, Theorem 2.6], there is a finite set $\Delta \subset \Gamma$ such that $\{a_i : i \geq i_0\} \subseteq \sum_{\gamma\in\Delta} \varphi_\gamma\{a_i : i \geq i_0\}$. □

Theorem 2.3. *The direct sum cone topology $(\mathcal{Q}, \mathcal{W}) = (\sum_{\gamma\in\Gamma} \mathcal{P}_\gamma, \times_{\gamma\in\Gamma} \mathcal{V}_\gamma)$ is weakly cone-complete, whenever for each $\gamma \in \Gamma$, $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ is separated and weakly cone-complete.*

Proof. Suppose $b \in \mathcal{Q}$, $b = \sum_{\gamma\in\Delta} b_\gamma$, $w \in \mathcal{W}$, $w = \times_{\gamma\in\Gamma} v_\gamma$ and let $(a_i)_{i\in\mathbb{N}} \subset w(b) \cap (b)w$ converges to b in the symmetric topology of $\mathcal{Q}$ and $\eta_i > 0$ such that $\sum_{i=1}^{\infty} \eta_i = 1$. Using the Lemma 2.2, we may assume that $a_i = \sum_{\gamma\in\Delta} \varphi_\gamma(a_i)$ for all $i \in \mathbb{N}$. By Proposition 2.1 (a), the sequence $(\varphi_\gamma(a_i))_{i\in\mathbb{N}} \subset (\alpha_\gamma v_\gamma)(b_\gamma) \cap (b_\gamma)(\alpha_\gamma v_\gamma)$ converges to b_γ in the symmetric topology of $\mathcal{P}_\gamma$ for all $\gamma \in \Delta$, where $\sum_{\gamma\in\Delta} \alpha_\gamma \leq 1$;

so from the weak cone-completeness of $\mathcal{P}_\gamma$ there exists $a_\gamma \in (\alpha_\gamma v_\gamma)(b_\gamma) \cap (b_\gamma)(\alpha_\gamma v_\gamma)$ such that

$$\mu_\gamma(a_\gamma) = \sum_{i=1}^{\infty} \eta_i \mu_\gamma(\varphi_\gamma(a_i))$$

for all $\mu_\gamma \in \mathcal{P}_\gamma^*$ with $\mu_\gamma(b_\gamma) < \infty$. Then $a := \sum_{\gamma\in\Delta} a_\gamma \in w(b) \cap (b)w$ and for every $\mu \in \mathcal{Q}^*$ with $\mu(b) < +\infty$, we have $\mu = \times_{\gamma\in\Gamma}\mu_\gamma$ by Proposition 2.1 (b), where $\mu_\gamma \in \mathcal{P}_\gamma^*$ such that $\mu_\gamma(b_\gamma) < +\infty$ for all $\gamma \in \Delta$. Thus

$$\begin{aligned} \mu(a) &= \sum_{\gamma\in\Delta}\sum_{i=1}^{\infty} \eta_i \mu_\gamma(\varphi_\gamma(a_i)) = \sum_{i=1}^{\infty} \eta_i \sum_{\gamma\in\Delta} \mu_\gamma(\varphi_\gamma(a_i)) \\ &= \sum_{i=1}^{\infty} \eta_i \mu(\sum_{\gamma\in\Delta} \varphi_\gamma(a_i)) = \sum_{i=1}^{\infty} \eta_i \mu(a_i), \end{aligned}$$

i.e., $(\mathcal{Q}, \mathcal{W})$ is weakly cone-complete. □

By combining Theorem 2.3 and [8, Theorem 2.3], we have:

Corollary 2.4. *A direct sum cone topology is barreled, whenever its components are separated and weakly cone-complete with the countable bases.*

Example 2.1. (i) If we consider $(\overline{\mathbb{R}}, \varepsilon)$, $\varepsilon = \{\epsilon \in \mathbb{R} : \epsilon > 0\}$, then $\overline{\mathbb{R}}^* = \mathbb{R}_+ \cup \{\overline{0}\}$; where $\overline{0}(x) = 0$ for all $x \in \mathbb{R}$ and $\overline{0}(+\infty) = +\infty$ [9, Example 2.2]. Let $b \in \overline{\mathbb{R}}$, $\epsilon > 0$, $(a_i)_{i\in\mathbb{N}} \subset \epsilon(b) \cap \epsilon(b)$ converges to b in the symmetric topology of $(\overline{\mathbb{R}}, \varepsilon)$ and let $\eta_i > 0$ such that $\sum_{i=1}^{\infty} \eta_i = 1$. If $b = +\infty$, then for $a = +\infty$ the assertion holds. If $b \in \mathbb{R}$ then $a := \sum_{i=1}^{\infty} \eta_i a_i \in \epsilon(b) \cap (b)\epsilon$ and for every $\mu \in \overline{\mathbb{R}}^*$, $\mu = \lambda$ for some $\lambda > 0$, hence $\mu(a) = \lambda(\sum_{i=1}^{\infty} \alpha_i a_i) = \sum_{i=1}^{\infty} \alpha_i \mu(a_i)$, i.e., $(\overline{\mathbb{R}}, \varepsilon)$ is weakly cone-complete.

