On Dual-Hyperbolic Numbers with Generalized Fibonacci and Lucas Numbers Components

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Article Info

Abstract

Dual-hyperbolic Fibonacci and Lucas numbers with Fibonacci and Lucas coefficients are introduced by Cihan et al. and some identities and theorems are given regarding modules and conjugates of these numbers. Later, generating function and Binet's formula with the help of this generating function have been derived. Also, Binet formula, Cassini's, Catalan's, d'Ocagne's, Honsberger and Tagiuri identities are found for dual-hyperbolic numbers with generalized Fibonacci and Lucas coefficients. While these operations are being done, we will benefit from the well-known Fibonacci and Lucas identitites. Moreover, it is seen that the results which are obtained for the values $p = 1$ and $q = 0$ corresponds to the theorems in the article by Cihan et al. [1].

1. Introduction

Italian mathematician Leonardo Fibonacci's Liber Abaci was one of the most important books on mathematics in the Middle Ages. Through this book mathematicians introduced Fibonacci number sequence concept. Several studies have been conducted with respect to Fibonacci numbers and Fibonacci quaternins [2]-[5].

Dual-hyperbolic numbers with Fibonacci and Lucas coefficients which is constitutes a new number system have been introduced by Cihan and her colleagues [1]. In this article, the dual-hyperbolic number system has been generalized based on the article [1].

Firstly, addition, multiplication, modules and conjugates of these numbers have been defined and the fundamental identities for these numbers regarding these operations have been proven. Then, we have defined generating function and this function helped us to find Binet's formula. Additionally, d'Ocagne's, Honsberger, Tagiuri, Catalan identities have been obtained and Cassini's identity has been given in case of $r = 1$ for the Catalan identity. Finally, we have discussed special cases and have given examples.

2. Preliminaries

The Fibonacci and Lucas numbers have many interesting properties and applications. Initial conditions for the Fibonacci and Lucas numbers are defined as follows respectively

$$F_0 = 0, \quad F_1 = 1,\ldots, \quad F_{n+1} = F_n + F_{n-1}, \quad n \geq 1$$

and

$$L_0 = 2, \quad L_1 = 1,\ldots, \quad L_{n+1} = L_n + L_{n-1}, \quad n \geq 1$$

where F_n and L_n denote the n-th Fibonacci and Lucas numbers, respectively.
Binet formula for the n-th Fibonacci and Lucas numbers are given by the following relation

$$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n), \qquad L_n = \alpha^n + \beta^n, \quad n \geq 1$$

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(see [2]).
On the other hand, Horadam introduced generalized Fibonacci numbers with the initial conditions as follows

$$H_1 = p, \quad H_2 = p+q, \quad p,q \in Z$$

where the recurrence relation is defined by

$$H_n = H_{n-1} + H_{n-2}, \quad n \geq 3,$$

or

$$H_n = (p-q)F_n + qF_{n+1}.$$

In the above equation, if we take $p = 1$ and $q = 0$, then the generalized Fibonacci number becomes Fibonacci number. If we take $p = 1$ and $q = 2$, then the generalized Lucas number becomes Lucas number. Furthermore, Horadam investigated Binet formula for the n-th generalized Fibonacci number such that

$$H_n = \frac{1}{2\sqrt{5}} \left(\alpha^n - \mu\beta^n \right)$$

(see [2]).

The set of dual-hyperbolic numbers is defined by

$$DH = \left\{ w = z_1 + z_2\varepsilon \,|\, z_1, z_2 \in H \, where, \, \varepsilon^2 = 0, \varepsilon \neq 0 \right\}.$$

If we consider two hyperbolic numbers $z_1 = x_1 + x_2 j$ and $z_2 = y_1 + y_2 j$, then any dual-hyperbolic number can be written as

$$w = x_1 + x_2 j + y_1\varepsilon + y_2 j\varepsilon.$$

There exist five different conjugates and these conjugates are given as follows

$$|\omega|^{\dagger_1} = \bar{z}_1 + \varepsilon\bar{z}_2, \quad \text{hyperbolic conjugation}$$
$$|\omega|^{\dagger_2} = z_1 - \varepsilon z_2, \quad \text{dual conjugation}$$
$$|\omega|^{\dagger_3} = \bar{z}_1 - \varepsilon\bar{z}_2, \quad \text{coupled conjugation}$$
$$|\omega|^{\dagger_4} = \bar{z}_1 \left(1 - \varepsilon\frac{z_2}{z_1} \right) \quad (\omega \in DH - A), \quad \text{dual} - \text{hyperbolic conjugation}$$
$$|\omega|^{\dagger_5} = z_2 - \varepsilon z_1, \quad \text{anti} - \text{dual conjugation}$$

where "$-$" denotes the standard complex conjugation and the zero divisors of DH is defined by the set A [6]. Namely, $DH - A$ is a multiplicative group. The dual hyperbolic numbers form a commutative ring with 0 characteristic. Unlike quaternions, the multiplication of dual-hyperbolic numbers with generalized Fibonacci and Lucas number has a commutative ring structure. However, multiplication of dual-hyperbolic numbers with generalized Fibonacci and Lucas number constitutes two-dimensional Complex Clifford and 4-dimensional Real Clifford algebra structure.

3. Properties of Dual-Hyperbolic numbers with generalized Fibonacci and Lucas coefficients

The dual-hyperbolic Fibonacci and Lucas numbers are defined as

$$DHF_n = F_n + F_{n+1}j + F_{n+2}\varepsilon + F_{n+3}j\varepsilon$$

and

$$DHL_n = L_n + L_{n+1}j + L_{n+2}\varepsilon + L_{n+3}j\varepsilon$$

respectively. Here, F_n and L_n are the n-th generalized Fibonacci number and Lucas numbers respectively and ε denotes dual unit $\left(\varepsilon^2 = 0, \, \varepsilon \neq 0 \right)$, j denotes imaginary unit $\left(j^2 = 1 \right)$, $j\varepsilon$ denotes imaginary-dual unit $\left(j\varepsilon^2 = 0 \right)$. After these numbers have been defined in the article [1], some identities regarding the modules, conjugates have been obtained for dual-hyperbolic Fibonacci and Lucas numbers. Then, negadual-hyperbolic Fibonacci, negadual-hyperbolic Lucas, d'Ocagne's, Cassini, Catalan identities and the correspondence of Binet formula have been given for these numbers. Now, Let's define the dual-hyperbolic number system with generalized Fibonacci and Lucas coefficients by considering the study [1].

