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RESEARCH ARTICLE

COMPARISONS OF SOME BIASED ESTIMATORS FOR LINEAR MEASUREMENT ERROR MODELS

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ABSTRACT

Measurement errors are very often come upon in data analysis. Classical statistical methods become disadvantageous and ordinary least squares estimator of parameters turns into inconsistent and biased, in the existence of measurement errors in the data. Although some methods are used, the performances of them are not good enough in the presence of multicollinearity and measurement errors in the data, simultaneously. That's why researchers have been inquiring about the estimation of the parameters if the measurement error models have multicollinearity, lately. Especially, biased estimation techniques have been researched in the existence of multicollinearity for measurement error models recently. In this paper, the ridge and Liu estimation approaches to the measurement error models in the existence of multicollinearity are investigated. The comparisons of the biased estimators' performances are analyzed theoretically and numerically.

Keywords: Measurement Error Models, Reliability Matrix, Ridge Estimator, Liu Estimator

1. MODEL AND ESTIMATORS

Let us consider the multiple linear measurement error (ME) models

$$Y = \beta_0 + x\beta + e, \quad X = x + u, \tag{1}$$

where β_0 is the slope and $\beta = (\beta_1, \beta_2, ..., \beta_p)'$ is the $p \times 1$ vector of coefficients, $x = (x_1, x_2, ..., x_p)'$ is the $p \times 1$ vector set of unobservable true p regressor variables that are observed as $X = (X_1, X_2, ..., X_p)'$ with $p \times 1$ ME vector $u = (u_1, u_2, ..., u_p)'$, u_i being the ME in the i-th regressor variable x_i, e is the $n \times 1$ vector of the response error in the observed variable $Y = (Y_1, Y_2, ..., Y_n)'$. We presume that

$$(x', e, u) \sim N_{2p+1}\left\{(\mu_x', 0, \mathbf{0}')', \, BlockDiag(\Sigma_{xx}, \sigma_{ee}, \Sigma_{uu})\right\}, \tag{2}$$

where μ_x is the mean vector of x, σ_{ee} is the variance of e's, Σ_{xx} and Σ_{uu} are the covariance matrices of x's and u's, consecutively. Owing to the assumption of normality, we have the follows:

$$\begin{pmatrix} Y \\ X \end{pmatrix} \sim N_{p+1}\left\{\begin{pmatrix} \beta_0 + \beta'\mu_x \\ \mu_x' \end{pmatrix}, \begin{pmatrix} \sigma_{ee} + \beta'\Sigma_{xx}\beta & \beta'\Sigma_{xx} \\ \Sigma_{xx}\beta & \Sigma_{xx} + \Sigma_{uu} \end{pmatrix}\right\}.$$

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At that case, the conditional distribution of Y given X is normally distributed with mean matrix $E(Y/X) = \gamma_0 + \gamma' X$ and variance-covariance matrix $\text{cov}(Y/X) = \sigma_{zz} = \sigma_{ee} + \beta' \Sigma_{xx}\left(I_p - K_{xx}'\right)\beta$, where $\gamma_0 = \beta_0 + \beta'\left(I_p - K_{xx}'\right)\mu_x$, $\gamma = K_{xx}\beta$, $\beta = K_{xx}^{-1}\gamma$, and

$$K_{xx} = \Sigma_{XX}^{-1}\Sigma_{xx} = \left(\Sigma_{xx} + \Sigma_{uu}\right)^{-1}\Sigma_{xx} \tag{3}$$

is the $p \times p$ matrix of reliability ratios of X (see, Gleser [1] and Cheng and Van Ness, [2]). The fundamental problem here is to obtain the consistent estimation of β under diverse cases presuming Σ_{uu} is known. Gleser [1] demonstrated that the maximum likelihood (ML) estimators of γ_0, γ and σ_{zz} are basically the OLS estimators, namely

$$\tilde{\gamma}_{0n} = \bar{Y} - \tilde{\gamma}_n{}'\bar{X}, \quad \tilde{\gamma}_n = S_{XX}^{-1}S_{XY}, \tag{4}$$

where $\bar{X}_i$ is the mean of X_i, $\bar{Y}$ is the mean of Y, $S_{XX} = \left(S_{X_iX_i}\right)$, $S_{X_iX_i} = \left(x_i - \bar{X}_i j_n\right)'\left(x_i - \bar{X}_i j_n\right)$, $S_{XY} = \left(S_{X_1Y}, S_{X_2Y}, \ldots, S_{X_pY}\right)'$, $S_{X_iY} = \left(X_i - \bar{X}_i j_n\right)'\left(Y_i - \bar{Y}_i j_n\right)$, $j_n = (1,1,\ldots,1)'$. Apparently, $\frac{1}{n-1}S_{XX}$ is an unbiased estimator of Σ_{XX} and $\frac{1}{n}S_{XX}$ is convergence in probability to Σ_{XX}. (4) ensured that

$$\tilde{\sigma}_{ee} = \tilde{\sigma}_{zz} - \tilde{\gamma}_n{}'K_{xx}^{-1}\Sigma_{uu}\tilde{\gamma}_n \geq 0, \tag{5}$$

where $\tilde{\sigma}_{zz} = \frac{1}{n}\left(Y - \tilde{\gamma}_{0n}j_n - \tilde{\gamma}_n{}'X\right)'\left(Y - \tilde{\gamma}_{0n}j_n - \tilde{\gamma}_n{}'X\right)$ is the estimate of error variance. When Σ_{uu} is known and K_{xx} is unknown, K_{xx} is estimated consistently by replacing Σ_{xx} and $\Sigma_{xx} + \Sigma_{uu}$ by their individual consistent estimators as below:

$$\hat{K}_{xx} = S_{XX}^{-1}\left(S_{XX} - n\Sigma_{uu}\right), \tag{6}$$

where $\frac{1}{n}S_{XX}$ is the ML estimate of $\Sigma_{xx} + \Sigma_{uu}$. Hence, the ML estimates of β_0, β and σ_{ee} are obtained as follows:

$$\tilde{\beta}_{0n} = \tilde{\gamma}_{0n} - \tilde{\beta}_n'\left(I_p - \hat{K}_{xx}'\right)\bar{x}, \quad \tilde{\beta}_n = \hat{K}_{xx}^{-1}\tilde{\gamma}_n \text{ and } \tilde{\sigma}_{ee} = \tilde{\sigma}_{zz} - \tilde{\beta}_n'\Sigma_{uu}\hat{K}_{xx}\tilde{\beta}_n. \tag{7}$$

At last, $\tilde{\beta}_{0n} = \bar{Y} - \tilde{\beta}_n'\bar{X}$ and

$$\tilde{\beta}_n = \left(S_{XX} - n\Sigma_{uu}\right)^{-1}S_{XY} \tag{8}$$

supplied to $\sigma_{ee} \geq 0$ which specified in (5). Thus, as $n \to \infty$, $\sqrt{n}\left(\tilde{\gamma}_n - \gamma\right) \sim N_p\left(0, \sigma_{zz}\Sigma_{XX}^{-1}\right)$ and

$$\sqrt{n}\left(\tilde{\beta}_n - \beta\right) \sim N_p\left(0, \sigma_{zz}C^{-1}\right), \tag{9}$$

where

$$C = K_{xx}'\Sigma_{XX}K_{xx} = \Sigma_{xx}'\Sigma_{XX}^{-1}\Sigma_{xx}. \tag{10}$$

A consistent estimator of C is stated as below:

$$C_n = \hat{K}_{xx}'\hat{\Sigma}_{XX}\hat{K}_{xx} = \left(S_{XX} - n\Sigma_{uu}\right)'S_{XX}^{-1}\left(S_{XX} - n\Sigma_{uu}\right) \tag{11}$$

(see Fuller [3], chapter 2).

