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AN INVERSE DIFFUSION-WAVE PROBLEM DEFINED IN HETEROGENEOUS MEDIUM WITH ADDITIONAL BOUNDARY MEASUREMENT

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Abstract. This paper deals with an inverse problem to determine a space-dependent coefficient in a one-dimensional time fractional diffusion-wave equation defined in heterogeneous medium with additional boundary measurement. Then, we construct the explicit finite difference scheme for the direct problem based on the equivalent partial integro-differential equation and Simpson's rule. Using the matrix analysis and mathematical induction, we prove that our scheme is stable and convergent . The least squares method with homotopy regularization is introduced to determine the space-dependent coefficient, and an inversion algorithm is performed by one numerical example. This inversion algorithm is effective at least for this inverse problem.

1. Introduction

In this paper, we consider the following equation:

$$ {}^{c}\mathcal{D}_t^{\alpha} u(x,t) = \frac{\partial}{\partial x}\left(D(x)\frac{\partial u(x,t)}{\partial x}\right) + f(x,t), \quad 0<x<L,\ 0<t\le T, \tag{1.1}$$

with the initial conditions

$$ u(x,0)=\psi(x), \quad u_t(x,0)=\varphi(x), \quad 0\le x\le L, \tag{1.2}$$

and the Neumann boundary conditions

$$ \frac{\partial u(0,t)}{\partial x} = \frac{\partial u(L,t)}{\partial x} = 0, \quad 0\le t\le T, \tag{1.3}$$

where $u(x,t)$ denotes state variable at space point x and time t, and $1<\alpha<2$ is called fractional order of the derivative in time, $D(x)$ is the space-dependent

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coefficient, $f(x,t)$ is a source term, and ${}^{c}\mathcal{D}_t^{\alpha}u(x,t)$ means the Caputo derivative defined by:

$$ {}^{C}\mathcal{D}_t^{\alpha}u(x,t)=\frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}\frac{\partial^2 u(x,s)}{\partial s^2}ds. \tag{1.4}$$

In this study, we are concerned with the inverse problem of approximating the unknown space-dependent coefficient $D(x)$, while the initial functions $\psi(x)$ and $\varphi(x)$ and the source term $f(x,t)$ are considered as known functions. To determine the set of functions (u,D) in the inverse problem (1.1)-(1.3), we need an over-specified condition:

$$u(x,T)=\eta(x), \quad 0<x<L, \tag{1.5}$$

is used.

2. The direct problem

The direct problem is composed by Eq. (1.1) with the initial and boundary value conditions (1.2) and (1.3).

2.1. **The explicit finite difference scheme.** Firstly, we have the following lemma:

Lemma 2.1. *([2, 3]) Let $\alpha\in\,]1,2[$ and $y\in\mathcal{C}^2([0,T])$ with $T>0$. Then, we have*

(1) ${}_0^{C}\mathcal{D}_t^{\alpha-1}\left(\mathcal{I}_t^{\alpha-1}y(t)\right)=y(t)$,
(2) $\mathcal{I}_t^{\alpha-1}\left({}_0^{C}\mathcal{D}_t^{\alpha}y(t)\right)=\mathcal{I}_t^{\alpha-1}\left[{}_0^{C}\mathcal{D}_t^{\alpha-1}\left(y'(t)\right)\right]=y'(t)-y'(0)$,

where ${}_0^{C}\mathcal{D}_t^{\alpha}$ is the Caputo fractional derivative operator defined in (1.4) and $\mathcal{I}_t^{\alpha-1}$ is the Riemann-Liouville integral operator defined as

$$\mathcal{I}_t^{\alpha-1}g(t)=\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}g(s)\,ds. \tag{2.1}$$

Based on this lemma, we have the following theorem.

