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THE STRUCTURE' VORTEX OF CONDENSED GAS IN TRAPPED CLOUD

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ABSTRACT

Bose-Einstein condensation, which has a very low particle density and a highly complex quantum structure, has been experimentally proven to date for rubidium, sodium, lithium, hydrogen, metastable helium, cesium and chromium atoms. Due to the differences between the properties and binary interactions of these atoms, experimental studies on condensation have found many impressive results. The structure of a vortex in a trapped environment and the factors affecting it were investigated in this study. Angular Momentum was calculated using Thomas-Fermi Approach.

Keywords: Bose-Einstein Condensation, Vortex, Dynamics of Condensation

1. INTRODUCTION

The Bose-Einstein concentration, first proposed by Einstein in 1925 for the first time, was observed experimentally in 1995 in the rubidium, sodium and lithium alkali atoms. Bose-Einstein estimates that condensation is for particles that do not interact under certain temperatures. A phase transition will occur with each other by condensing the macroscopic distribution of the gas.

At the center of the atomic cloud of the Bose-Einstein condensate, the particle density is in the order of 10^{13}-10^{15} cm^{-3}. This can be interpreted as the dilution of the Bose-Einstein condensation when compared to the density of air molecules at room temperature and atmospheric pressure, 10^{19} cm^{-3}. In systems with such low density, the temperature must be in the order of 10^{-5} K for the quantum phenomenon to be examined.

Bose-Einstein condensate has been experimentally obtained to date for rubidium, sodium, lithium, hydrogen, metastable helium, cesium and chromium atoms. Due to the differences between the properties of these atoms and their binary interactions, many impressive results have been obtained from experimental studies on condensation. When nuclear and electronic spin rates are included in the system, the system content is richer [1].

2. FORMALISM

Bose-Einstein gas properties that do not interact with each other in the trap can be determined by statistical mechanics. The Bose distribution function is given by

$$f^0(\in_\upsilon) = \frac{1}{e^{(\in_\upsilon - \mu)/kT} - 1} \quad (1)$$

for the thermal balanced uninterrupted bosons, where the average number of settlements of the single-particle state υ is the single-particle state energy $\in_\upsilon$ for a given trapped potential. For a free particle

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in a given 3-dimensional internal state, there is an average state in each volume $(2\pi h)^3$ of the phase space. The momentum space region, which is less than the *p* momentum, has a cubic volume ($4\pi p^3/3$) with a *p* radius, since it is the energy of the particle with momentum $\vec{p}$ is given by $\in_p = p^2/2m$, the total number of states $G(\in)$ with lower energies than the energy $\in$ is given by

$$G(\in) = V\frac{4\pi}{3}\frac{(2m\in)^{3/2}}{(2\pi\hbar)^3} = V\frac{2^{1/2}}{3\pi^2}\frac{(m\in)^{3/2}}{\hbar^3} \quad (2)$$

where V is the volume in this system.

The number of states between $\in$ and $\in + d\in$ energetic levels is usually given by $g(\in)$ the state density:

$$g(\in) = \frac{dG(\in)}{d\in} \quad (3)$$

The T_c transition temperature can also be defined as the maximum temperature at which the macroscopic settlement occurs in the lowest-energy state. The number of particles in excited levels is given by

$$N_{ex} = \int_0^\infty d\in g(\in) f^0(\in) \quad (4)$$

When $\mu = 0$, it takes its maximum value. The transition temperature is T_c when examined for the case where the total number of particles is in the evoked levels:

$$N = N_{ex}(T_c, \mu = 0) = \int_0^\infty d\in g(\in)\frac{1}{e^{\in/kT_c} - 1} \quad (5)$$

The number density ($n = N/V$) defined as the number of particles per unit volume for uniform Bose gas in the *V* volume three-dimensional box can be expressed by the Equation (6).

$$kT_c = \frac{2\pi}{[\zeta(3/2)]^{2/3}}\frac{\hbar^2 n^{2/3}}{m} \approx 3{,}31\frac{\hbar^2 n^{2/3}}{m} \quad (6)$$