(ii) We consider $\mathcal{Q} = \sum_{n\in\mathbb{N}} \overline{\mathbb{R}}$ with the countable neighborhood system $\mathcal{W} = \times_{n\in\mathbb{N}}\varepsilon$. For elements $a, b \in \mathcal{Q}$, $a = \sum_{n\in\Delta} a_n$, $b = \sum_{n\in\Theta} b_n$, the direct sum neighborhood $w \in \mathcal{W}$, $w = \times_{n\in\mathbb{N}}\epsilon_n$ on $\mathcal{Q}$ is defined by

$$a \leq_{\mathbb{N}} b + w \quad \text{if} \quad a_n \leq b_n + \alpha_n \varepsilon_n \quad (\text{for all} \quad n \in \Delta \cup \Theta)$$

where $\sum_{n\in\Delta\cup\Theta} \alpha_n \leq 1$. Suppose $b \in \mathcal{Q}$, $b = \sum_{n\in\Delta_{\mathbb{R}}} b_n + \sum_{n\in\Delta\setminus\Delta_{\mathbb{R}}}(+\infty)$, where $\Delta = \{n \in \mathbb{N} : b_n \neq 0\}$, $\Delta_{\mathbb{R}} = \{n \in \Delta : b_n \in \mathbb{R}\}$ and let $w \in \mathcal{W}$, $w = \times_{n\in\mathbb{N}}\epsilon_n$. Let $(a_i)_{i\in\mathbb{N}} \subset w(b) \cap (b)w$, $a_i = \sum_{n\in\Delta^i_{\mathbb{R}}} a^i_n + \sum_{n\in\Delta_i\setminus\Delta^i_{\mathbb{R}}}(+\infty)$ for all $i \in \mathbb{N}$ such that $(a_i)_{i\in\mathbb{N}}$ converges to b in the symmetric topology of $\mathcal{Q}$ and for $\eta_i > 0$, let $\sum_{i=1}^{\infty} \eta_i = 1$. Without loss of generality we may assume that $\Delta_i = \Delta$ and $\Delta^i_{\mathbb{R}} = \Delta_{\mathbb{R}}$ for all $i \in \mathbb{N}$. Then

$$a := \sum_{n\in\Delta_{\mathbb{R}}} \sum_{i=1}^{\infty} \eta_i a^i_n + \sum_{n\in\Delta\setminus\Delta_{\mathbb{R}}} \sum_{i=1}^{\infty} \eta_i(+\infty) \in w(b) \cap (b)w$$

and for every $\mu \in \mathcal{Q}^*$ with $\mu(b) < +\infty$, we have $\mu = \times_{n\in\mathbb{N}}\mu_n$ by Proposition 2.1 (b), where

$$\mu_n = \begin{cases} \lambda_n \ (\text{some } \lambda_n > 0) & \text{if} \quad b_n \in \mathbb{R}, \\ 0_{\overline{\mathbb{R}}^*} & \text{if} \quad b = +\infty. \end{cases}$$

Thus

$$\begin{aligned}\mu(a) &= \sum_{n\in\Delta_{\mathbb{R}}}\lambda_n(\sum_{i=1}^{\infty}\eta_i a_n^i) + \sum_{n\in\Delta\setminus\Delta_{\mathbb{R}}} 0_{\overline{\mathbb{R}}^*}(\sum_{i=1}^{\infty}\eta_i(+\infty)) \\ &= \sum_{i=1}^{\infty}\eta_i\lambda_n(\sum_{n\in\Delta_{\mathbb{R}}} a_n^i) + \sum_{i=1}^{\infty}\eta_i 0_{\overline{\mathbb{R}}^*}(\sum_{n\in\Delta\setminus\Delta_{\mathbb{R}}}(+\infty)) \\ &= \sum_{i=1}^{\infty}\eta_i\mu(a_i),\end{aligned}$$

i.e., $(\sum_{n\in\mathbb{N}}\overline{\mathbb{R}}, \times_{n\in\mathbb{N}}\varepsilon)$ is weakly cone-complete.

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