Definition 3.1. *H_n is called as n-th Fibonacci number which have either $H_n = H_{n-1} + H_{n-2}$, $n \geq 3$ or $H_n = (p-q)F_n + qF_{n+1}$ the recurrence relations and depending on the initial values such that*

$$H_1 = p, \; H_2 = p+q, \; H_3 = 2q+3p, \dots \qquad (p,q \in Z).$$

Then, the sets of generalized Fibonacci and Lucas sequences are defined

$$DHX = \{ DHX_n = R_n + \varepsilon R_n^* = (H_n + jH_{n+1}) + \varepsilon(H_{n+2} + jH_{n+3}) \mid H_n \text{ Generalized Fibonacci Number} \}$$

and

$$DHY = \{ DHY_n = P_n + \varepsilon P_n^* = (V_n + jV_{n+1}) + \varepsilon(V_{n+2} + jV_{n+3}) \mid V_n \text{ Generalized Lucas Number} \}$$

where $\varepsilon \left(\varepsilon^2 = 0, \; \varepsilon \neq 0 \right)$, $j \left(j^2 = 1 \right)$ and $i\varepsilon \left((j\varepsilon)^2 = 0 \right)$, denote dual unit, imaginary unit and dual-imaginary unit, respectively. So the base elements of dual-hyperbolic numbers with generalized Fibonacci and Lucas coefficients are $(1, \, j, \, \varepsilon, \, j\varepsilon)$. Multiplication scheme of these base elements are given in Table 1.

×	1	j	ε	$j\varepsilon$
1	1	j	ε	$j\varepsilon$
j	j	1	$j\varepsilon$	ε
ε	ε	$j\varepsilon$	0	0
$j\varepsilon$	$j\varepsilon$	ε	0	0

Table 1: Multiplication scheme of dual-hyperbolic units

If two dual-hyperbolic numbers with generalized Fibonacci coefficients are $DHX_n^1 = R_n + R_n^* \varepsilon = H_n + H_{n+1} j + H_{n+2} \varepsilon + H_{n+3} j\varepsilon$ and $DHX_n^2 = K_n + K_n^* \varepsilon = G_n + G_{n+1} j + G_{n+2} \varepsilon + G_{n+3} j\varepsilon$ then the addition, subtraction and multiplication operations of these numbers are defined as

$$
\begin{aligned}
DHX_n^1 \pm DHX_n^2 &= (R_n + R_n^* \varepsilon) \pm (K_n + K_n^* \varepsilon) \\
&= (H_n + H_{n+1} j + H_{n+2} \varepsilon + H_{n+3} j\varepsilon) \pm (G_n + G_{n+1} j + G_{n+2} \varepsilon + G_{n+3} j\varepsilon) \\
&= (H_n \pm G_n) + (H_{n+1} \pm G_{n+1}) j + (H_{n+2} \pm G_{n+2}) \varepsilon + (H_{n+3} \pm G_{n+3}) j\varepsilon
\end{aligned}
\tag{3.1}
$$

and

$$
\begin{aligned}
DHX_n^1 \times DHX_n^2 &= (R_n + R_n^* \varepsilon) \times (K_n + K_n^* \varepsilon) \\
&= (H_n + H_{n+1} j + H_{n+2} \varepsilon + H_{n+3} j\varepsilon) \times (G_n + G_{n+1} j + G_{n+2} \varepsilon + G_{n+3} j\varepsilon) \\
&\quad + (H_n G_n + H_{n+1} G_{n+1}) + (H_n G_{n+1} + H_{n+1} G_n) j \\
&\quad + (H_n G_{n+2} + H_{n+1} G_{n+3} + H_{n+3} G_{n+1} + H_{n+2} G_n) \varepsilon \\
&\quad + (H_n G_{n+3} + H_{n+1} G_{n+2} + H_{n+2} G_{n+1} + H_{n+3} G_n) j\varepsilon
\end{aligned}
\tag{3.2}
$$

respectively. Also, any dual-hyperbolic number with generalized Fibonacci coefficient can be expressed as follows

$$
DHX_n = R_n + R_n^* \varepsilon = (H_n + H_{n+1} j) + (H_{n+2} + H_{n+3} j) \varepsilon.
\tag{3.3}
$$

This yields five different conjugates. Thus, these five different conjugates can be defined as follows

$$
DHX_n^{\dagger_1} = (H_n - H_{n+1} j) + (H_{n+2} - H_{n+3} j)\,\varepsilon, \qquad \text{hyperbolic conjugation}
\tag{3.4}
$$

$$
DHX_n^{\dagger_2} = (H_n + H_{n+1} j) - (H_{n+2} + H_{n+3} j)\,\varepsilon, \qquad \text{dual conjugation}
\tag{3.5}
$$

$$
DHX_n^{\dagger_3} = (H_n - H_{n+1} j) - (H_{n+2} - H_{n+3} j)\,\varepsilon, \qquad \text{coupled conjugation}
\tag{3.6}
$$

$$
DHX_n^{\dagger_4} = (H_n - H_{n+1} j) \times \left(1 - \frac{H_{n+2} + H_{n+3}\,j}{H_n + H_{n+1}\,j} \varepsilon \right), \qquad \text{dual} - \text{hyperbolic conjugation}
\tag{3.7}
$$

$$
DHX_n^{\dagger_5} = (H_{n+2} + H_{n+3}\,j) - (H_n + H_{n+1}\,j)\,\varepsilon, \qquad \text{anti} - \text{dual conjugation}.
\tag{3.8}
$$

Five different norms can be given for dual-hyperbolic numbers with generalized Fibonacci coefficients thanks to the definition of conjugates.

Definition 3.2. *Let DHX_n be a dual-hyperbolic number with generalized Fibonacci coefficient. In this case, j-th modulus of DHX_n are denoted by $|DHX_n|^2_{\dagger_i}$, $(j = 1, 2, 3, 4, 5)$ and are given as follows*

$$
\begin{aligned}
|DHX_n|^2_{\dagger_1} &= DHH_n \times DHH_n^{\dagger_1} \\
|DHX_n|^2_{\dagger_2} &= DHH_n \times DHH_n^{\dagger_2} \\
|DHX_n|^2_{\dagger_3} &= DHH_n \times DHH_n^{\dagger_3} \\
|DHX_n|^2_{\dagger_4} &= DHH_n \times DHH_n^{\dagger_4} \\
|DHX_n|^2_{\dagger_5} &= DHH_n \times DHH_n^{\dagger_5}
\end{aligned}
$$

