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Geraghty Contractions in Ordered Uniform Spaces

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Abstract: Banach contraction principle is first and most essential result in the fixed point theory. There are many generalisations of this principle in the literature. One of them is Geraghty contraction. In this work, Geraghty type contraction was defined via E-distance and common fixed point theorems were proved for two mappings satisfying Geraghty type contraction in ordered uniform spaces. Also, some results of these theorems were obtained.

Sıralı Düzgün Uzaylarda Geraghty Büzülmeler

Anahtar Kelimeler
Sabit nokta,
E-uzaklık fonksiyonu,
Geraghty büzülme,
Düzgün uzay

Özet: Banach büzülme prensibi, sabit nokta teorinin ilk ve en önemli sonucudur. Bu prensibin literatürde pek çok genelleştirmesi vardır. Bunlardan biri de Geraghty dönüşümüdür. Bu çalışmada, sıralı düzgün uzaylarda, E-uzaklık fonksiyonu yardımıyla, Geraghty tipli büzülme tanımlanmış ve Geraghty tipli büzülmeyi sağlayan iki dönüşüm için ortak sabit nokta teoremleri ispatlanmıştır. Ayrıca bu teoremlerin bazı sonuçları elde edilmiştir.

1. Introduction

Let M be any set and the diagonal set $\Delta = \{(u,u): u \in M\}$ in M×M. Let Ѣ and Ђ are sets in M × M then,

$$Ѣ o Ђ = \{(\sigma,t): (\sigma,u) \in Ђ, (u,t) \in Ѣ\}.$$

Let ß be nonempty family of subsets of M×M,

(i) if Ѣ ∈ ß, then $\Delta \subseteq$ Ѣ,
(ii) if Ѣ ∈ ß and Ѣ ⊆ Ђ ⊆M×M, then Ђ ∈ ß,
(iii) if Ѣ, Ђ ∈ ß then Ѣ ∩ Ђ ∈ ß,
(iv) if Ѣ ∈ ß then there exists Ђ ∈ ß and if (σ,t), $(t,u) \in$ Ђ , then $(\sigma,u) \in$ Ѣ,
(v) if Ѣ ∈ ß then $\{(t,\sigma): (\sigma,t) \in Ѣ\} \in$ ß.

Then (M, ß) is a uniform space. If Ѣ ∈ ß and $(\sigma,t) \in$ Ѣ, $(t,\sigma) \in$ Ѣ then σ and t are called Ѣ-close. Also a sequence $\{u_n\}$ in M, is said to be a Cauchy sequence if for any Ѣ ∈ ß , there exists $N \geq 1$ such that u_n and u_m are Ѣ-close for $m,n \geq N$. An uniformity ß defines a unique topology $\tau($ ß $)$ on M for which the neighborhoods of $u \in M$ are the sets $Ѣ(u) = \{t \in M: (u,t) \in Ѣ\}$ when Ѣ runs over Ɣ [1].

If ∩ {Ѣ: Ѣ ∈ ß }=Δ, then (M,ß) is said to be Hausdorff (we Show H.U.S.). In Hausdorff uniform spaces, limit of a convergent sequence is unique.

Aamri and El-Moutawakil defined E-distance in uniform spaces and proved new fixed point theorems for weakly compatible contractive and expansive mappings [2]. Altun and Imdad proved fixed point theorems using a partial ordering in uniform spaces [3]. Ozturk and Turkoglu and Ozturk and Ansari gived some generalized results in ordered uniform spaces [4,5]. Recently, Olisama et.al. proved best proximity results in uniform spaces [6,7]. (Also see [8,9,10,11]).

In last years, lost of fixed point theorems obtained using concept of partially ordered relation [12,13,14].

On the other hand, Geraghty introduced a generalization of Banach contraction principle in metric spaces [15]. In this work, we will defined generalized Geraghty contraction in ordered uniform spaces for commuting mappings. In main theorem, we will prove a common fixed point theorem for two continuous mappings satisfying this type contraction. Secondly, we will use S-completeness without continuity of mappings. In last section, we will give fixed point result of main theorems for a mapping.

Definition 1.1 Let $(M,\preccurlyeq)$ be a partially ordered set and J,L: $M \rightarrow M$ be mappings. J is called L-nondecreasing if for u,v∈ M, Lu≼Lv implies Ju≼Jv [13].