Multicollinearity is a very popular problem in the linear regression model and has been studied on a large scale. With the intent to get over the multicollinearity, biased estimators, which are alternative to the OLS estimator, have been suggested by many researchers. The most popular techniques used in the existence of the multicollinearity in linear regression analysis are ridge regression suggested by Hoerl and Kennard [4] and Liu estimator suggested by Liu [5].

Lately, researchers have investigated the estimation of the parameters for the ME models in the event of the multicollinearity that comes into being in the data. Saleh and Shalabh [6] considered the ridge regression estimation approach, which is suggested by Hoerl and Kennard [4] in linear regression, for the ME models. They derived the ridge estimator for the ME models with known reliability matrix K_{XX}, by minimizing

$$\left(\tilde{\gamma}_{0n} j_n + XK_{XX}\beta - Y\right)'\left(\tilde{\gamma}_{0n} j_n + XK_{XX}\beta - Y\right) + k\beta'\beta, \tag{12}$$

where k is a Lagrangian multiplier. Then the solution of (12) gives the ridge estimator:

$$\hat{\beta}(k) = \left(C + kI\right)^{-1} C\tilde{\beta}, \tag{13}$$

where $k \geq 0$ is the biasing parameter. Subsequently, replacing the consistent estimator of $\tilde{\beta}$ and C supplied with (8) and (11), respectively, the ridge estimator of β for the ME model is stated as follows:

$$\hat{\beta}_n(k) = \left(C_n + kI\right)^{-1} C_n\tilde{\beta}_n, \tag{14}$$

where $\tilde{\beta}_n$ and C_n are supplied with (8) and (11), sequentially and $\left(C_n + kI\right)^{-1} C_n$ is a consistent estimator of $\left(C + kI\right)^{-1} C$ with $C = K'_{xx}\Sigma_{XX}K_{xx}$. Although the ridge estimator is powerful in the applications, it is a complicated function of the biasing parameter.

Üstündağ Şiray [7] moved the Liu estimation attitude, which suggested by Liu [5] in the linear regression, to the ME models in the presence of multicollinearity. She considered the extension $d\tilde{\beta} = \beta + \varepsilon'$ to the model and attained the new model is as below:

$$\begin{pmatrix} Y - \gamma_0 j_n \\ d\tilde{\beta} \end{pmatrix} = \begin{pmatrix} XK_{xx} \\ I \end{pmatrix}\beta + \begin{pmatrix} \varepsilon \\ \varepsilon' \end{pmatrix}, \quad 0 < d < 1, \tag{15}$$

where d is a constant. By implementing the least-squares method to the model (15) and after some algebraic operations Liu estimator for ME models obtained as follows:

$$\hat{\beta}(d) = \left(K'_{xx}X'XK_{xx} + I\right)^{-1}\left(K'_{xx}X'XK_{xx} + dI\right)\tilde{\beta}. \tag{16}$$

In this way, replacing the consistent estimator of $\tilde{\beta}$ and C given in (8) and (11), sequentially, the Liu estimator of β for the ME model is got as below:

$$\hat{\beta}_n(d) = \left(C_n + I\right)^{-1}\left(C_n + dI\right)\tilde{\beta}_n \tag{17}$$

where $0 < d < 1$ and $\left(C_n + I\right)^{-1}\left(C_n + dI\right)$, which is a consistent estimator of $\left(C + I\right)^{-1}\left(C + dI\right)$ with $C = K'_{xx}\Sigma_{XX}K_{xx}$. The Liu estimator is advantageous to the ridge estimator inasmuch as it is a linear function of d and therefore choosing the biasing parameter d is more convenient.

In this paper, we compare the ridge and Liu estimators for the ME models by the criterion of mean squared error. In section 2, firstly asymptotic matrix mean squared error (MMSE) comparisons, secondly, asymptotic scalar mean squared error (SMSE) comparisons of the ridge and Liu estimators for the ME models are done. We point out that the comparison of the ridge and Liu estimators are very troublesome in the sense of the SMSE and MMSE criteria. In section 3, we perform a simulation analysis with the intent of comparing the ridge and Liu estimators' superiority of each other for the ME model by the criterion SMSE. Also, we do a numerical illustration to demonstrate the theoretical results, in section 4. Lastly, we give a summary and conclusions in section 5.

2. MEAN SQUARED ERROR COMPARISONS OF THE RIDGE AND LIU ESTIMATORS

2.1. Mean Squared Errors of The Ridge and Liu Estimators

It is very well-known that MMSE of $\widehat{\beta}$ that is an estimator of β is as follows:

$$\begin{aligned} M\left(\widehat{\beta}\right) &= E\left(\lim_{n\to\infty}\left\{n\left(\widehat{\beta}-\beta\right)\left(\widehat{\beta}-\beta\right)'\right\}\right) \\ &= \operatorname{cov}\left(\widehat{\beta}\right)+bias\left(\widehat{\beta}\right)bias\left(\widehat{\beta}\right)', \end{aligned} \tag{18}$$

where $bias\left(\widehat{\beta}\right)$ is the asymptotic bias of $\widehat{\beta}$ and $\operatorname{cov}\left(\widehat{\beta}\right)$ is the asymptotic covariance matrix of $\widehat{\beta}$. Also, the SMSE is achieved by implementing the trace operator to the MMSE:

$$m\left(\widehat{\beta}\right)=tr\left(\operatorname{cov}\left(\widehat{\beta}\right)\right)+bias\left(\widehat{\beta}\right)'bias\left(\widehat{\beta}\right) \tag{19}$$

Let $\widehat{\beta}_1$ and $\widehat{\beta}_2$ are yielded as two estimators of $\widehat{\beta}$, it is said that the estimator $\widehat{\beta}_2$ outperforms the estimator $\widehat{\beta}_1$ in terms of MMSE criterion iff the difference $M\left(\widehat{\beta}_1\right)-M\left(\widehat{\beta}_2\right)$ is a nonnegative definite (nnd) matrix. If $m\left(\widehat{\beta}_1\right)\geq m\left(\widehat{\beta}_2\right)$ then the estimator $\widehat{\beta}_2$ is better than the estimator $\widehat{\beta}_1$ in terms of the SMSE criterion.