Proposition 3.3. *Let DHX_n be a dual-hyperbolic number with generalized Fibonacci coefficient. Then, the following identities are satisfied:*

$$
DHX_n + DHX_n^{\dagger_1} = 2 \left(H_n + H_{n+2} \varepsilon \right)
\tag{3.9}
$$

$$
DHX_n \times DHX_n^{\dagger_1} = (H_n^2 - H_{n+1}^2) + 2\varepsilon \left(H_n H_{n+2} - H_{n+1} H_{n+3} \right)
\tag{3.10}
$$

$$
DHX_n + DHX_n^{\dagger_2} = 2 \left(H_n + H_{n+1} j \right)
\tag{3.11}
$$

$$
DHX_n \times DHX_n^{\dagger_2} = \left[(2p - q) H_{2n+1} - eF_{2n+1} \right] + 2 H_n H_{n+1} j
\tag{3.12}
$$

$$DHX_n + DHX_n^{\dagger_3} = 2\left(H_n + H_{n+3}\,j\varepsilon\right) \tag{3.13}$$

$$DHX_n \times DHX_n^{\dagger_3} = (H_n^2 - H_{n+1}^2) + j\varepsilon\left[2e(-1)^{n+1}\right] \tag{3.14}$$

$$DHX_n + DHX_n^{\dagger_4} = 2H_n + \varepsilon\,\frac{2H_{n+1}}{H_n^2 - H_{n+1}^2}\left[(H_{n+3}H_n - H_{n+1}H_{n+2}) + j\left(H_{n+2}H_n - H_{n+1}H_{n+3}\right)\right] \tag{3.15}$$

$$DHX_n \times DHX_n^{\dagger_4} = H_n^2 - H_{n+1}^2 \tag{3.16}$$

$$DHX_n + DHX_n^{\dagger_5} = (H_n + H_{n+2}) + (H_{n+1} + H_{n+3})\,j + H_{n+1}\varepsilon + H_{n+2}\,j\varepsilon \tag{3.17}$$

$$\begin{aligned}DHX_n \times DHX_n^{\dagger_5} &= (H_nH_{n+2} + H_{n+1}H_{n+3}) + j\left(H_nH_{n+3} + H_{n+1}H_{n+2}\right)\\ &\quad + \varepsilon\left(H_{n+2}^2 + H_{n+3}^2 - H_{n+1}^2 - H_n^2\right) + 2j\varepsilon\left(H_{n+3}H_{n+2} - H_{n+1}H_n\right)\end{aligned} \tag{3.18}$$

Proof. (3.9): Using equations (3.1), (3.3) and (3.4), we obtain

$$DHX_n + DHX_n^{\dagger_1} = 2\left(H_n + H_{n+2}\varepsilon\right).$$

Here, If the values $p = 1, q = 0$ are specially taken in the generalized Fibonacci number H_n, then it is concluded that $DHX_n + DHX_n^{\dagger_1} = 2\left(F_n + F_{n+2}\varepsilon\right)$.

(3.10): Considering equations (3.2), (3.3) and (3.4), the result is found by

$$\begin{aligned}DHX_n \times DHX_n^{\dagger_1} &= \left(H_n^2 + H_{n+1}^2\right) + 2\left(H_nH_{n+2} + H_{n+1}H_{n+3}\right)\varepsilon\\ &= (H_n^2 - H_{n+1}^2) + 2\varepsilon\left(H_nH_{n+2} - H_{n+1}H_{n+3}\right).\end{aligned}$$

Here, If the values $p = 1, q = 0$ are specially taken in the generalized Fibonacci number H_n, then it is concluded that $DHX_n \times DHX_n^{\dagger_1} = F_{2n+1} + 2F_{2n+3}\varepsilon$.

(3.11): From the equations (3.1), (3.3) and (3.5), we can reach the following identity

$$DHX_n + DHX_n^{\dagger_2} = 2\left(H_n + H_{n+1}\,j\right).$$

Here, If the values $p = 1, q = 0$ are specially taken in the generalized Fibonacci number H_n, $DHX_n + DHX_n^{\dagger_1} = 2\left(F_n + F_{n+2}\,j\right)$.

(3.12): Using the equations (3.2), (3.3), (3.5), using the identity $H_{n-1}^2 + H_n^2 = (2p - q)H_{2n-1} - eF_{2n-1}$ (see.ref. [2]) and simplifying we have

$$DHX_n \times DHX_n^{\dagger_2} = \left[(2p - q)H_{2n+1} - eF_{2n+1}\right] + 2H_nH_{n+1}\,j.$$

Here, If the values $p = 1, q = 0$ are specially taken in the generalized Fibonacci number H_n, then it is concluded that $DHX_n \times DHX_n^{\dagger_2} = F_{2n+1} + 2F_nF_{n+1}\,j$.

(3.13): We can write the following equation by using the equations (3.1), (3.3) and (3.6)

$$DHX_n + DHX_n^{\dagger_3} = 2\left(H_n + H_{n+3}\,j\varepsilon\right).$$

Here, If the values $p = 1, q = 0$ are specially taken in the generalized Fibonacci number H_n, then it is concluded that $DHX_n + DHX_n^{\dagger_3} = 2\left(F_n + F_{n+3}\,j\varepsilon\right)$.

(3.14): From equations (3.2), (3.3) and (3.6), we have

$$DHX_n \times DHX_n^{\dagger_3} = (H_n^2 - H_{n+1}^2) + j\varepsilon\left[2e(-1)^{n+1}\right].$$

While we are obtaining the above equation, the identity $H_nH_{n+r+1} - H_{n-s}H_{n+r+s+1} = (-1)^{n+s}eF_sF_{r+s+1}$ has been used [2]. Here, If the values $p = 1, q = 0$ are specially taken in the generalized Fibonacci number H_n, then it is concluded that $DHX_n \times DHX_n^{\dagger_3} = -F_{n-1}F_{n+2} - 2(-1)^n\,j\varepsilon$.

(3.15): If we take into account the equations (3.1), (3.3) and (3.7), then the following identity can be easily seen

$$DHX_n + DHX_n^{\dagger_4} = 2H_n + \varepsilon\,\frac{2H_{n+1}}{H_n^2 - H_{n+1}^2}\left[(H_{n+3}H_n - H_{n+1}H_{n+2}) + j\left(H_{n+2}H_n - H_{n+1}H_{n+3}\right)\right].$$

Here, If the values $p=1, q=0$ are specially taken in the generalized Fibonacci number H_n, then it is concluded that

$$DHX_n + DHX_n^{\dagger_4} = 2F_n + \varepsilon \frac{2F_{n+1}}{F_n^2 - F_{n+1}^2} \left[(F_{n+3}F_n - F_{n+1}F_{n+2}) + j(F_{n+2}F_n - F_{n+1}F_{n+3}) \right].$$

(3.16): By making the necessary operations with the help of the equations (3.2), (3.3), (3.7) and rearranging the last equation, the following identity can be given

$$DHX_n \times DHX_n^{\dagger_4} = H_n^2 - H_{n+1}^2.$$

Here, If the values $p=1, q=0$ are specially taken in the generalized Fibonacci number H_n, then it is concluded that $DHX_n \times DHX_n^{\dagger_4} = F_n^2 - F_{n+1}^2$.