N_{ex} the number of particles in the excited levels below the T_c transition temperature, as in Equation (4) ($\mu = 0$):

$$N_{ex}(T) = C_\alpha \int_0^\infty d\in \in^{\alpha-1}\frac{1}{e^{\in/kT} - 1} \quad (7)$$

For particles ($\alpha = 3/2$) in the three-dimensional box, the number of excited particles n_{ex} in the unit volume is given by Equation (7):

$$n_{ex} = \frac{N_{ex}}{V} = \zeta(3/2)\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} \quad (8)$$

Utilizing the thermodynamic properties of the ideal Bose gas, energy, entropy and other properties of condensed phase, can be determined. Since the number of basal augmentations and the interactions between particles are low in the T_c temperatures below, the chemical potential is lifted off and the internal energy is given by

$$E = C_\alpha \int_0^\infty d \in \in^{\alpha-1} \frac{\in}{e^{\in/kT} - 1} = C_\alpha \Gamma(\alpha+1)\zeta(\alpha+1)(kT)^{\alpha+1} \tag{9}$$

The specific heat $C = \partial E / \partial T$ is therefore given by;

$$C = (\alpha+1)\frac{E}{T} \tag{10}$$

The intrinsic heat is also found Equation (11), because it can be expressed by using the term entropy S by $C = T\partial S / \partial T$:

$$S = \frac{C}{\alpha} = \frac{\alpha+1}{\alpha}\frac{E}{T} \tag{11}$$

It should be noted here that under T_c temperature, energy, entropy and specific heat are not dependent on the total number of particles.

When these results are examined in the classical limit, the result that the Bose-Einstein distribution transforms into the Boltzmann distribution at high temperatures is reached. The total number of particles and energy is Equation (12) and Equation (13) in this case:

$$N = C_\alpha \int_0^\infty d \in \in^{\alpha-1} e^{(\mu-\in)/kT} \tag{12}$$

$$E = C_\alpha \int_0^\infty d \in \in^{\alpha} e^{(\mu-\in)/kT} \tag{13}$$

In the case of full condensation, all bosons are at the same $\phi(\mathbf{r})$ single-particle level, and for this reason the wave function is given by

$$\psi(\mathbf{r_1}, \mathbf{r_2}, ..., \mathbf{r_N}) = \prod_{i=1}^{N} \phi(\mathbf{r}_i) \tag{14}$$

for the system consisting of N particles. If the single-particle wave function is normalized and written energy, then

$$E(\psi) = \int d\mathbf{r} \left[\frac{\hbar^2}{2m} |\nabla \psi(\mathbf{r})|^2 + V(\mathbf{r})|\psi(\mathbf{r})|^2 + \frac{1}{2}U_0 |\psi(\mathbf{r})|^4 \right] \tag{15}$$

is found. By equating the $E - \mu N$ variation to zero according to ψ^*, the time-independent Gross-Pitaevski equation in the form of

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) + U_0|\psi(\mathbf{r})|^2\psi(\mathbf{r}) = \mu\psi(\mathbf{r}) \tag{16}$$

is reached [2].

3. DYNAMICS OF CONDENSATION

The continuity equation is derived to understand the nature of the velocity of the condensate. When the time-dependent Gross-Pitaevski equilibrium is multiplied by $\psi^*(\mathbf{r},t)$, the complex conjugate of the equation obtained is subtracted and

$$\frac{\partial|\psi|^2}{\partial t}+\nabla\cdot\left[\frac{\hbar}{2mi}\left(\psi^*\nabla\psi-\psi\nabla\psi^*\right)\right]=0 \tag{17}$$

result is reached. This is the same as that obtained from the linear Schrödinger equation because the nonlinear potential in the Gross-Pitaevski equation is the reel.

4. ROTATING CONDENSATION

One of the properties of superfluidity is the response to rotation. In addition, for charged superfluids, the response to the magnetic field is also a evidence. One of the important properties of superfluids is the result of the forced movements that occur due to the rate of coagulation being proportional to the gradient of the phase of the wave function. The speed of a condensate is the gradient of the ϕ scalar:

$$\mathbf{v}=\frac{\hbar}{m}\nabla\phi \tag{18}$$

This equation is one of the consequences of the condensation's possible actions, affecting many features of the condensation.