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Definition 1.2 Let $(M,ß)$ be a uniform space. A function $p: M \times M \to \mathbb{R}^+$ is called an A-distance if for any Ʊ ∈ ß there exists $c>0$, such that $p(y_1,y_2)\leq c$ and $p(y_1,y_3)\leq c$ for some $y_1\in M$ imply $(y_2, y_3) \in$ Ʊ [2].

Definition 1.3 Let $(M,ß)$ be a uniform space. If

(p1) p is an A- distance,
(p2) $p(y_1,y_2)\leq p(y_1,y_3)+p(y_3,y_2)$, $\forall y_1, y_2, y_3 \in M$.

Then, $p: M \times M \to \mathbb{R}^+$ is called an E-distance [2].

Lemma 1.4 Let $(M,ß)$ be a H.U.S., p be E-distance, $\{x_n\}$,$\{y_n\} \subseteq M$ be arbitrary sequences and $\{\delta_n\}$,$\{\gamma_n\}\subseteq \mathbb{R}^+$ be convergent sequences to zero. For u,w,z$\in M$ and n$\in\mathbb{N}$,

(i) If $p(x_n, w) \leq \delta_n$ and $p(x_n, z) \leq \beta_n$ then w=z and if $p(u, w) = 0$ and $p(u, z) = 0$ then w=z.
(ii) If $p(x_n, y_n) \leq \delta_n$ and $p(x_n, z) \leq \beta_n$ then $y_n \to z$.
(iii) If $p(x_n, x_l) \leq \delta_n$ every l>n, then $\{x_n\} \subseteq M$ is a Cauchy sequence [2][3].

A sequence in M is called p-Cauchy if it satisfies usual metric condition.

Definition 1.5 Let $(M,ß)$ be a uniform space.

i) M is called S-complete if for every p-Cauchy sequence $\{u_n\}$ there exists u$\in M$ with $\lim_{n\to\infty} p(u_n, u) = 0$,
ii) M is called p-Cauchy complete if for every p-Cauchy sequence $\{u_n\}$ there exists u$\in M$ with $\lim_{n\to\infty} u_n = u$ with respect to τ(ß),
iii) H: $M \to M$ is p-continuous if $\lim_{n\to\infty} p(u_n, u) = 0$ implies $\lim_{n\to\infty} p(Hu_n, Hu) = 0$,
iv) H: $M \to M$ is τ(ß)-continuous if $\lim_{n\to\infty} u_n = u$ implies $\lim_{n\to\infty} Hu_n = Hu$ respect to τ(ß) [2][3].

Remark 1.6 S-completeness implies p-Cauchy completeness [2].

2. Fixed Point Results

In this work, we suppose (M,Ɣ,≼) be an ordered H.U.S. , p be an E-distance on S-complete space M and Γ= $\{\beta: [0, \infty) \to [0,1) : \beta(t_n) \to 1 \Rightarrow\ t_n \to 0\}$.

Theorem 2.1 Let $J, L: M \to M$ be two commuting selfmappings with $J(M) \subseteq L(M)$ such that

(i) J,L are p-continuous or τ(Ɣ)-continuos,
(ii) J is L-nondecreasing,
(iii) there exists $u_0\in$M such that $Lu_0 \preccurlyeq Ju_0$,
(iv) $p(Ju, Jv) \leq \beta\big(p(Lu, Lv)\big)p(Lu, Lv)$
for all $u, v \in M$ with $Lu \preccurlyeq Lv$. Then, J and L have a unique common fixed point.

Proof. Let $u_0 \in M$ be as in (iii). Since $J(M) \subseteq L(M)$, for $u_1 \in X$, $Ju_0 = Lu_1$. Then $Lu_0 \preccurlyeq Ju_0 = Lu_1$. Using (ii), $Ju_0 \preccurlyeq Ju_1$. Continuing this way,

$$Lu_n = Ju_{n-1} \tag{1}$$

for which

$$Lu_0 \preccurlyeq Ju_0 = Lu_1 \preccurlyeq Ju_1 = Lu_2$$
$$\preccurlyeq \cdots \preccurlyeq Ju_{n-1} = Lu_n \preccurlyeq \cdots.$$

From (iv),

$$\begin{aligned} &p(Ju_n, Ju_{n+1}) \\ &\leq \beta\big(p(Lu_n, Lu_{n+1})\big)p(Lu_n, Lu_{n+1}) \\ &\leq \beta\big(p(Ju_{n-1}, Ju_n)\big)p(Ju_{n-1}, Ju_n) \\ &\leq p(Ju_{n-1}, Ju_n). \end{aligned}$$