(3.17): Considering the equations (3.1), (3.3) and (3.8), we have

$$DHX_n + DHX_n^{\dagger_5} = (H_n + H_{n+2}) + (H_{n+1} + H_{n+3})j + H_{n+1}\varepsilon + H_{n+2}j\varepsilon.$$

Here, If the values $p=1, q=0$ are specially taken in the generalized Fibonacci number H_n, then it is concluded that $DHX_n + DHX_n^{\dagger_5} = F_n + F_{n+2} + (F_{n+1} + F_{n+3})J + F_{n+1}\varepsilon + F_{n+2}j\varepsilon$.

(3.18): If we use equations (3.2), (3.3), (3.8) and make the necessary calculations, then the rearranged equation yields

$$DHX_n \times DHX_n^{\dagger_5} = (H_nH_{n+2} + H_{n+1}H_{n+3}) + j(H_nH_{n+3} + H_{n+1}H_{n+2}) \\ + \varepsilon (H_{n+2}^2 + H_{n+3}^2 - H_{n+1}^2 - H_n^2) + 2j\varepsilon (H_{n+3}H_{n+2} - H_{n+1}H_n).$$

Here, If the values $p=1, q=0$ are specially taken in the generalized Fibonacci number H_n, then it is concluded that

$$DHX_n \times DHX_n^{\dagger_5} = (F_nF_{n+2} + F_{n+1}F_{n+3}) + j(F_nF_{n+3} + F_{n+1}F_{n+2}) \\ + \varepsilon (F_{n+2}^2 + F_{n+3}^2 - F_{n+1}^2 - F_n^2) + 2j\varepsilon (F_{n+3}F_{n+2} - F_{n+1}F_n).$$

$\square$

Theorem 3.4. *Let DHX_n and DHX_{n-1} be two dual-hyperbolic numbers with generalized coefficients. There exist the following identities for these numbers and their conjugates:*

i) $\left(DHX_n \times DHX_n^{\dagger_1} \right) + \left(DHX_{n-1} \times DHX_{n-1}^{\dagger_1} \right) = -\left[(2p-q)H_{2n} - eF_{2n} \right] + 2\varepsilon \left(-H_{n+1}^2 \right)$

ii) $DHX_n^2 = 2H_nDHX_n - \left(DHX_n \times DHX_n^{\dagger_1} \right) + 2\varepsilon \left(H_{n+2}^2 - H_{n+1}H_{n+3} \right) + 2j\varepsilon (H_{n+1}H_{n+2})$

iii) $DHX_n^2 + DHX_{n-1}^2 = 2(2p-q)DHX_{2n-1} - DHX_n \times DHX_n^{\dagger_1} - DHX_{n-1} \times DHX_{n-1}^{\dagger_1} + (2p-q)(2H_{2n+3}\varepsilon + 2H_{2n+2}j\varepsilon)$
$\qquad -e(2F_{2n-1} + 2F_{2n}j + 2(F_{2n+3} + F_{2n+1})\varepsilon + 4F_{2n+2}i\varepsilon) - 2H_{n+1}^2\varepsilon$

iv) $DHY_n \times DHX_n^{\dagger_1} - DHY_n^{\dagger_1} \times DHX_n = (-1)^n \left[(4p^2 - 8pq + 8q^2) + (-6p^2)j\varepsilon \right].$

Proof. **i)** By using identity $H_{n-1}^2 + H_n^2 = (2p-q)H_{2n-1} - eF_{2n-1}$ [2] and considering the equations (3.2), (3.3) and the above equations which have been defined by Horadam, the proof can be seen easily.

ii) Considering the equation (3.1), the proof can be easily seen.

iii) From the identity $H_nH_m + H_{n+1}H_{m+1} = (2p-q)H_{m+n+1} - eF_{m+n+1}$ [3] and equation (3.1), the proof is completed.

iv) Using the equation (3.1) and the identity $L_nF_m = F_{m+n} + (-1)^m F_{m-n}$ [2], the desired result is obtained. Also, the equations given in Proposition 2.2. in the article [1] are specially obtained by giving values $p=1, q=0$ in the equations we have found above.

i) $\left(DHX_n \times DHX_n^{\dagger_1} \right) + \left(DHX_{n-1} \times DHX_{n-1}^{\dagger_1} \right) = -F_{2n} + 2\varepsilon \left(-F_{n+1}^2 \right)$

ii) $DHX_n^2 = 2H_nDHX_n - \left(DHX_n \times DHX_n^{\dagger_1} \right) + 2\varepsilon \left(F_{n+2}^2 - F_{n+1}F_{n+3} \right) + 2j\varepsilon (F_{n+1}F_{n+2})$

iii) $DHX_n^2 + DHX_{n-1}^2 = 4DHX_{2n-1} - DHX_n \times DHX_n^{\dagger_1} - DHX_{n-1} \times DHX_{n-1}^{\dagger_1} + [-2F_{2n-1} + 2(F_{2n+3} - F_{2n+1})\varepsilon - 2F_{2n}j] - 2F_{n+1}^2\varepsilon$

iv) $DHY_n \times DHX_n^{\dagger_1} - DHY_n^{\dagger_1} \times DHX_n = (-1)^n [4 + (-6)j\varepsilon]$

$\square$

Theorem 3.5. *Let DHX_n be a dual-hyperbolic number with generalized coefficient. Then, the following identities are valid:*

1) $DHX_n + DHX_{n+1} = DHX_{n+2}$

2) $(DHX_n)^2 = 2(H_nDHX_n) + 2(H_{n+1}DHX_{n+1}) - \left[(H_n^2 + H_{n+1}^2) + 2(H_{n+1}H_{n+2})j + 2(H_{n+1}H_{n+3})j\varepsilon \right]$

3) $-DHX_n + DHX_{n+1}j + DHX_{n+2}\varepsilon - DHX_{n+3}j\varepsilon = H_{n+1}$

4)
$$(DHX_n \times DHX_m) + (DHX_{n+1} \times DHX_{m+1}) = (2p-q)\left[(H_{m+n+1} + H_{m+n+3}) + 2H_{m+n+2}j + 2(H_{m+n+3} + H_{m+n+5})\varepsilon + 4H_{m+n+4}i\varepsilon \right]$$
$$-e\left[(F_{m+n+1} + F_{m+n+3}) + 2F_{m+n+2}j + 2(F_{m+n+3} + F_{m+n+5})\varepsilon + 4F_{m+n+4}j\varepsilon \right]$$

5)
$$DHX_n^2 + DHX_{n-1}^2 = [(2p-q)(H_{2n+1} + H_{2n-1}) - e(F_{2n+1} + F_{2n-1})] + 2j[(2p-q)H_{2n} - eF_{2n}]$$
$$+ 2\varepsilon[(2p-q)(H_{2n+3} + H_{2n+1}) - e(F_{2n+3} + F_{2n+1})] + 4j\varepsilon[(2p-q)H_{2n+2} - eF_{2n+2}]$$

Proof. **1)** Let DHX_n and DHX_{n+1} be two dual-hyperbolic numbers with generalized coefficients. In this case, taking into account that the equation (3.1), we get

$$DHX_n + DHX_{n+1} = H_{n+2} + H_{n+3}j + H_{n+4}\varepsilon + H_{n+5}j\varepsilon = DHX_{n+2}.$$

Here, if the values $p = 1$ and $q = 0$ are specially taken in the generalized Fibonacci number H_n, then it is concluded that $DHX_n + DHX_{n+1} = DHF_{n+2}$.