In general, the condensate wave function is monovalent. The change in $\Delta\phi$ in the wave function phase around a closed contour must be a multiple of 2π. Or it must have

$$\Delta\phi=\oint\nabla\phi\cdot d\mathbf{l}=2\pi l \tag{19}$$

conditions, an integer l. Thus, G circulation around a closed contour:

$$\Gamma=\oint\mathbf{v}\cdot d\mathbf{l}=\frac{\hbar}{m}2\pi l=l\frac{h}{m} \tag{20}$$

Equation (20) shows that the circulation is quantized with h/m. The value of the quantum circulation is about $(4{,}0\times10^{-7}/A)$ m^2 s^{-1}, which A is the mass number. As a simple example of such a flow, a completely azimuthal flow in the invariant trap under rotation around the axis can be considered. In order to provide a single valued condition, the condensation wave function $e^{il\varphi}$ must be changed to φ with the azimuthal angle. In order to satisfy the condition of being a mono-valued condition, the condensate wave function must be changed by $e^{il\varphi}$. Here φ is azimuth angle.

ρ is the distance from the axis of the trap, velocity equation;

$$v_\varphi=l\frac{h}{2\pi m\rho} \tag{21}$$

is reached. The angular momentum per particle is not quantized, but the circulation of the other levels is quantized. The generalized state is

$$\nabla \times \mathbf{v} = \hat{z}\frac{lh}{m}\delta^2(\boldsymbol{\rho}) \tag{22}$$

where $\nabla \times \mathbf{v} = 0$ is for the level having the vortex extending along the z-axis. Where δ^2 is the two-dimensional Dirac delta function in the xy plane, and $\boldsymbol{\rho} = (x, y)$ and $\hat{z}$ are the unit vector in the z direction. When there are a large number of vortices, the right side of the above equation is the sum of the two-dimensional delta functions on the perpendicular surfaces in the direction of the vortex line. The intensity of the delta function is a vector oriented along the vortex line, and the value is equal to the value of the circulation associated with the vortex [3].
If the wave function for the trap with axial symmetry is $e^{il\varphi}$, the wave function of the condensation is

$$\psi(\mathbf{r}) = f(\rho, z)e^{il\varphi} \qquad (23).$$

in the spherical polar coordinates. f is a real number. Equation (15) uses for the energy value:

$$E = \int d\mathbf{r}\left\{\frac{\hbar^2}{2m}\left[\left(\frac{\partial f}{\partial \rho}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2\right] + \frac{\hbar^2}{2m} l^2 \frac{f^2}{\rho^2} + V(\rho, z) f^2 + \frac{U_0}{2} f^4\right\} \tag{24}$$

The difference between the energy obtained from Equation (24) and the energy obtained when the phase is not dependent on the position of the condensation is the addition of the term $1/\rho^2$. This leads to a $mf^2 v_\varphi^2 / 2 = \hbar^2 l^2 f^2 / 2m\rho^2$ increase in the kinetic energy density and is a result of the azimuthal movement of the condensate. The value of the amplitude of the vortex wave function can be obtained from the Gross-Pitaevski equation in Equation (16):

$$-\frac{\hbar^2}{2m}\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{df}{d\rho}\right) + \frac{d^2 f}{dz^2}\right] + \frac{\hbar^2}{2m\rho^2} l^2 f + V(\rho, z) f + U_0 f^3 = \mu f \tag{25}$$