In this section, we compare the MMSE and SMSEs of the ridge and Liu estimators. We presume that as $n\to\infty$ the limit of C_n exists and the parameter β is identifiable. As $n\to\infty$ the MMSE of the ridge estimator, which is given by Saleh and Shalabh [6], is written as follows:

$$M\left(\hat{\beta}_n(k)\right)=\sigma_{zz}C'\left(C+kI\right)^{-1}C^{-1}\left(C+kI\right)^{-1}C+k^2\left(C+kI\right)^{-1}\beta\beta'\left(C+kI\right)^{-1}. \tag{20}$$

Also, as $n\to\infty$ SMSE of ridge estimator is expressed as below:

$$m\left(\hat{\beta}_n(k)\right)=\sigma_{zz}tr\left[\left(C+kI\right)^{-2}C\right]+k^2\beta'\left(C+kI\right)^{-2}\beta. \tag{21}$$

The asymptotic covariance matrix of $\tilde{\beta}_n$ is $\sigma_{zz}C^{-1}$, where $C=K'_{xx}\Sigma_{XX}K_{xx}$ is a positive definite (pd) matrix. So, there is an orthogonal matrix Γ such that

$$\Gamma'C\Gamma=\Gamma'\left(K'_{xx}\Sigma_{XX}K_{xx}\right)\Gamma=diag\left(\lambda_1,\lambda_2,...,\lambda_p\right), \tag{22}$$

where $\lambda_1\geq\lambda_2\geq...\geq\lambda_p>0$ are eigenvalues of the matrix C. Therefore, the SMSE of the ridge estimator is derived as follows:

$$m\left(\hat{\beta}_n(k)\right)=\sigma_{zz}\sum_{i=1}^{p}\frac{\lambda_i}{\left(\lambda_i+k\right)^2}+k^2\sum_{i=1}^{p}\frac{\theta_i^2}{\left(\lambda_i+k\right)^2}, \tag{23}$$

where $\theta=\Gamma'\beta=\left(\theta_1,\theta_2,...,\theta_p\right)'$.

As $n\to\infty$ the MMSE of the Liu estimator, which is given by Üstündağ Şiray [7], is expressed as below:

$$M\left(\hat{\beta}_n(d)\right)=\sigma_{zz}\left(C+I\right)^{-1}\left(C+dI\right)C^{-1}\left(C+dI\right)\left(C+I\right)^{-1}+\left(d-1\right)^2\left(C+I\right)^{-1}\beta\beta'\left(C+I\right)^{-1} \tag{24}$$

Also, as $n\to\infty$ the SMSE of the Liu estimator is expressed as follows:

$$m\left(\hat{\beta}_n(d)\right)=\sigma_{zz}tr\left(\left(C+I\right)^{-1}\left(C+dI\right)C^{-1}\left(C+dI\right)\left(C+I\right)^{-1}\right)+\left(d-1\right)^2\beta'\left(C+I\right)^{-2}\beta. \tag{25}$$

By using orthogonal decomposition given in (22), the SMSE of the Liu estimator is achieved as below:

$$m\left(\hat{\beta}_n(d)\right)=\sigma_{zz}\sum_{i=1}^{p}\frac{\left(\lambda_i+d\right)^2}{\lambda_i\left(\lambda_i+1\right)^2}+\left(d-1\right)^2\sum_{i=1}^{p}\frac{\theta_i^2}{\left(\lambda_i+1\right)^2}, \tag{26}$$

where $\theta=\Gamma'\beta=\left(\theta_1,\theta_2,...,\theta_p\right)'$.

2.2. MMSE Comparisons of The Ridge and Liu Estimators

In this section, we show that the comparison of the ridge and Liu estimators are very troublesome in the sense of the MMSE criterion. Despite the comparisons are made, it is very hard to ensure the conditions achieved. Hence, we do not enounce the superiority comparison results as theorems.

Let $\Delta_1=C'\left(C+kI\right)^{-1}C^{-1}\left(C+kI\right)^{-1}C-\left(C+I\right)^{-1}\left(C+dI\right)C^{-1}\left(C+dI\right)\left(C+I\right)^{-1}$ is the pd matrix.

$$M\left(\hat{\beta}_n(k)\right)-M\left(\hat{\beta}_n(d)\right)=\sigma_{zz}\Delta_1+k^2\left(C+kI\right)^{-1}\beta\beta'\left(C+kI\right)^{-1}-\left(d-1\right)^2\left(C+I\right)^{-1}\beta\beta'\left(C+I\right)^{-1},$$

where Δ_1 is defined as above. It is obvious that $k^2\left(C+kI\right)^{-1}\beta\beta'\left(C+kI\right)^{-1}$ is a pd matrix. Therefore, we are interested in $\sigma_{zz}\Delta_1-\left(d-1\right)^2\left(C+I\right)^{-1}\beta\beta'\left(C+I\right)^{-1}$. If $\sigma_{zz}\Delta_1$ is pd matrix, then by utilizing Lemma 1 (given in the appendix) $M\left(\hat{\beta}_n(k)\right)-M\left(\hat{\beta}_n(d)\right)$ is nnd iff $\sigma_{zz}^{-1}\left(d-1\right)^2\beta'\left(C+I\right)^{-1}\Delta_1^{-1}\left(C+I\right)^{-1}\beta<1$. After the algebraic simplification, it implies $\beta'\left[\left(C+I\right)\Delta_1\left(C+I\right)\right]^{-1}\beta<\frac{\sigma_{zz}}{\left(d-1\right)^2}$. Hence, $\hat{\beta}_n(d)$ is better than $\hat{\beta}_n(k)$ iff $\beta'\left[\left(C+I\right)\Delta_1\left(C+I\right)\right]^{-1}\beta<\frac{\sigma_{zz}}{\left(d-1\right)^2}$ by the criterion of MMSE.

Let $\Delta_2=\left(C+I\right)^{-1}\left(C+dI\right)C^{-1}\left(C+dI\right)\left(C+I\right)^{-1}-C'\left(C+kI\right)^{-1}C^{-1}\left(C+kI\right)^{-1}C$ is the pd matrix

$$M\left(\hat{\beta}_n(d)\right)-M\left(\hat{\beta}_n(k)\right)=\sigma_{zz}\Delta_2-k^2\left(C+kI\right)^{-1}\beta\beta'\left(C+kI\right)^{-1}+\left(d-1\right)^2\left(C+I\right)^{-1}\beta\beta'\left(C+I\right)^{-1},$$

where Δ_2 is defined as above. It is obvious that $\left(d-1\right)^2\left(C+I\right)^{-1}\beta\beta'\left(C+I\right)^{-1}$ is a pd matrix. Therefore, we deal with $\sigma_{zz}\Delta_2-k^2\left(C+kI\right)^{-1}\beta\beta'\left(C+kI\right)^{-1}$. If $\sigma_{zz}\Delta_2$ is pd matrix, then by utilizing Lemma 1 (attended in the appendix) $M\left(\hat{\beta}_n(d)\right)-M\left(\hat{\beta}_n(k)\right)$ is nnd iff $\sigma_{zz}^{-1}k^2\beta'\left(C+kI\right)^{-1}\Delta_1^{-1}\left(C+kI\right)^{-1}\beta<1$. After the algebraic facilitation, it becomes

$\beta'\left[\left(C+kI\right)\Delta_2\left(C+kI\right)\right]^{-1}\beta<\frac{\sigma_{zz}}{k^2}$. That's why $\hat{\beta}_n(k)$ is better than $\hat{\beta}_n(d)$ iff $\beta'\left[\left(C+kI\right)\Delta_2\left(C+kI\right)\right]^{-1}\beta<\frac{\sigma_{zz}}{k^2}$ in terms of the criterion of MMSE.