2) Let DHX_n be dual-hyperbolic numbers with generalized coefficients. If the equation (3.2) is used, then the following equality is obtained

$$DHX_n^2 = [(H_n + H_{n+1}j) + (H_{n+2} + H_{n+3}j)\varepsilon] \times [(H_n + H_{n+1}j) + (H_{n+2} + H_{n+3}j)\varepsilon]$$
$$= 2(H_n DHX_n) + 2(H_{n+1} DHX_{n+1}) - \left[(H_n^2 + H_{n+1}^2) + 2(H_{n+1}H_{n+2})j + 2(H_{n+1}H_{n+3})j\varepsilon \right].$$

Here, if the values $p = 1$ and $q = 0$ are specially taken in the generalized Fibonacci number H_n, then $(DHX_n)^2 = 2(F_n DHF_n) + 2(F_{n+1} DHF_{n+1}) - \left[(F_n^2 + F_{n+1}^2) + 2(F_{n+1}F_{n+2})j + 2(F_{n+1}F_{n+3})j\varepsilon \right]$ is found.

3) By considering the equation (3.1) and doing some algebraic calculations, we obtain

$$-DHX_n + DHX_{n+1}j + DHX_{n+2}\varepsilon - DHX_{n+3}j\varepsilon = -[(H_n + H_{n+1}j) + (H_{n+2} + H_{n+3}j)\varepsilon]$$
$$+ [(H_{n+1} + H_{n+2}j) + (H_{n+3} + H_{n+4}j)\varepsilon]j$$
$$+ [(H_{n+2} + H_{n+3}j) + (H_{n+4} + H_{n+5}j)\varepsilon]\varepsilon$$
$$- [(H_{n+3} + H_{n+4}j) + (H_{n+5} + H_{n+6}j)\varepsilon]i\varepsilon$$
$$= H_{n+1}.$$

Here, if the values $p = 1$ and $q = 0$ are specially taken in the generalized Fibonacci number H_n,

$$-DHX_n + DHX_{n+1}j + DHX_{n+2}\varepsilon - DHX_{n+3}j\varepsilon = F_{n+1}$$

is found.

4) Follows from the identity $H_n H_m + H_{n+1}H_{m+1} = (2p-q)H_{m+n+1} - eF_{m+n+1}$ (see ref. [3]) and using the equation (3.2), we achieve that

$$(DHX_n \times DHX_m) + (DHX_{n+1} \times DHX_{m+1}) = [(H_n + H_{n+1}j) + (H_{n+2} + H_{n+3}j)\varepsilon] \times [(H_m + H_{m+1}j) + (H_{m+2} + H_{m+3}j)\varepsilon]$$
$$+ [(H_{n+1} + H_{n+2}j) + (H_{n+3} + H_{n+4}j)\varepsilon] \times [(H_{m+1} + H_{m+2}j) + (H_{m+3} + H_{m+4}j)\varepsilon]$$
$$= (2p-q)\left[\begin{array}{l} (H_{m+n+1} + H_{m+n+3}) + 2H_{m+n+2}j + 2(H_{m+n+3} + H_{m+n+5})\varepsilon \\ + 4H_{m+n+4}i\varepsilon \end{array} \right]$$
$$- e[(F_{m+n+1} + F_{m+n+3}) + 2F_{m+n+2}j + 2(F_{m+n+3} + F_{m+n+5})\varepsilon + 4F_{m+n+4}j\varepsilon]$$

Here, If the values $p = 1$ and $q = 0$ are specially taken in the generalized Fibonacci number H_n

$$(DHX_n \times DHX_m) + (DHX_{n+1} \times DHX_{m+1}) = (F_{m+n+1} + F_{m+n+3}) + 2F_{m+n+2}j + 2(F_{m+n+3} + F_{m+n+5})\varepsilon + 4F_{m+n+4}j\varepsilon$$

5) Considering the identity $H_n H_m + H_{n+1}H_{m+1} = (2p-q)H_{m+n+1} - eF_{m+n+1}$ (see ref. [3]) and the equation (3.2), we reach the result. Here, If the values $p = 1$ and $q = 0$ are specially taken in the generalized Fibonacci number $DHX_n^2 + DHX_{n-1}^2 = (F_{2n+1} + F_{2n-1}) + 2F_{2n}j + 2(F_{2n+3} + F_{2n+1})\varepsilon + 4F_{2n+2}j\varepsilon$. $\square$

Theorem 3.6. *Let DHX_n and DHL_n be dual-hyperbolic Fibonacci and dual-hyperbolic Lucas numbers with generalized Fibonacci and Lucas coefficients, respectively. For $n \geq 0$, there exist the following relationships between these numbers:*
1) $DHX_{n+1} + DHX_{n-1} = pDHL_n + qDHL_n$

2) $DHX_{n+2} - DHX_{n-2} = pDHL_n + qDHL_n$

Proof. Equations **1)** and **2)** are found by taking the identity $H_{n+1} + H_{n-1} = pL_n + qL_{n-1}$ (see ref. [4]) and using the recurrence relation $H_n = (p-q)F_n + qF_{n+1}$, respectively.

$$DHX_{n+1} + DHX_{n-1} = (H_{n+1} + H_{n+2}j + H_{n+3}\varepsilon + H_{n+4}j\varepsilon) + (H_{n-1} + H_n j + H_{n+1}\varepsilon + H_{n+3}j\varepsilon)$$
$$= (H_{n+1} + H_{n-1}) + (H_{n+2} + H_n)j + (H_{n+3} + H_{n+1})\varepsilon + (H_{n+4} + H_{n+3})j\varepsilon$$
$$= (pL_n + qL_{n-1}) + (pL_{n-1} + qL_n)j + (pL_{n+2} + qL_{n+1})\varepsilon + (qL_{n+3} + qL_{n+2})j\varepsilon$$
$$= pDHL_n + qDHL_n$$

$$DHX_{n+2} - DHX_{n-2} = (H_{n+2} + H_{n+3}j + H_{n+4}\varepsilon + H_{n+5}j\varepsilon) - (H_{n-2} + H_{n-1}j + H_n\varepsilon + H_{n+1}j\varepsilon)$$
$$= (H_{n+2} - H_{n-2}) + (H_{n+3} - H_{n-1})j + (H_{n+4} - H_n)\varepsilon + (H_{n+5} - H_{n+1})j\varepsilon$$
$$= (pL_n + qL_{n-1}) + (pL_{n-1} + qL_n)j + (pL_{n+2} + qL_{n+1})\varepsilon + (qL_{n+3} + qL_{n+2})j\varepsilon$$
$$= pDHL_n + qDHL_n$$