Then, $\{p(Ju_n, Ju_{n+1})\}$ is a nonincreasing and bounded below. So converging to some r≥0. Suppose that r>0. Then it follows

$$\frac{p(Ju_n, Ju_{n+1})}{p(Ju_{n-1}, Ju_n)} \leq \beta\big(p(Ju_{n-1}, Ju_n)\big) \leq 1$$

passing to the limit when n→∞, we have

$$\beta\big(p(Ju_{n-1}, Ju_n)\big) = 1$$

By definition of β,

$$\lim_{n\to\infty} p(Ju_{n-1}, Ju_n) = 0 \tag{2}$$

Similarly we can show that $\lim_{n\to\infty} p(Ju_n, Ju_{n-1}) = 0$.

Now, suppose that $\{Ju_n\}$ is not a Cauchy sequence in M. Then there exists an ε>0 for which we can find two sequences of positive integers $\{m(\omega)\}$ and $\{n(\omega)\}$ such that, $n(\omega) > m(\omega) > \omega$ for all positive integers ω,

$$p(Ju_{n(\omega)}, Ju_{m(\omega)}) \geq \varepsilon,$$
$$p\big(Ju_{n(\omega)-1}, Ju_{m(\omega)}\big) < \varepsilon.$$

Using (p2),

$$\varepsilon \leq\ p\big(Ju_{n(\omega)}, Ju_{m(\omega)}\big)$$
$$\leq p\big(Ju_{n(\omega)}, Ju_{n(\omega)-1}\big) + p\big(Ju_{n(\omega)-1}, Ju_{m(\omega)}\big).$$

Thus,

$$\varepsilon \leq\ p\big(Ju_{n(\omega)}, Ju_{m(\omega)}\big) \leq p\big(Ju_{n(\omega)}, Ju_{n(\omega)-1}\big) + \varepsilon$$

letting ω→∞ in the above inequality, from (2)

$$\lim_{\omega\to\infty} p\big(Ju_{n(\omega)}, Ju_{m(\omega)}\big) = \varepsilon.$$

On the other hand, from (p2), we get

$$p(Ju_{n(\omega)},Ju_{m(\omega)}) \leq p(Ju_{n(\omega)},Ju_{n(\omega)-1}) + p(Ju_{n(\omega)-1},Ju_{m(\omega)-1}) + p(Ju_{m(\omega)-1},Ju_{m(\omega)})$$

and

$$p(Ju_{n(\omega)-1},Ju_{m(\omega)-1}) \leq p(Ju_{n(\omega)-1},Ju_{n(\omega)}) + p(Ju_{n(\omega)},Ju_{m(\omega)}) + p(Ju_{m(\omega)},Ju_{m(\omega)-1}).$$

As ω→∞,

$$\lim_{\omega\to\infty} p(Ju_{n(\omega)-1},Ju_{m(\omega)-1}) = \varepsilon.$$

From (iv),

$$\begin{aligned} &p(Ju_{n(\omega)},Ju_{m(\omega)}) \\ &\leq \beta\left(p(Lu_{n(\omega)},Lu_{m(\omega)})\right) p(Lu_{n(\omega)},Lu_{m(\omega)}) \\ &= \beta\left(p(Ju_{n(\omega)-1},Ju_{m(\omega)-1})\right) p(Ju_{n(\omega)-1},Ju_{m(\omega)}. \end{aligned}$$

Thus, we have

$$\frac{p(Ju_{n(\omega)},Ju_{m(\omega)})}{p(Ju_{n(\omega)-1},Ju_{m(\omega)-1})} \leq \beta\left(p(Ju_{n(\omega)-1},Ju_{m(\omega)-1})\right) < 1$$

and

$$\lim_{\omega\to\infty} \beta\left(p(Ju_{n(\omega)-1},Ju_{m(\omega)-1})\right) = 1.$$

Using definiton of β

$$\lim_{\omega\to\infty} p(Ju_{n(\omega)-1},Ju_{m(\omega)-1}) = 0$$

which is a contradiction with ε>0. Hence $\{Ju_n\}$ is a Cauchy sequence in M.