If it is paid attention, $\sigma_{zz}\Delta_1$ signifies the difference between the covariance matrices of the ridge and Liu estimators and further $\sigma_{zz}\Delta_2$ signifies the difference between the covariance matrices of the Liu and ridge estimators, respectively. The comparisons presented here are based on the positive definiteness of the difference between the covariance matrices of the mentioned estimators.

2.3. SMSE Comparisons of The Ridge and Liu Estimators

In this section, we show that the comparison of the ridge and Liu estimators can be made, however, it is very troublesome to ensure the conditions procured. We express the comparisons result as theorems, but they are not practical, which can be seen. Instead of using the theorems, the comparison of these estimators ' superiority of each other for the ME model is achieved by a simulation analysis in section 3 in terms of the SMSE criterion.

Now we compare the ridge and Liu estimators for the ME models by the criterion of SMSE. For this purpose, we first assume that d is fixed and make the comparison. We latter assume that k is fixed and make the comparison.

Theorem 2.1 Let d be fixed.

i. If $\dfrac{\sum_{i=1}^{p}\frac{\theta_i^2}{(\lambda_i+1)^2}}{\sum_{i=1}^{p}\left(\frac{\lambda_i}{(\lambda_i+k)^2}-\frac{(\lambda_i+d)^2}{\lambda_i(\lambda_i+1)^2}\right)}<\dfrac{\sigma_{zz}}{(d-1)^2}$, then $\hat{\beta}_n(d)$ is better than $\hat{\beta}_n(k)$ for $0<k<\frac{\lambda_i(1-d)}{\lambda_i+d}$, $i=1,2,...p$ by the criterion of SMSE.

ii. If $\dfrac{\sum_{i=1}^{p}\frac{\theta_i^2}{(\lambda_i+k)^2}}{\sum_{i=1}^{p}\left(\frac{(\lambda_i+d)^2}{\lambda_i(\lambda_i+1)^2}-\frac{\lambda_i}{(\lambda_i+k)^2}\right)}<\dfrac{\sigma_{zz}}{k^2}$, then $\hat{\beta}_n(k)$ is better than $\hat{\beta}_n(d)$ for $0<\frac{\lambda_i(1-d)}{\lambda_i+d}<k$, $i=1,2,...,p$ by the criterion of SMSE.

Proof: i. By using (23) and (26)

$$m\left(\hat{\beta}_n(k)\right)-m\left(\hat{\beta}_n(d)\right)=\sigma_{zz}\left(\sum_{i=1}^{p}\frac{\lambda_i}{(\lambda_i+k)^2}-\sum_{i=1}^{p}\frac{(\lambda_i+d)^2}{\lambda_i(\lambda_i+1)^2}\right)+k^2\sum_{i=1}^{p}\frac{\theta_i^2}{(\lambda_i+k)^2}-(d-1)^2\sum_{i=1}^{p}\frac{\theta_i^2}{(\lambda_i+1)^2}. \tag{27}$$

It is easily seen that $k^2\sum_{i=1}^{p}\frac{\theta_i^2}{(\lambda_i+k)^2}$ is positive. So, $m\left(\hat{\beta}_n(k)\right)-m\left(\hat{\beta}_n(d)\right)$ will be positive if

$$\sigma_{zz}\left(\sum_{i=1}^{p}\frac{\lambda_i}{(\lambda_i+k)^2}-\sum_{i=1}^{p}\frac{(\lambda_i+d)^2}{\lambda_i(\lambda_i+1)^2}\right)-(d-1)^2\sum_{i=1}^{p}\frac{\theta_i^2}{(\lambda_i+1)^2}>0. \tag{28}$$

Firstly, we consider the positivity of

$$\sum_{i=1}^{p}\left(\frac{\lambda_i}{(\lambda_i+k)^2}-\frac{(\lambda_i+d)^2}{\lambda_i(\lambda_i+1)^2}\right)=\sum_{i=1}^{p}\frac{\left[-k(\lambda_i+d)+\lambda_i(1-d)\right]\left[\lambda_i(\lambda_i+1)+(\lambda_i+d)(\lambda_i+k)\right]}{\lambda_i(\lambda_i+1)^2(\lambda_i+k)^2} \tag{29}$$

(29) is positive for $0<k<\frac{\lambda_i(1-d)}{\lambda_i+d}$, $i=1,2,...,p$. Because (29) is positive for $0<k<\frac{\lambda_i(1-d)}{\lambda_i+d}$, $i=1,2,...p$, inequality (28) can be stated as $\frac{\sum_{i=1}^{p}\frac{\theta_i^2}{(\lambda_i+1)^2}}{\sum_{i=1}^{p}\left(\frac{\lambda_i}{(\lambda_i+k)^2}-\frac{(\lambda_i+d)^2}{\lambda_i(\lambda_i+1)^2}\right)}<\frac{\sigma_{zz}}{(d-1)^2}$. The proof is finished.

ii. By using (23) and (26)

$$m\left(\hat{\beta}_n(d)\right)-m\left(\hat{\beta}_n(k)\right)=\sigma_{zz}\left(\sum_{i=1}^{p}\frac{(\lambda_i+d)^2}{\lambda_i(\lambda_i+1)^2}-\sum_{i=1}^{p}\frac{\lambda_i}{(\lambda_i+k)^2}\right)$$
$$-k^2\sum_{i=1}^{p}\frac{\theta_i^2}{(\lambda_i+k)^2}+(d-1)^2\sum_{i=1}^{p}\frac{\theta_i^2}{(\lambda_i+1)^2}. \tag{30}$$

It is easily seen that $(d-1)^2\sum_{i=1}^{p}\frac{\theta_i^2}{(\lambda_i+1)^2}$ is positive. So, $m\left(\hat{\beta}_n(d)\right)-m\left(\hat{\beta}_n(k)\right)$ will be positive if

$$\sigma_{zz}\left(\sum_{i=1}^{p}\frac{(\lambda_i+d)^2}{\lambda_i(\lambda_i+1)^2}-\sum_{i=1}^{p}\frac{\lambda_i}{(\lambda_i+k)^2}\right)-k^2\sum_{i=1}^{p}\frac{\theta_i^2}{(\lambda_i+k)^2}>0. \tag{31}$$

Firstly, we consider the positivity of

$$\sum_{i=1}^{p}\left(\frac{(\lambda_i+d)^2}{\lambda_i(\lambda_i+1)^2}-\frac{\lambda_i}{(\lambda_i+k)^2}\right)=\sum_{i=1}^{p}\frac{\left[k(\lambda_i+d)-\lambda_i(1-d)\right]\left[\lambda_i(\lambda_i+1)+(\lambda_i+d)(\lambda_i+k)\right]}{\lambda_i(\lambda_i+1)^2(\lambda_i+k)^2}. \tag{32}$$

(32) is positive for $0<\frac{\lambda_i(1-d)}{\lambda_i+d}<k$, $i=1,2,...,p$. Since (32) is positive for $0<\frac{\lambda_i(1-d)}{\lambda_i+d}<k$, $i=1,2,...p$, inequality (31) can be expressed as $\frac{\sum_{i=1}^{p}\frac{\theta_i^2}{(\lambda_i+k)^2}}{\sum_{i=1}^{p}\left(\frac{(\lambda_i+d)^2}{\lambda_i(\lambda_i+1)^2}-\frac{\lambda_i}{(\lambda_i+k)^2}\right)}<\frac{\sigma_{zz}}{k^2}$. The proof is completed.