Theorem 2.2. *([2, 5]). The equation (1.1) is equivalent to the following partial integro-differential equation*

$$u_t(x,t)=\varphi(x)+\mathcal{I}_t^{\alpha-1}\left[D'(x)\,u_x(x,t)+D(x)\,u_{xx}(x,t)\right]+\mathcal{I}_t^{\alpha-1}f(x,t), \tag{2.2}$$

where $F(x,t)=\mathcal{I}_t^{\alpha-1}f(x,t)$.

We consider $\Omega_\tau=\{t_n:t_n=n\tau, 0\le n\le N\}$ a uniform mesh of the interval $[0,T]$ with $\tau=T/N$ and using Simpson's rule [3], we obtain the following lemma.

Lemma 2.3. *If $g\in\mathcal{C}^4([0,T])$ and $\alpha\in\,]1,2[$, then*

$$\mathcal{I}_t^{\alpha-1}g(t_n)=\frac{\tau^{\alpha-1}}{3\Gamma(\alpha-1)}\sum_{k=1}^{n}\omega_k g(t_{n-k})+O\left(\tau^5\right),$$

where $\omega_1=5,\ \omega_k=6k^{\alpha-2}, k=2,\dots,n-2,\ \omega_{n-1}=5(n-1)^{\alpha-2},\ \omega_n=n^{\alpha-2}$.

Let $\Omega_h = \{x_i / x_i = ih, 0 \leq i \leq M\}$ is a uniform mesh of the interval $[0, L]$ with $h = L/M$ and $M \in \mathbb{N}^*$. Suppose $u = \{u_i^n / 0 \leq i \leq M, 0 \leq n \leq N\}$ is a grid function on $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$. Considering the Eq.(2.2) at the point (x_i, t_n) and with Lemma 2.3, we obtain an explicit scheme for (2.2) in the following matrix form:

$$\begin{cases} U^0 = \psi, \\ U^1 = (I + 5A)\, U^0 + \tau\varphi + \frac{\tau^\alpha}{3\Gamma(\alpha-1)} f^0, \\ U^n = (I + 5A)\, U^{n-1} + \tau\varphi + \frac{\tau^\alpha}{3\Gamma(\alpha-1)} \sum_{k=1}^n \omega_k f^{n-k} + \displaystyle\sum_{k=2}^{n} \omega_k A U^{n-k}. \end{cases} \tag{2.3}$$

Here I is the $M-1$ order identity matrix. Where $U^n = \left(u_1^n, u_2^n, \ldots, u_{M-1}^n\right)^T, \varphi = \left(\varphi_1, \varphi_2, \ldots, \varphi_{M-1}\right)^T, \psi = \left(\psi(x_1), \ldots, \psi(x_{M-1})\right)^T$, $f^n = \left(f_1^n, f_2^n, \ldots, f_{M-1}^n\right)^T$ and $A = (a_{ij})\,, i, j = 1, 2, \ldots, M-1$ is defined by

$$A = \begin{pmatrix} -p_1 & p_1 & 0 & \ldots & 0 & 0 \\ p_2 - q_2 & q_2 - 2p_2 & p_2 & \ldots & 0 & 0 \\ 0 & p_3 - q_3 & q_3 - 2p_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & q_{M-2} - 2p_{M-2} & p_{M-2} \\ 0 & 0 & 0 & \ldots & p_{M-1} - q_{M-1} & q_{M-1} - p_{M-1} \end{pmatrix} \tag{2.4}$$

2.2. **Stability and convergence.** Firstly, we have the following lemma.

Lemma 2.4. *Suppose that $D : [0, L] \to \mathbb{R}_+$ is a continuously differentiable function on $]\,0, L\,[$. Then, the matrix A given by* (2.4) *is negative definite, and we have*

$$a_{ii} = -\sum_{j=1, j\neq i}^{M-1} |a_{ij}|\,, \quad \|A\| \leq \frac{4\tau^\alpha}{3h^2\Gamma(\alpha-1)} \max_{0\leq x\leq L} D(x)\,. \tag{2.5}$$

By utilizing linear difference scheme (2.3), we can easily get

$$\begin{cases} E^0 = \tilde{\psi} - \psi \\ E^1 = (I + 5A)\, E^0, \\ E^n = (I + 5A)\, E^{n-1} + \displaystyle\sum_{k=2}^{n} \omega_k A E^{n-k}, \end{cases} \tag{2.6}$$

where $\tilde{\psi}$ denotes the initial function with noises, $E^n = \tilde{U}^n - U^n$ denotes the solutions error for the n-th step iteration, and $n = 1, \ldots, N$.