Since M is S-complete, then there exists a z∈M such that

$$\lim_{n\to\infty} p(Ju_n,z) = 0, \qquad \lim_{n\to\infty} p(Lu_n,z) = 0.$$

Moreover, the p-continuity of J and L implies that

$$\lim_{n\to\infty} p(LJu_n,Lz) = \lim_{n\to\infty} p(JLu_n,Jz) = 0.$$

Since J and L are commuting, then $LJ = JL$. So we have

$$\lim_{n\to\infty} p(JLu_n,Lz) = \lim_{n\to\infty} p(JLu_n,Jz) = 0.$$

By Lemma 1.4 (i), $Jz = Lz$.

Now we will prove that, J and L have common fixed point.

Since, $JL = LJ$ we have $JJz = JLz = LJz = LLz$. Suppose that $p(Jz,JJz) \neq 0$. From (iv) and definition of β,

$$\begin{aligned} p(Jz,JJz) &\leq \beta(p(Lz,LJz))p(Lz,LJz) \\ &= \beta(p(Jz,JJz))p(Jz,JJz) < p(Jz,JJz) \end{aligned} \tag{3}$$

which is a contradiction. Thus $p(Jz,JJz) = 0$. Suppose $p(Jz,Jz) \neq 0$. From (iv), we have

$$\begin{aligned} p(Jz,Jz) &\leq \beta(p(Lz,Lz))p(Lz,Lz) \\ &= \beta(p(Jz,Jz))p(Jz,Jz) < p(Jz,Jz) \end{aligned} \tag{4}$$

which is a contradiction. Thus by (3), (4) and Lemma 1.4 (i), we have $JJz = Jz$. Hence Jz is a common fixed point of J and L.

Now, we show uniqueness.

Assume that there exists μ_1, μ_2,∈M such that $J\mu_1 = L\mu_1 = \mu_1$ and $J\mu_2 = L\mu_2 = \mu_2$. If $p(\mu_1,\mu_2) \neq 0$, then by (iv),

$$\begin{aligned} p(J\mu_1,J\mu_2) &\leq \beta(p(L\mu_1,L\mu_2))p(L\mu_1,L\mu_2) \\ &< p(L\mu_1,L\mu_2), \end{aligned}$$

this is a contradiction. Thus $p(\mu_1,\mu_2) = 0$. Similarly, we can prove $p(\mu_2,\mu_1) = 0$. By (p2)

$$p(\mu_1,\mu_1) \leq p(\mu_1,\mu_2) + p(\mu_2,\mu_1)$$

and therefore $p(\mu_1,\mu_1) = 0$. Since, $p(\mu_1,\mu_2) = 0$ and $p(\mu_1,\mu_1) = 0$, from Lemma 1.4 (i), $\mu_1 = \mu_2$.

The proof is similar when J and L are τ(ß)-continuous.

Theorem 2.2 Let J, L:M→M be two commuting selfmappings with $J(M) \subseteq L(M)$ such that

(i) $L(M)$ is S-complete,
(ii) J is L-nondecreasing,
(iii) there exists u_0∈M such that $Lu_0 \preccurlyeq Ju_0$,
(iv) $p(Ju,Jv) \leq \beta(p(Lu,Lv))p(Lu,Lv)$
for all u,v∈M with $Lu \preccurlyeq Lv$.

Then, J and L are have a common fixed point.

Proof. Following the proof of Theorem 2.1, we know that $\{Ju_n\}$ is a p-Cauchy sequence. Since by (1), we have $\{Ju_n\}$=$\{Lu_{n+1}\}\subseteq L(M)$ and by (i)

$$\lim_{n\to\infty} p(Lu_{n+1},Lz) = \lim_{n\to\infty} p(Ju_n,Lz) = 0. \tag{5}$$

We show that $Jz = Lz$. Using (5) and (iv),

$$p(Ju_n,Jz) \leq \beta(p(Lu_n,Lz))p(Lu_n,Lz).$$

Letting n→∞, we have

$$\lim_{n\to\infty} p(Ju_n, Jz) = 0. \qquad (6)$$

Hence by (5), (6) and Lemma 1.4 (i), $Jz = Lz$. Proof proceeds similarly with Theorem 2.1.