Theorem 2.2 Let k be fixed as $0<k<1$.

i. If $\dfrac{\sum_{i=1}^{p}\dfrac{\theta_i^2}{(\lambda_i+1)^2}}{\sum_{i=1}^{p}\left(\dfrac{\lambda_i}{(\lambda_i+k)^2}-\dfrac{(\lambda_i+d)^2}{\lambda_i(\lambda_i+1)^2}\right)}<\dfrac{\sigma_{zz}}{(d-1)^2}$, then $\hat{\beta}_n(d)$ is better than $\hat{\beta}_n(k)$ for $0<d<\dfrac{\lambda_i(1-k)}{\lambda_i+k}<1$, $i=1,2,...,p$ by the criterion of SMSE.

ii. If $\dfrac{\sum_{i=1}^{p}\dfrac{\theta_i^2}{(\lambda_i+k)^2}}{\sum_{i=1}^{p}\left(\dfrac{(\lambda_i+d)^2}{\lambda_i(\lambda_i+1)^2}-\dfrac{\lambda_i}{(\lambda_i+k)^2}\right)}<\dfrac{\sigma_{zz}}{k^2}$, then $\hat{\beta}_n(k)$ is better than $\hat{\beta}_n(d)$ for $0<\dfrac{\lambda_i(1-k)}{\lambda_i+k}<d<1$, $i=1,2,...,p$ by the criterion of SMSE.

Proof: i. By using (28), we investigate the positivity of

$$\sum_{i=1}^{p}\left(\frac{\lambda_i}{(\lambda_i+k)^2}-\frac{(\lambda_i+d)^2}{\lambda_i(\lambda_i+1)^2}\right)=\sum_{i=1}^{p}\frac{\left[-d(\lambda_i+k)+\lambda_i(1-k)\right]\left[\lambda_i(\lambda_i+1)+(\lambda_i+d)(\lambda_i+k)\right]}{\lambda_i(\lambda_i+1)^2(\lambda_i+k)^2} \quad (33)$$

(33) is positive for $0<d<\dfrac{\lambda_i(1-k)}{\lambda_i+k}<1$, $i=1,2,...,p$. Because (33) is positive for $0<d<\dfrac{\lambda_i(1-k)}{\lambda_i+k}<1$, $i=1,2,...,p$, inequality (28) can be expressed as $\dfrac{\sum_{i=1}^{p}\dfrac{\theta_i^2}{(\lambda_i+1)^2}}{\sum_{i=1}^{p}\left(\dfrac{\lambda_i}{(\lambda_i+k)^2}-\dfrac{(\lambda_i+d)^2}{\lambda_i(\lambda_i+1)^2}\right)}<\dfrac{\sigma_{zz}}{(d-1)^2}$. The proof is finished.

ii. By using (30), we consider the positivity of

$$\sum_{i=1}^{p}\left(\frac{(\lambda_i+d)^2}{\lambda_i(\lambda_i+1)^2}-\frac{\lambda_i}{(\lambda_i+k)^2}\right)=\sum_{i=1}^{p}\frac{\left[d(\lambda_i+k)-\lambda_i(1-k)\right]\left[\lambda_i(\lambda_i+1)+(\lambda_i+d)(\lambda_i+k)\right]}{\lambda_i(\lambda_i+1)^2(\lambda_i+k)^2} \quad (34)$$

(34) is positive for $0<\dfrac{\lambda_i(1-k)}{\lambda_i+k}<d<1$, $i=1,2,...p$. Because (34) is positive $0<\dfrac{\lambda_i(1-k)}{\lambda_i+k}<d<1$, $i=1,2,...,p$, inequality (31) can be stated as $\dfrac{\sum_{i=1}^{p}\dfrac{\theta_i^2}{(\lambda_i+k)^2}}{\sum_{i=1}^{p}\left(\dfrac{(\lambda_i+d)^2}{\lambda_i(\lambda_i+1)^2}-\dfrac{\lambda_i}{(\lambda_i+k)^2}\right)}<\dfrac{\sigma_{zz}}{k^2}$. The proof is completed.

3. SIMULATION ANALYSIS

In this section, we do a simulation analysis by utilizing the Matlab R2014a program with an eye to comparing the ridge and Liu estimators' superiority of each other for the ME model. By complying with McDonald and Galarneau [8], the regressor variables are reckoned by

$$x_{ij} = \left(1-\rho^2\right)^{1/2} z_{ij} + \rho z_{ip+1}, \quad i = 1,2,\ldots,n, \quad j = 1,2,\ldots,p, \tag{35}$$

where ρ is described as the correlation between any two regressor variables is stated by ρ^2, z_{ij} independent standard normal pseudorandom numbers. We consider ρ = 0.85, 0.95, and 0.99 in order to examine the effects of different degrees of multicollinearity on the mentioned estimators. The number of explanatory variables is used as $p = 5$ in the event of n = 100. After that, regressor variables standardized such that $x'x$ is in correlation form.

Independent e_i 's ($i = 1,2,\ldots,n$) are generated from the normal distribution with mean 0 and variance σ_{ee}, in each replication on the experiment. The variances of e_i 's are taken into consideration in the simulation analysis σ_{ee} = 1, 5, and 10. The parameter vector is used as the normalized eigenvector corresponding to the largest eigenvalue of the matrix $x'x$, which is procured from the restriction $\beta'\beta = 1$. In addition, independent u_i 's ($i = 1,2,\ldots,n$) are generated from the normal distribution with mean vector 0 and variance-covariance matrix $\sigma_{uu}I$. The variances of u_i 's used in the simulation analysis are σ_{uu} = 1, 3, and 5. The values of k and d are specified in the interval (0, 1). Observations on the dependent variable are generated by

$$\begin{aligned} Y_i &= \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_p x_{ip} + e_i \\ X_i &= x_i + u_i, \qquad\qquad i = 1,2,\ldots,n \end{aligned} \tag{36}$$

where $Y_k = \left(Y_{1k}, Y_{2k}, \ldots, Y_{nk}\right)'$, $x = \left(x_1, x_2, \ldots, x_n\right)'$.

The estimated SMSE (ESMSE) for any estimator $\widehat{\beta}^*$ is obtained as follows:

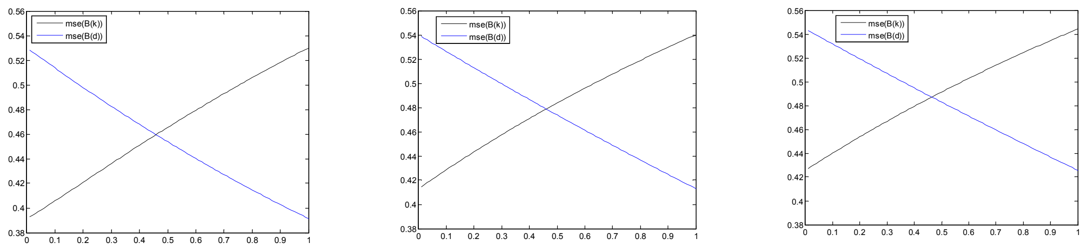
$$m\hat{s}e\left(\widehat{\beta}^*\right) = \frac{1}{MCN} \sum_{i=1}^{MCN} \left(\widehat{\beta}_i^* - \beta\right)'\left(\widehat{\beta}_i^* - \beta\right), \tag{37}$$

where $\widehat{\beta}_i^*$ is the calculated value of $\widehat{\beta}^*$ for the i-th replication of the experiment and MCN is the number of replications, which is specified as 3000 for this experiment. The consequences of the simulation analysis are presented in Figures 1a-9c. In the Figures, the values of biasing parameters on the horizontal axis and ESMSE values on the vertical axis are shown.
From the figures, we can decide that the most crucial characteristic that impacts the performance of the estimators is the variances of the MEs. Hence, we make comments according to the variances of the MEs.