Here, if the values $p = 1, q = 0$ are specially taken in the generalized Fibonacci number H_n, then the desired results are obtained.   □

Theorem 3.7. *The sums of the dual-hyperbolic numbers with generalized Fibonacci coefficients satisfy the following relations:*

1) $\sum\limits_{s=1}^{n} DHX_s = DHX_{n+2} - DHX_2$

2) $\sum\limits_{s=0}^{p} DHX_{s+n} + DHX_{x+1} = DHX_{n+p+2}$

3) $\sum\limits_{s=1}^{n} DHX_{2s-1} = DHX_{2n} - DHX_0$

4) $\sum\limits_{s=1}^{n} DHX_{2s} = DHX_{2n+1} - DHX_1$

Proof. Using the identity $\sum\limits_{t=a}^{n} H_t = H_{n+2} - H_{a+1}$ (see ref. [4]), the proof can be seen easily as follows

1) $\sum\limits_{s=1}^{n} DHX_s = \sum\limits_{s=1}^{n} H_s + j\sum\limits_{s=1}^{n} H_{s+1} + \varepsilon\sum\limits_{s=1}^{n} H_{s+2} + j\varepsilon\sum\limits_{s=1}^{n} H_{s+3} = DHX_{n+2} - DHX_2$

2) $\sum\limits_{s=0}^{p} DHX_{n+s} + DHX_{n+1} = \sum\limits_{s=1}^{n} H_{n+s} + H_{n+1} + j\sum\limits_{s=1}^{n} H_{n+s+1} + H_{n+2} + \varepsilon\sum\limits_{s=1}^{n} H_{n+s+2} + H_{n+3} + j\varepsilon\sum\limits_{s=1}^{n} H_{n+s+3} + H_{n+4} = DHX_{n+p+2}$

3) $\sum\limits_{s=1}^{n} DHX_{2s-1} = \sum\limits_{s=1}^{n} H_{2s-1} + j\sum\limits_{s=1}^{n} H_{2s} + \varepsilon\sum\limits_{s=1}^{n} H_{2s+1} + j\varepsilon\sum\limits_{s=1}^{n} H_{2s+2} = DHX_{2n} - DHX_0$

4) $\sum\limits_{s=1}^{n} DHX_{2s} = \sum\limits_{s=1}^{n} H_{2s} + j\sum\limits_{s=1}^{n} H_{2s+1} + \varepsilon\sum\limits_{s=1}^{n} H_{2s+2} + j\varepsilon\sum\limits_{s=1}^{n} H_{2s+3} = DHX_{2n+1} - DHX_1$

Also, if we consider the values $p = 1, q = 0$ in the generalized Fibonacci number H_n, then the above equations becomes as follows:

1) $\sum\limits_{s=1}^{n} DHX_s = DHF_{n+2} - DHF_2$

2) $\sum\limits_{s=0}^{p} DHX_{s+n} + DHX_{x+1} = DHF_{n+p+2}$

3) $\sum\limits_{s=1}^{n} DHX_{2s-1} = DHF_{2n} - DHF_0$

4) $\sum\limits_{s=1}^{n} DHX_{2s} = DHF_{2n+1} - DHF_1$ □

Now, let's find correspondence of the Binet formula for the dual-hyperbolic Fibonacci numbers which helps to find golden ratio.

Theorem 3.8. *Let DHX_n be dual-hyperbolic number with generalized Fibonacci coefficient. For $m, n \geq 1$, the Binet formula for this number is given by*

$$DHX_n = \frac{\bar{\alpha}\,\alpha^n - \bar{\beta}\,\beta^n}{\alpha - \beta}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$ and the coefficients $\bar{\alpha}$, $\bar{\beta}$ are as follows

$$\bar{\alpha} = (p - q\beta) + [p(1 - \beta) + q]j + [p(2 - \beta) + q(1 - \beta)]\varepsilon + [p(3 - 2\beta) + q(2 - \beta)]j\varepsilon$$

and

$$\bar{\beta} = (p - q\alpha) + [p(1 - \alpha) + q]j + [p(2 - \alpha) + q(1 - \alpha)]\varepsilon + [p(3 - 2\alpha) + q(2 - \alpha)]j\varepsilon.$$

Proof. If t_1 and t_2 denote the roots of characteristic equation $t^2 - t - 1 = 0$ associated to the recurrence relation $DHX_n + DHX_{n+1} = DHX_{n+2}$. Then, these roots can be found as $\alpha = t_1 = \frac{1+\sqrt{5}}{2}$ and $\beta = t_2 = \frac{1-\sqrt{5}}{2}$. Note that, $\alpha + \beta = 1, \alpha.\beta = -1$ and $\alpha - \beta = \sqrt{5}$. Therefore, the general term of the dual-hyperbolic number sequence with generalized Fibonacci coefficients may be expressed in the form:

$$DHX_n = A\alpha^n + B\beta^n$$

for some coefficients A and B. For $n = 0$ and $n = 1$, the following equalities can be written

$$DHX_0 = (q,\ p,\ p+q,\ 2p+q)$$

and

$$DHX_1 = (p,\ p+q,\ 2p+q,\ 3p+2q).$$

Also, if we give to n the values $n = 0$ and $n = 1$, we get

$$DHX_0 = A + B$$

and

$$DHX_1 = \alpha A + \beta B.$$

Then, solving this system of linear equations, we have

$$A = \frac{DHX_1 - \beta DHX_0}{\alpha - \beta} \qquad and \qquad B = \frac{\alpha DHX_0 - DHX_1}{\alpha - \beta}$$

where some coefficients $\bar{\alpha}$ and $\bar{\beta}$ are

$$\bar{\alpha} = (p - q\beta) + [p(1 - \beta) + q] j + [p(2 - \beta) + q(1 - \beta)] \varepsilon + [p(3 - 2\beta) + q(2 - \beta)] j\varepsilon$$

and

$$\bar{\beta} = (p - q\alpha) + [p(1 - \alpha) + q] j + [p(2 - \alpha) + q(1 - \alpha)] \varepsilon + [p(3 - 2\alpha) + q(2 - \alpha)] j\varepsilon.$$