Example 2.3 Let $M = [0,1]$equipped with $d(\mu_1, \mu_2) = |\mu_1 - \mu_2|$ and suppose $\mu_1 \preccurlyeq \mu_2 \Leftrightarrow \mu_2 \leq \mu_1$ and ß = {Ƕ ⊂ $M \times M$: $\Delta \subset$ Ƕ}. Define the function p as $p(\mu_1, \mu_2) = \mu_2$ for all μ_1, μ_2 in M and

$$\beta{:}[0,\infty)\to[0,1)\,,\ \beta(t) = \begin{cases} \frac{1}{2}, if\ t = 0 \\ \frac{1}{1+t}, if\ t > 0 \end{cases}$$

$J,L{:}M\to M$ defined by $J(t) = \frac{t}{8}$ and $L(t) = \frac{t}{2}$.

Thus, M is S-complete and $\cap_{V\in ß} V = \Delta$ and (M, ß) is Hausdorff uniform space. p is an E-distance. J,L are commuting, p-continuous and J is L-nondecreasing. If $\mu_2 = 0$, then all conditions of Theorem 2.1 are satisfy. If $\mu_2 \neq 0$, then

$$p(J\mu_1, J\mu_2) = p\left(\frac{\mu_1}{8}, \frac{\mu_2}{8}\right) = \frac{\mu_2}{8}$$
$$\leq \frac{2}{2+\mu_2}\frac{\mu_2}{2} = \beta(p(L\mu_1, L\mu_2))p(L\mu_1, L\mu_2).$$

And zero is the unique common fixed point of J and L.

3. Discussion and Conlusions

If L is idendity function in main theorem we give following result.

Corollary 3.1 Let $J{:}M\to M$ be a p-continuous or τ(ß)-continuos, nondecreasing selfmapping such that for all comparable u,v∈M with

$$p(Ju, Jv) \leq \beta\big(p(u, v)\big)p(u, v).$$

If there exists $u_0 \in M$ with $u_0 \preccurlyeq J(u_0)$, then J has a unique fixed point.

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References

[1] Willard, S. 1970. General Topology. Addison-Wesley Publishing.

[2] Aamri, M., El Moutawakil, D. 2004. Common fixed point theorems for E-contractive or E-expansive maps in uniform spaces. Acta Mathematica Academiae Peadegogicae Nyiregyhaziensis, 20, 83-91.

[3] Altun, I., Imdad, M. 2009. Some fixed point theorems on ordered uniform spaces. Filomat, 23(3), 15-22.

[4] Turkoglu, D., Ozturk, V. 2014. (psi-phi)-weak contraction on ordered uniform spaces. Filomat, 28(6), 1265—1269.

[5] Ozturk, V., Ansari, A.H. 2017. Fixed point theorems for (F,psi,phi)-contractions on ordered S-complete Hausdorff uniform spaces. New Trends in Mathematical Sciences, 5(1), 243-249.

[6] Olisama, V., Olaleru, J., Akewe, H., 2017. Best proximity point results for some contractive mappings in uniform spaces. International Journal of Analysis, 2017, Article ID 6173468.

[7] Olisama, V., Olaleru, J., Akewe, H., 2018. Best proximity point results for Hardy–Rogers p-proximal cyclic contraction in uniform spaces. Fixed Point Theory Appl., 2018, Article ID 18.

[8] Olatinwo, M.O. 2007. Some common fixed point theorems for self-mappings in uniform space. Acta Mathematica Academiae Peadegogicae Nyiregyhaziensis, 23, 47-54.

[9] Aamri, M., Bennari, S., El Moutawakil, D. 2006. Fixed points and variational principle in uniform spaces. Siberian Electronic Mathematical Reports, 3, 137-142.

[10] Olatinwo, M.O. 2008. On some common fixed point theorems of Aamri and El Moutawakil in uniform spaces. Applied Mathematics E-Notes, 8, 254-262.

[11] Shobkolaei, N., Sedghi, S., 2016. Suzuki-type fixed point results for E-contractive maps in uniform spaces. Thai Journal of Mathematics, 14(3), 575-583.

[12] Ran, A.C.M., Reurings, M.C.B. 2004. A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Amer. Math. Soc., 132, 1435-1443.

[13] Nieto, J.J., Lopez, R.R. 2005. Contractive mapping theorems in partially ordered sets applications to ordinary differantial equations. Order., 22, 223-239.

[14] Ciric, L.J., Cakić, N., Rajović, M., Ume, J.S. 2008. Monotone generalized nonlinear contractions in partially ordered metric spaces. Fixed Point Theory Appl., 2008, Article ID 131294.

[15] Geraghty, M. 1973. On contractive mappings. Proc Amer Math Soc., 40, 604-608