In the case of $\Sigma_{uu} = I$, in the event of biasing parameters soar, the ESMSE values of the Liu estimator diminish, and ESMSE values of the ridge estimator soar in the interval (0,1), generally. The increase in σ_{ee} does not alter the superiority comparison of the estimators but soars the ESMSE values of the estimators. Interestingly, the soar in ρ diminishes the ESMSE values of the estimators.

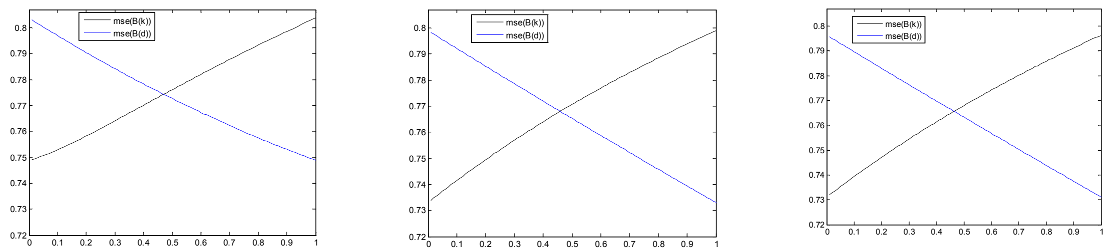
In the case of $\Sigma_{uu} = 3I$ and $\Sigma_{uu} = 5I$, in the event of biasing parameters soar, the ESMSE values of the ridge estimator diminish, and ESMSE values of the Liu estimator soar in the interval (0,1). The soar in σ_{ee} does not change the superiority comparison of the estimators but diminishes the ESMSE values of the estimators. Similarly, the soar in ρ does not alter the superiority comparison of the estimators but soars the ESMSE values of the estimators.

As a result of the simulation analysis, we can state that in the event of variances of MEs are about 1, we should prefer the values of *d* close to 1 and values of k close to 0. Also, for variances of MEs are bigger than 1, we should prefer the values of *d* close to 0 and values of k close to 1.



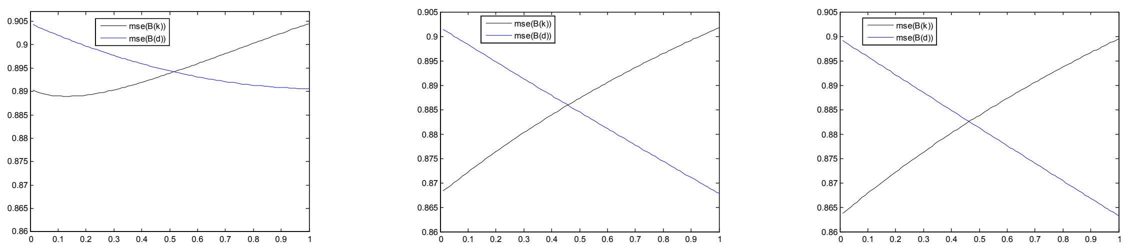
(a) (b) (c)

Figure 1. ESMSE values of Ridge and Liu estimators when σ_{ee} = 1, $\Sigma_{uu} = I$ for the ME models (a) when ρ = 0.85; (b) when ρ = 0.95 ; (c) when ρ = 0.99



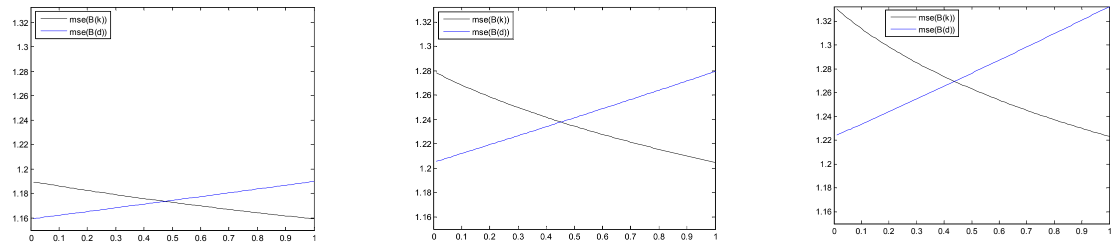
(a) (b) (c)

Figure 2. ESMSE values of Ridge and Liu estimators when σ_{ee} = 5, $\Sigma_{uu} = I$ for the ME models (a) when ρ = 0.85; (b) when ρ = 0.95 ; (c) when ρ = 0.99



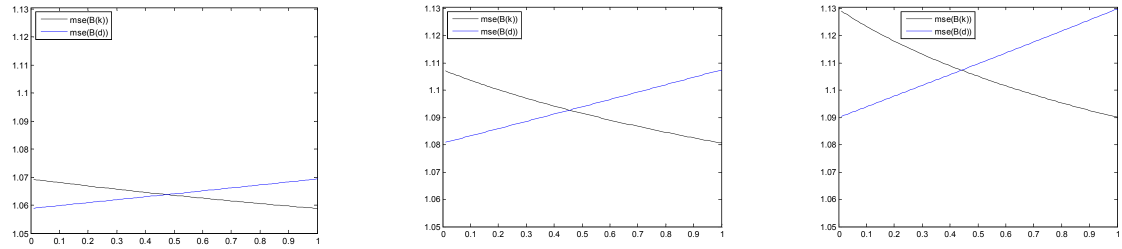
(a) (b) (c)

Figure 3. ESMSE values of Ridge and Liu estimators when σ_{ee} = 10, $\Sigma_{uu} = I$ for the ME models (a) when ρ = 0.85; (b) when ρ = 0.95 ; (c) when ρ = 0.99



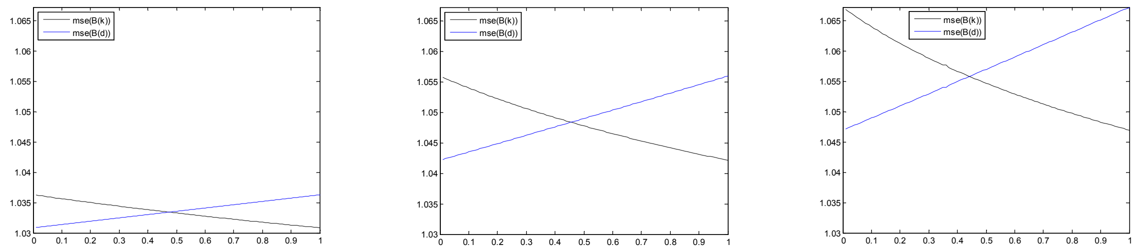
(a) (b) (c)