5. VORTEX IN UNIFORM MEDIUM

When the infinite mean of the uniform potential $V(\rho, z) = 0$ is taken into account, the wave function in the base state is not dependent on z, and therefore the derivatives related to z are off. For distances farther distally, the radial derivative and the centrifugal barrier in the form of $\propto 1/\rho^2$ lose their significance and therefore the value of the choke wave function becomes $f = f_0 \equiv (\mu/U_0)^{1/2}$. The derivative and centrifugal superposition dominates near the axis and the appropriate solution on the axis behaves like r in a free particle with unit angular momentum in two dimensions. The terms in the Gross-Pitaevski equation in Equation 25 show that, in the course of the distance from the axis to a certain distance, the transition between the two states occurs. For this reason, it is possible to scale the lengths taking advantage of a certain length of ξ [3]:

$$\frac{\hbar^2}{2m\xi^2} = nU_0 = \mu \tag{26}$$

Where $n = f_0^2$ is the intensity in the distant regions of the vortex. f_0 is the amplitude of the vortex wave function in distant regions from the vortices, and . f_0 is the energy density when $\chi = f / f_0$ is transformed:

$$\varepsilon = n^2 U_0 \left[\left(\frac{d\chi}{dx} \right)^2 + \frac{\chi^2}{x^2} + \frac{1}{2}\chi^4 \right] \tag{27}$$

Gross-Pitaevski's expression in Equation (27)

$$-\frac{1}{x}\frac{d}{dx}\left(x\frac{d\chi}{dx} \right) + \frac{\chi}{x^2} + \chi^3 - \chi = 0 \tag{28}$$

is found [4].

Another magnitude that must be taken into account for calculating the energies of the annulus is the extra energy that represents the energy of the particle and the enthalpy of the same number in uniform level. The $\in$ energy in the unit length of the vortex is:

$$\in = \int_0^b 2\pi\rho d\rho \left[\frac{\hbar^2}{2m}\left(\frac{df}{d\rho} \right)^2 + \frac{\hbar^2}{2m}\frac{f^2}{\rho^2} + \frac{U_0}{2} f^4 \right] \tag{29}$$

The number of particles per unit length is

$$\nu = \int_0^b 2\pi\rho d\rho f^2 = \pi b^2 f_0^2 - \int_0^b 2\pi\rho d\rho \left(f_0^2 - f^2 \right) \tag{30}$$

Thus, the uniform system has a unit-length energy

$$\in_0 = \frac{1}{2}\pi b^2 f_0^4 U_0 - f_0^2 U_0 \int_0^b 2\pi\rho d\rho \left(f_0^2 - f^2 \right) \tag{31}$$

The energy $\in_\nu$ at unit length pertaining to the vortex is the difference between Equations 28 and 30:

$$\in_\nu = \int_0^b 2\pi\rho d\rho \left[\frac{\hbar^2}{2m}\left(\frac{df}{d\rho} \right)^2 + \frac{\hbar^2}{2m}\frac{f^2}{\rho^2} + \frac{U_0}{2}\left(f_0^2 - f^2 \right)^2 \right] = \frac{\pi\hbar^2}{m} n \int_0^{b/\xi} x dx \left[\left(\frac{d\chi}{dx} \right)^2 + \frac{\chi^2}{x^2} + \frac{1}{2}\left(1 - \chi^2 \right)^2 \right] \tag{32}$$

The above expression is found in

$$\in_\nu = \pi n \frac{\hbar^2}{m} \ln\left(1{,}464 \frac{b}{\xi} \right) \tag{33}$$

when it is used in the numerical solution of the Gross-Pitaevski equation. This result was obtained by Ginzburg and Pitaevski in the phenomenological theory of liquid ^{4}He around T_λ. The mathematical form of the theory is identical to the Gross-Pitaevsky theory for zero temperature coherence; but the physical appearance of constants is different. Equation (32) can be used as a basis for the variational solution of the vortex wave function. In general, the energy is minimized by using the test form for f, taking into account the parameters of the test function. When the trial wave function at source [4] is used, the result

$$\chi = \frac{x}{\left(\alpha + x^2 \right)^{1/2}} \tag{34}$$

is reached.

The correct results are achieved for both small and large length values, with a being the optimum value of 2. μ the chemical potential is kept constant. As in Equation (32), the vortex energy is equal to the minimization of $E-\mu N$. Where E is the total energy. $\in_{\nu}=\pi\left(n\hbar^{2}/m\right)\ln\left(1{,}497b/\xi\right)$ is a value close to the original result in the form of variant $\alpha=2$. According to the chatter wave function in Equation (23), each particle carries one unit of angular momentum and is therefore given by the total angular momentum

$$\mathcal{L}=\nu\hbar \tag{35}$$

in the unit length.