$\square$

Theorem 3.9. *The generating function for dual-hyperbolic number with generalized coefficients is*

$$g(x) = \frac{1}{1 - x - x^2} \sum_{s=0}^{3} (DHX_s + DHX_{s-1}x)e_s.$$

Proof. Assuming that the generating function for dual-hyperbolic number with generalized coefficients becomes

$$g(x) = \sum_{n=0}^{\infty} P_n x^n.$$

such that

$$P_n = (DHX_n, \ DHX_{n+1}, \ DHX_{n+2}, DHX_{n+3}).$$

Multiplying the generating function by x and x^2, the following equalities can be written

$$\begin{aligned} xg(x) &= P_0 x + P_1 x^2 + \ldots + P_{n-1} x^n + \ldots \\ x^2 g(x) &= P_0 x^2 + P_1 x^3 + \ldots + P_{n-2} x^n + \ldots \end{aligned}$$

After some algebraic calculations, we obtain

$$g(x) = \frac{1}{1 - x - x^2} \sum_{s=0}^{3} (P_0 + (P_1 - P_0)x).$$

This completes the proof. $\square$

Now, let's write the Binet formula in terms of the generating function which has been obtained in Theorem 3.9.

Theorem 3.10. *Binet formula for the dual-hyperbolic numbers with generalized Fibonacci coefficients is*

$$P_n = P_1 H_n + P_0 H_{n-1}.$$

Proof. Let's we take the relation

$$P_n = A\alpha^n + B\beta^n.$$

Putting $n = 0$ and $n = 1$ in the above equation, A and B are obtained by

$$A = \frac{P_1 - \beta P_0}{\alpha - \beta}, \quad B = \frac{\alpha P_0 - P_1}{\alpha - \beta}$$

In this case, P_n can be rewritten as

$$P_n = \frac{1}{\alpha - \beta} \left[(P_1 - \beta P_0) \alpha^n + (\alpha P_0 - P_1) \beta^n \right].$$

When the equalities of P_0 and P_1 is written in Theorem 3.9 and is arranged, P_n is found as

$$P_n = \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) \sum_{s=0}^{3} DHX_{s+1} e_s + \left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} \right) \sum_{s=0}^{3} DHX_s e_s.$$

Finally

$$P_n = P_1 H_n + P_0 H_{n-1}$$

is obtained. $\square$

Let us express the Catalan identity which is one of the most known identities of Fibonacci numbers.

Theorem 3.11. *(Catalan's Identity)*
For $n \geq r$, the relation

$$DHX_n^2 - DHX_{n+r} \times DHX_{n-r} = (-1)^{n-r}\mu F_r^2 [j+3j\varepsilon]$$

is verified.

Proof. Squaring DHX_n, multiplying DHX_{n+r} and DHX_{n-r} and noting that $H_{m+k}H_{n-k} - H_m H_n = (-1)^{n-k+1}\mu F_k F_{m+k-n}$ [7], the following equalities are obtained

$$DHX_n^2 = H_n^2 + H_{n+1}^2 + 2H_n H_{n+1} j + 2(H_n H_{n+2} + H_{n+1}H_{n+3})\varepsilon + 2(H_{n+1}H_{n+2} + H_n H_{n+3})j\varepsilon$$

and

$$\begin{aligned}
DHX_{n+r} \times DHX_{n-r} = & \, H_{n+r}H_{n-r} + H_{n+r+1}H_{n-r+1} + (H_{n+r+1}H_{n-r} + H_{n+r}H_{n-r+1})j \\
& + (H_{n+r}H_{n-r+2} + H_{n+r+1}H_{n-r+3} + H_{n-r+2}H_{n-r} + H_{n-r+3}H_{n-r+1})\varepsilon \\
& + (H_{n+r+1}H_{n-r+2} + H_{n+r}H_{n-r+3} + H_{n+r+3}H_{n-r} + H_{n+r+2}H_{n-r+1})j\varepsilon.
\end{aligned}$$

Adding the above equations gives us the proof. Writting $p = 1$ and $q = 0$ in the Catalan identity for dual-hyperbolic numbers with generalized coefficients, Catalan identity for dual-hyperbolic numbers with Fibonacci coefficients is found. Namely

$$DHX_n^2 - DHX_{n+r} \times DHX_{n-r} = (-1)^{n-r}F_r^2(j+3j\varepsilon).$$

$\square$

Let's give Cassini identity for generalized dual-hyperbolic numbers as a special case of Catalan identity.

Theorem 3.12. *(Cassini's Identity)*
Let DHX_n be the dual-hyperbolic number with generalized Fibonacci coefficients. For $n \geq 1$, we have

$$DHX_n^2 - (DHX_{n+1} \times DHX_{n-1}) = (-1)^{n-1}\mu(j+3j\varepsilon).$$

Proof. For $r = 1$, we see that the identity in Theorem 3.11 becomes the desired identity. Putting $p = 1$ and $q = 0$ in the above identity, we get

$$DHX_n^2 - (DHX_{n+1} \times DHX_{n-1}) = (-1)^{n-1}(j+3j\varepsilon).$$

This identity is Cassini formula for dual-hyperbolic numbers.

$\square$

Theorem 3.13. *(Honsberger Identity)*
For $n,m \geq 0$, the Honsberger identity for the dual-hyperbolic number with generalized coefficient DHX_n is given by

$$\begin{aligned}
(DHX_{k-1} \times DHX_n) + (DHX_k \times DHX_{n+1}) = & \, [(2p-q)(H_{k+n} + H_{k+n+2}) - e(F_{k+n} + F_{k+n+2})] \\
& + 2j[(2p-q)H_{k+n+1} - eF_{k+n+1}] \\
& + 2\varepsilon[(2p-q)(H_{k+n+2} + H_{k+n+4}) - e(F_{k+n+2} + F_{k+n+4})] \\
& + 4j\varepsilon[(2p-q)H_{k+n+3} - eF_{k+n+3}].
\end{aligned}$$

Proof. If we take into consider the equations (3.1), (3.2) and use the identity $H_n H_m + H_{n+1}H_{m+1} = (2p-q)H_{m+n+1} - eF_{m+n+1}$ (see ref. [3]), we complete the proof. If the values $p = 1, q = 0$ are specially taken in the generalized Fibonacci number H_n, then the following identity is found

$$\begin{aligned}
(DHX_{k-1} \times DHX_n) + (DHX_k \times DHX_{n+1}) = & \, (F_{k+n} + F_{k+n+2}) + 2F_{k+n+1}j \\
& + 2(F_{k+n+2} + F_{k+n+4})\varepsilon + 4F_{k+n+3}j\varepsilon.
\end{aligned}$$