Figure 4. ESMSE values of Ridge and Liu estimators when σ_{ee} = 1, $\Sigma_{uu} = 3I$ for the ME models (a) when ρ = 0.85; (b) when ρ = 0.95 ; (c) when ρ = 0.99



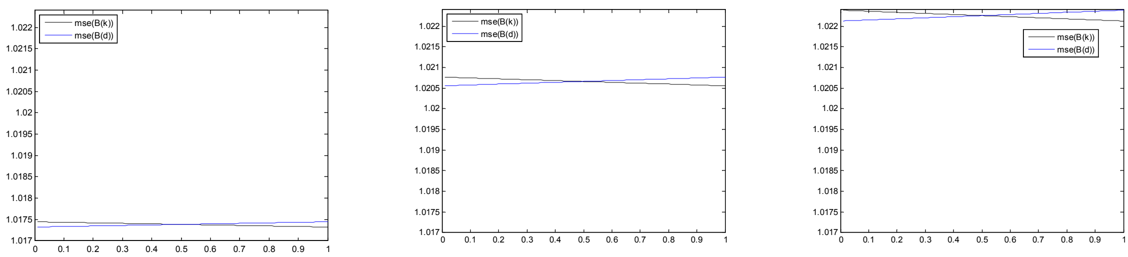
(a) (b) (c)

Figure 5. ESMSE values of Ridge and Liu estimators when σ_{ee} = 5, $\Sigma_{uu} = 3I$ for the ME models (a) when ρ = 0.85; (b) when ρ = 0.95 ; (c) when ρ = 0.99



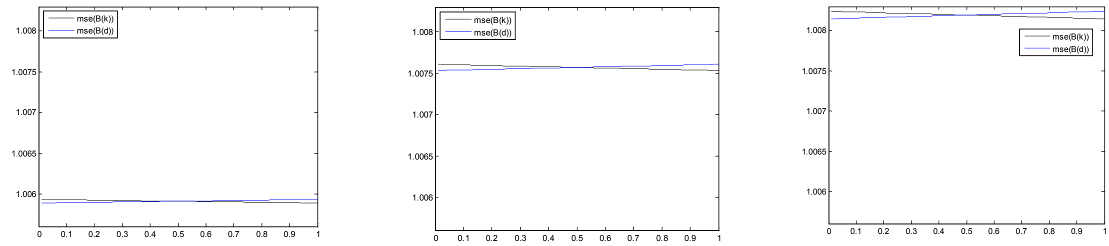
(a) (b) (c)

Figure 6. ESMSE values of Ridge and Liu estimators when σ_{ee} = 10, $\Sigma_{uu} = 3I$ for the ME models (a) when ρ = 0.85; (b) when ρ = 0.95 ; (c) when ρ = 0.99



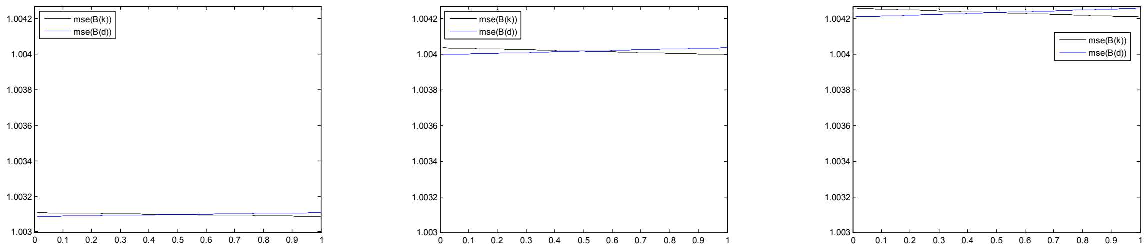
(a) (b) (c)

Figure 7. ESMSE values of Ridge and Liu estimators when σ_{ee} = 1, $\Sigma_{uu} = 5I$ for the ME models (a) when ρ = 0.85; (b) when ρ = 0.95 ; (c) when ρ = 0.99



(a) (b) (c)

Figure 8. ESMSE values of Ridge and Liu estimators when σ_{ee} = 5, $\Sigma_{uu} = 5I$ for the ME models (a) when ρ = 0.85; (b) when ρ = 0.95 ; (c) when ρ = 0.99



(a) (b) (c)

Figure 9. ESMSE values of Ridge and Liu estimators when σ_{ee} = 10, $\Sigma_{uu} = 5I$ for the ME models (a) when ρ = 0.85; (b) when ρ = 0.95 ; (c) when ρ = 0.99

4. NUMERICAL EXAMPLE

We use the Longley dataset to exemplify the theoretical findings. The Longley dataset contains diverse macroeconomic variables that are known to be highly multicollinear. Longley [9] essentially used the dataset to test the computational accuracy of regression programs. The variables are Y : total derived employment (in thousands), X_1 : gross national product implicit price deflator (in tenths), X_2 : gross national product (GNP) (in millions), X_3 : unemployment (in thousands), X_4 : size of armed forces, X_5 : noninstitutional population 14 years of age and over (in thousands), X_6 : year. Beaton et al. [10] used this data in a discussion of the effect ME on regression coefficients. They claimed that rounding error provides a lower bound for the ME model. A possible covariance matrix for ME obtained by Beaton et al. [10] is $\Sigma_{uu} = \frac{1}{12} diag\left\{1,\ 10^{-2},\ 1,\ 1,\ 1,\ 1\right\}$.

With the aim of comparing the ridge and Liu estimators in the ME models, we use the criterion of SMSE. We obtained the SMSE values for the mentioned estimators in the Matlab R2014a program. The SMSE values are given in Figure 10. In figure 10, the values of biasing parameters on the horizontal axis and ESMSE values on the vertical axis are shown. As seen in Figure 10, for values between [0, 0.4] of k the ridge estimator and for values between [0.4, 1] of d is the Liu estimator show good performance for this dataset, by the criterion of SMSE.

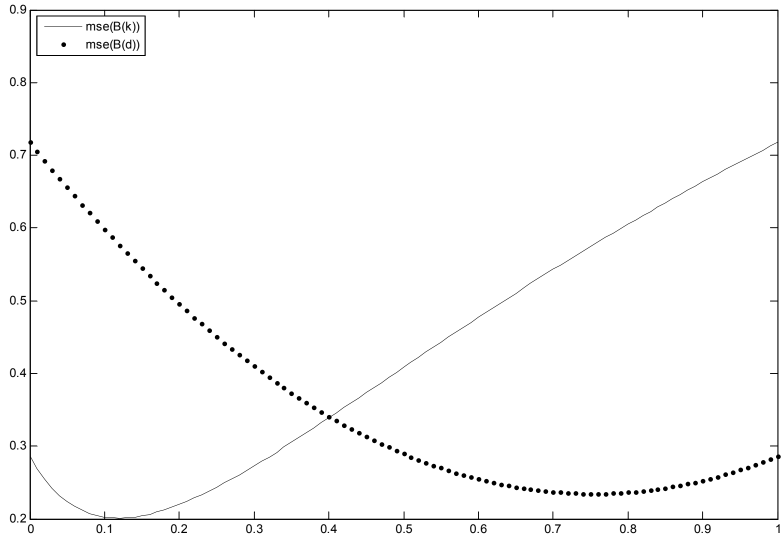


Figure 10. SMSE values of the ridge and Liu estimators for the values of *k* and *d*.