6. VORTEX IN A TRAPED CLOUD

The vortex energy in the Bose-Einstein condensate cloud in the trap is important for predicting the lowest angular velocity that is appropriate for energy in order to generate vortex in the cloud. For this purpose, considering the rotationally invariant harmonic trap potential around the z-axis and the number of atoms is sufficiently high for the Thomas-Fermi approach to be good, the localized vortex core radius on the z-axis of the trap determined by the consistent length is short compared to the cloud size.

Considering the two-dimensional problem by neglecting the trap force in the z-direction, the cloud is cylindrical to have the radius ρ_2 and the energy of unit length

$$\in_{\nu}=\pi n_0\frac{\hbar^{2}}{m}\ln\left(1{,}464\frac{\rho_1}{\xi_0}\right)+\frac{1}{2}\int_{\rho_1}^{\rho_2}mn(\rho)\nu^{2}(\rho)2\pi\rho d\rho \tag{36}$$

where n_0 is the particle density when there is no vortex for $\rho\to 0$, and ξ_0 is the consistent length determined for this density value. Since the value of the velocity is $v=h/2\pi\rho m$, the density in the harmonic trap varies with $1-\rho^{2}/R^{2}$ in the Thomas-Fermi approach.

The angular momentum $\mathcal{L}$ per unit length is obtained by multiplying the total number of particles $\hbar$ per unit length. The second case for $\rho\rangle\rangle\xi$ is found using the Thomas-Fermi approach as follows:

$$\mathcal{L}=n_0\hbar\int_{0}^{\rho_2}\left(1-\frac{\rho^{2}}{\rho_2^{2}}\right)2\pi\rho d\rho=\frac{1}{2}n_0\pi\rho_2^{2}\hbar \tag{37}$$

When the three-dimensional problem is considered, z-axis is formed in the Z-direction and the size of the cloud in this direction is greater than the consistent length. In this case, the energy of the cloud can be asserted as the sum of the horizontal cloud slice energies, so that kinetic energy from the vertical gradient of the condensation wave function is neglected. The total kinetic energy and is integrated over the vertical axis of the cloud and found:

$$E=\frac{\pi\hbar^{2}}{m}\int_{-Z}^{Z}dzn_0(z)\ln\left[0{,}888\frac{\rho_2(z)}{\xi_0(z)}\right] \tag{38}$$

Thus, energy is obtained as follows:

$$E = \frac{\pi\hbar^2 n(0)}{m} \int_{-Z}^{Z} dz \left(1 - \frac{z^2}{Z^2}\right) \ln\left[0{,}888 \frac{R}{\xi_0}\left(1 - \frac{z^2}{Z^2}\right)\right] \tag{39}$$

$\int_0^1 dy(1-y^2)\ln(1-y^2) = (12\ln 2 - 10)/9$ is used,

$$E = \frac{4\pi n(0)}{3} \frac{\hbar^2}{m} Z \ln\left(0{,}671 \frac{R}{\xi_0}\right) \tag{40}$$

is reached. This result is well consistent with numerical calculations for large clouds.
Total angular momentum:

$$L = \hbar \int d\mathbf{r} n(\mathbf{r}) = n(0)\hbar \int_{-Z}^{Z} dz \int_0^{\rho_2(z)} 2\pi\rho d\rho \left(1 - \frac{\rho^2}{R^2} - \frac{z^2}{Z^2}\right) \tag{41}$$

$$= \frac{8\pi}{15} n(0) R^2 Z\hbar \tag{42}$$

7. CONCLUSION

The vortex energy in the Bose-Einstein condensate cloud in the trap is important for predicting the lowest angular velocity that is appropriate for energy in order to generate vortex in the cloud. Angular momentum for a vortex in a uniform environment is a result of rotational symmetry. The angle of the angular momentum for the cloud that is not in the input trap axis depends on the position of the input. In addition, the angular momentum is not maintained for an invariant trap by rotating about the axis of rotation, and therefore has no particular value. The expression of total angular momentum is simply a velocity-dependent magnitude.

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