Theorem 2.5. *The explicit difference scheme defined by* (2.3) *is unconditionally stable.*

We denote $e_i^n = u(x_i, t_n) - u_i^n$, $i = 1, \ldots, M-1$, $n = 1, \ldots, N$, where $u(x_i, t_n)$ is the exact solution of the direct problem (1.1)-(1.3) at mesh point (x_i, t_n) and u_i^n is the solution of the difference scheme (2.3) also at (x_n, t_n), and $e^n = \left(e_1^n, e_2^n, \ldots, e_{M-1}^n\right)^T$. Note that $e_i^0 = u(x_i, 0) - \psi(x_i) = 0$. We have

$$\begin{cases} e^1 = R^1, \\ e^n = (I + 5A)\, e^{n-1} + \displaystyle\sum_{k=2}^{n} \omega_k A e^{n-k} + R^n, \end{cases} \tag{2.7}$$

where $R^n = \left(R_1^n, R_2^n, \ldots, R_{M-1}^n\right)^T$ denotes the truncated term.

Theorem 2.6. *The solution of the explicit difference scheme* (2.3) *is convergent to the exact solution of the direct problem* (1.1)-(1.3) *as* $h, \tau \to 0$ *for finite time domain.*

3. The inverse problem

The inverse problem is formulated by: the fractional diffusion-wave equation (1.1), the initial conditions (1.2), the boundary conditions (1.3) and the additional condition (1.4). For the solution of the inverse problem, suppose that the function $D \in \mathcal{C}(0, L)$. Let V be a subspace of $\mathcal{C}(0, L)$ of finite dimension s and $(\eta_i(x))$, $i = 1, \ldots, s$ une base de V. We can write the diffusion-wave coefficient $D(x)$ by:

$$D(x) = \sum_{i=1}^{s} p_i \eta_i(x). \tag{3.1}$$

For $D(x)$ given, the direct problem (1.1)-(1.3) admits a unique solution noted by $u(x, t, D)$. To find $D(x)$ just find the vector $P = (p_1, p_2, \ldots, p_s)^T \in \mathbb{R}^s$. Let $\beta > 0$, we notice $S_\beta = \{P \in \mathbb{R}^s : \|P\| \leq \beta\}$ the admissible set of unknowns P.

3.1. Nonlinear least squares problem.

To solve the inverse problem we solve a nonlinear least squares problem:

$$\begin{cases} \min \Phi(P) = \|u(L, t; P) - \psi(t)\|_2^2, \quad 0 < t \leq T. \\ P \in S_\beta \end{cases} \tag{3.2}$$

The objective function Φ continuous and convex on the set S_β closed and bounded. Therefore, according to Weierstrass theorem, the problem (3.2) admits at least one solution. On the other hand the problem (3.2) is ill-posed so that the problem admits several solutions. For uniqueness, using Homotopy regularization [1], we consider the following regularized problem:

$$\begin{cases} \min \Phi_\lambda(P) = (1-\lambda)\|u(L, t; P) - \psi(t)\|_2^2 + \lambda \|P\|_2^2, \\ P \in S_\beta, \end{cases} \tag{3.3}$$

where $0 < \lambda < 1$ is the regularization parameter. To get P^j, we assume that $P^{j+1} = P^j + \delta P^j$, $j = 0, 1, \ldots$. We need to determine a regularized vector $\delta P^j = \left(\delta p_1^j, \delta p_2^j, \ldots, \delta p_s^j\right)^T$. Using Taylor's approximation to order one, we find:

$$u(L, t; P + \delta P) \approx u(L, t; P) + \nabla_P^T u(L, t; P) \cdot \delta P. \tag{3.4}$$