Now, we check the conditions of Theorems 2.1 and 2.2 according to Figure 10. Firstly we specify fixed *d* and we select the value of *k* according to fixed *d* value, after that we specify fixed *k* and we select the value of *d* according to fixed *k* value. Let *d* is fixed as 0.75. We choose the value of *k* as $k=\min\left\{\frac{\lambda_i\left(1-d\right)}{\lambda_i+d}\right\}-0.01\cong 0.001$. In this case, $\frac{\sum_{i=1}^{p}\frac{\theta_i^2}{\left(\lambda_i+1\right)^2}}{\sum_{i=1}^{p}\left(\frac{\lambda_i}{\left(\lambda_i+k\right)^2}-\frac{\left(\lambda_i+d\right)^2}{\lambda_i\left(\lambda_i+1\right)^2}\right)}<\frac{\sigma_{zz}}{\left(d-1\right)^2}$ is provided. By using these biasing parameters we get $m\left(\hat{\beta}_n\left(d\right)\right)=0.2381$ and $m\left(\hat{\beta}_n\left(k\right)\right)\cong 0.28$, which signifies the Liu estimator is better than the ridge estimator. This result supports the Theorem 2.1.i. Secondly, we choose the value of *k* as $k=\max\left\{\frac{\lambda_i\left(1-d\right)}{\lambda_i+d}\right\}+0.01\cong 0.26$. In this case, $\frac{\sum_{i=1}^{p}\frac{\theta_i^2}{\left(\lambda_i+k\right)^2}}{\sum_{i=1}^{p}\left(\frac{\left(\lambda_i+d\right)^2}{\lambda_i\left(\lambda_i+1\right)^2}-\frac{\lambda_i}{\left(\lambda_i+k\right)^2}\right)}<\frac{\sigma_{zz}}{k^2}$ is supplied. By using these biasing parameters we get $m\left(\hat{\beta}_n\left(d\right)\right)=0.2437$ and $m\left(\hat{\beta}_n\left(k\right)\right)\cong 0.2348$, which signifies the ridge estimator is better than the Liu estimator. This result endorses the Theorem 2.1.ii.

Let *k* is fixed as 0.013 and we choose the value of *d* as $d=\min\left\{\frac{\lambda_i\left(1-k\right)}{\lambda_i+k}\right\}-0.01\cong 0.8097$, which ensures the condition $\frac{\sum_{i=1}^{p}\frac{\theta_i^2}{\left(\lambda_i+1\right)^2}}{\sum_{i=1}^{p}\left(\frac{\lambda_i}{\left(\lambda_i+k\right)^2}-\frac{\left(\lambda_i+d\right)^2}{\lambda_i\left(\lambda_i+1\right)^2}\right)}<\frac{\sigma_{zz}}{\left(d-1\right)^2}$. By using these biasing parameters

we get $m\left(\hat{\beta}_n\left(d\right)\right) \cong 0.2365$ and $m\left(\hat{\beta}_n\left(k\right)\right) \cong 0.27$, which signifies the Liu estimator is better than the ridge estimator. This result supports the Theorem 2.2.i.
Again we get k is fixed as 0.013 and we choose the value of d as $d = \max\left\{\frac{\lambda_i\left(1-k\right)}{\lambda_i + k}\right\} + 0.01 \cong 0.9969$, which supplies the condition

$$\frac{\sum_{i=1}^{p} \frac{\theta_i^2}{\left(\lambda_i + k\right)^2}}{\sum_{i=1}^{p}\left(\frac{\left(\lambda_i + d\right)^2}{\lambda_i\left(\lambda_i + 1\right)^2} - \frac{\lambda_i}{\left(\lambda_i + k\right)^2}\right)} < \frac{\sigma_{zz}}{k^2}$$

. By using these biasing parameters we get $m\left(\hat{\beta}_n\left(d\right)\right) \cong 0.28$ and $m\left(\hat{\beta}_n\left(k\right)\right) \cong 0.27$, which signifies the ridge estimator is better than the Liu estimator. This result endorses the Theorem 2.2.ii.

5. SUMMARY AND CONCLUSIONS

In this paper, we examine the biased estimation approach to the ME models in the event of the multicollinearity exists in the data. We compare two estimators, which are the ridge and Liu estimators by theoretical and numerical evaluations. We demonstrated that, although the comparisons are made theoretically by the criterion of MMSE, it is very troublesome to ensure the conditions obtained. Furthermore, we give as the theorems which are obtained the comparisons of the above-mentioned estimators by the criterion of SMSE. Nevertheless, these theorems are not practical because of the difficulties of providing the conditions. But then, we show the conditions of the theorems provide by a numerical example. Also, we perform a simulation analysis for yielding the comparisons of the ridge and Liu estimators in the sense of the SMSE criterion. Consequently, we indicate that the most crucial factor that has an impact on the performance of the estimators is the variances of the MEs. Additionally, we point out that, when the variances of the MEs are close to 1, we should prefer the values of d close to 1 and values of k close to 0, when the variances of the MEs are bigger than 1 we should prefer the values of d close to 0 and values of k close to 1.

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APPENDIX

We give Lemma 1 used for yielding the MMSE comparisons of estimators.

Lemma 1. Let G be a pd matrix, namely $G > 0$, α be some vector, then $G - \alpha\alpha' \geq 0$ iff $\alpha' G^{-1}\alpha \leq 1$. (Farebrother, [11])

REFERENCES

[1] Gleser LJ. The importance of assessing measurement reliability in multivariate regression, Journal of the American Statistical Association 1992; 87 (419): 696-707.

[2] Cheng CL, Van Ness JW. Statistical Regression with Measurement Error, Arnold, London, 1999.

[3] Fuller WA. Measurement Error Models. New York: Wiley, 1987.

[4] Hoerl AE. Kennard RW. Ridge regression: Biased estimation for non-orthogonal Problems, Technometrics 1970; 12: 55-67.

[5] Liu K. A new class of biased estimate in linear regression, Commun. Statist. Theor. Meth. 1993; 22: 393-402.

[6] Saleh A.K.M.D. Shalabh, A ridge regression estimation approach to the measurement error model, Journal of Multivariate Analysis 2014; 123: 68-84.

[7] Üstündağ Şiray G. Liu estimation approach to the measurement error models. Journal of Statistical Computation and Simulation 2019; 89(18): 3453-3496.

[8] McDonald GC, Galarneau DI. A. Monte Carlo evaluation of some ridge-type estimators, J. Amer. Statist. Assoc. 1975; 20: 407-416.

[9] Longley JW. An appraisal of least squares programs for the electronic computer from the point of view of the user. Journal of the American Statistical Association 1967; 62: 819-841.

[10] Beaton AE, Rubin DB and Barone JL. The Acceptability of Regression Solutions: Another look at Computational Accuracy. Journal of the American Statistical Association 1976; 71(353): 158-168.

[11] Farebrother RW. Further Results on the Mean Square Error of Ridge Regression. Journal of the Royal Statistical Society 1976; 38: 248-250.