From (3.3) and (3.4) the objective function of the regularized problem becomes:

$$F_\lambda(\delta P) = (1-\lambda)\left\|\nabla_P^T u(L, t; P).\delta P - (\psi(t) - u(L, t; P))\right\|_2^2 + \lambda \|\delta P\|_2^2. \tag{3.5}$$

By the finite difference method, we obtain:

$$\nabla_P^T u(L, t_n; P).\delta P \approx \sum_{i=0}^{s} \frac{u(L, t_n; (p_0, \ldots, p_i + \tau, \ldots, p_s)) - u(L, t_n; p)}{\tau}.\delta p_i. \tag{3.6}$$

We define the matrix $H = (h_{ni})_{N \times (s+1)}$ by:

$$h_{ni} = \frac{u(L, t_n; (p_0, \ldots, p_i + \tau, \ldots, p_s)) - u(L, t_n; p)}{\tau}. \tag{3.7}$$

Let $U = (u(L,t_1;p), u(L,t_2;p), \ldots, u(L,t_N;p))^T$, $\Psi = (\psi(t_1), \psi(t_2), \ldots, \psi(t_N))^T$. Using (3.6) and (3.7), we can write (3.5) in the form:

$$F_\lambda(\delta P) = (1-\lambda)\left\|H\delta P - (\Psi - U)\right\|_2^2 + \lambda \left\|\delta P\right\|_2^2. \tag{3.8}$$

We have the following equivalence result:

Proposition 3.1. *([4, 5]).*

✓ *δP a minimum point of F_λ if only if δP solution of the normal equation:*

$$(1-\lambda)H^T H\delta P + \lambda\delta P = H^T(\Psi - U). \tag{3.9}$$

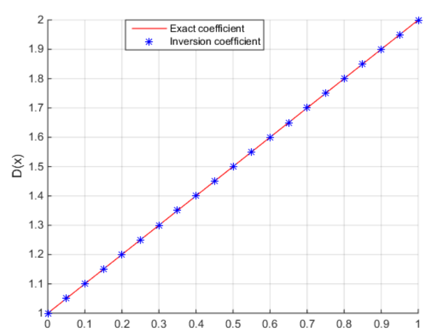
✓ *For all $0 < \lambda < 1$, the normal equation (3.9) has a unique solution.*

Algorithm 1 (Inversion algorithm, [4, 5])

1: Give an initial value P, the step τ, α, λ and ε,
2: Solve the scheme (2.3) to get $u(\ell, t_n; P)$ and $u(\ell, t_n; (p_0, \ldots, p_i + \tau, \ldots, p_s))$, for all $n = 1, 2, \ldots, N$ et $i = 0, 1, \ldots, s$
3: Calculate the matrix H and the vectors U, Ψ,
4: Calculate a regularization vector δP by: $\delta P = \left[(1-\lambda)\, H^T H + \lambda I\right]^{-1} H^T(\Psi - U)$.
5: If $\|\delta P\|_2 \leq \varepsilon$ stop, and $P + \delta P$ is considered a solution. Otherwise, go to step 2 by replacing P with $P + \delta P$.

3.2. **Numerical test.** ([4, 5])

- $T = 1$, $L = 1$, $\varphi(x) = x^2(1-x)^2$, $\psi(x) = 0$, $\lambda = 0.01$,
- $f(x,t) = \dfrac{2x^2(1-x)^2 t^{2-\alpha}}{\Gamma(3-\alpha)} - (1+t^2)(16x^3 - 6x^2 - 8x + 2)$,
- $D(x) = 1 + x$, $P^0 = (1, 1)$. $\tau = 0.4$, $\varepsilon = 10^{-6}$, $\alpha = 1.8$,
- $M = 20$, $N = 1000$.



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