$\square$

Theorem 3.14. *(Tagiuri Identity)*
Let DHX_n be the dual-hyperbolic number with generalized Fibonacci coefficients. For m $n,m \geq 1$, Tagiuri's identity is as follows:

$$(DHX_{m+k} \times DHX_{n-k}) - (DHX_m \times DHX_n) = (-1)^{n-k-1}\mu F_k F_{m+k-n}(j+3j\varepsilon).$$

Proof. The proof can be easily seen by using the identity $H_{m+k}H_{n-k} - H_m H_n = (-1)^{n-k+1}\mu F_k F_{m+k-n}$ [7] and equations (3.1) and (3.2). If the values $p = 1, q = 0$ are specially taken in the generalized Fibonacci number H_n, then it is concluded that

$$(DHX_{m+k} \times DHX_{n-k}) - (DHX_m \times DHX_n) = (-1)^{n-k-1}F_k F_{m+k-n}(j+3j\varepsilon).$$

$\square$

Theorem 3.15. *(d'Ocagne's Identity)*
Let DHX_n be the dual-hyperbolic number with generalized Fibonacci coefficients. For $m > n$, $m \in N$ and $n \in Z$, we have

$$(DHX_{m+k} \times DHX_{n-k}) - (DHX_m \times DHX_n) = \mu F_{m-n}(-1)^n(j+3j\varepsilon).$$

Proof. Using identity $H_{m+k}H_{n-k} - H_m H_n = (-1)^{n-k+1}\mu F_k F_{m+k-n}$ [7] and the equations (3.1) and (3.2), the proof is completed. If the values $p = 1, q = 0$ are specially taken in the generalized Fibonacci number H_n, then the following identity is found

$$(DHX_{m+k} \times DHX_{n-k}) - (DHX_m \times DHX_n) = F_{m-n}(-1)^n(j+3j\varepsilon).$$

$\square$

4. Conclusion

Our main aim in this study was to generalize the study which was done on dual-hyperbolic numbers. It was seen that, some theorems were obtained as a result of this generalization and they corresponded to the theorems in the article [1] for $p = 1$, $q = 0$. Also, the generating function was obtained and the Binet formula was given with the help of the generating function. Unlike the identities which was given in the article [1], Honsberger and Tagiuri identities were proved. At the same time, special cases of these identities were discussed. Because of the fact that generalized Fibonacci and Lucas coefficient dual-hyperbolic number system have commutative algebra structure, five different conjugates can be defined. As a result, in addition to the identities related to the conjugates which we obtained in Proposition 3.4, the following identities are given.

i)

$$\left(DHX_n \times DHX_n^{\dagger_2}\right) + \left(DHX_{n-1} \times DHX_{n-1}^{\dagger_2}\right) = (2p-q)\left[H_{2n-1} + H_{2n+2}\right] - e\left[F_{2n-1} + F_{2n+2}\right] + 2jH_nH_{n-1}$$

$$\left(DHX_n \times DHX_n^{\dagger_3}\right) + \left(DHX_{n-1} \times DHX_{n-1}^{\dagger_3}\right) = -(1+2j)\left[(2p-q)H_{2n} - eF_{2n}\right]$$

$$\left(DHX_n \times DHX_n^{\dagger_4}\right) + \left(DHX_{n-1} \times DHX_{n-1}^{\dagger_4}\right) = -\left[(2p-q)H_{2n} - eF_{2n}\right]$$

$$\left(DHX_n \times DHX_n^{\dagger_5}\right) + \left(DHX_{n-1} \times DHX_{n-1}^{\dagger_5}\right) = \left[(2p-q)\left(H_{2n+3} + H_{2n+1}\right) - e\left(F_{2n+3} + F_{2n+1}\right)\right]$$
$$+ 2j\left[(2p-q)H_{2n+2} - eF_{2n+2}\right]$$
$$+ \varepsilon\left[(2p-q)\left(2H_{2n} + H_{2n+5}\right) - e\left(2F_{2n+3} + F_{2n+5}\right)\right]$$
$$+ 2j\varepsilon\left[(2p-q)\left(H_{2n+2} + H_{2n}\right) - e\left(F_{2n+2} + F_{2n}\right)\right]$$

ii)

$$DHX_n^2 = 2H_nDHX_n - DHX_nDHX_n^{\dagger_2} + 2H_{n+1}\left(H_{n+1} + H_nj + H_{n+2}\varepsilon + H_{n+2}j\varepsilon\right)$$

$$DHX_n^2 = 2H_nDHX_n - DHX_nDHX_n^{\dagger_3} + 2\left(H_{n+2}H_{n+1}\varepsilon + H_nH_{n+3}j\varepsilon\right)$$

$$DHX_n^2 = 2H_nDHX_n - DHX_nDHX_n^{\dagger_4} + 2H_{n+2}\left(H_n\varepsilon + H_{n+1}j\varepsilon\right)$$

$$DHX_n^2 = 2H_nDHX_n - DHX_nDHX_n^{\dagger_5} + \left(H_{n+2}H_{n-1} + H_{n+2}H_n + H_{n+1}H_{n+3}\right)$$
$$+ \left(H_{n+3}H_n + H_{n+1}H_{n+2}\right)j + \left(2H_{n+2}H_n + H_{n+2}^2 + H_{n+3}^2 - H_{n+1}^2 - H_n^2\right)\varepsilon$$
$$+ 2\left(H_{n+2}H_{n+1} + H_{n+2}H_{n+3} - H_{n+1}H_n\right)j\varepsilon$$

iii)

$$DHX_n^2 + DHX_{n-1}^2 = 2(2p-q)DHX_{2n-1} - DHX_nDHX_n^{\dagger_2} - DHX_{n-1}DHX_{n-1}^{\dagger_2}$$
$$+ (2p-q)\left(2H_{2n+1} + 2H_{2n+3}\varepsilon + 2H_{2n+2}j\varepsilon\right)$$
$$- e\left(2F_{2n-1} + 2F_{2n+1} + F_{2n}j + 2\left(F_{2n+3} + F_{2n+1}\right)\varepsilon + 4F_{2n+2}j\varepsilon\right) + 2\left(H_nH_{n-1}\right)j$$

$$DHX_n^2 + DHX_{n-1}^2 = 2(2p-q)DHX_{2n-1} - DHX_nDHX_n^{\dagger_3} - DHX_{n-1}DHX_{n-1}^{\dagger_3}$$
$$+ (2p-q)\left(-2H_{2n}j + 2H_{2n+3}\varepsilon + 2H_{2n+2}j\varepsilon\right)$$
$$- e\left(2F_{2n-1} + 2\left(F_{2n+3} + F_{2n+1}\right)\varepsilon + 4F_{2n+2}j\varepsilon\right)$$

$$DHX_n^2 + DHX_{n-1}^2 = 2(2p-q)DHX_{2n-1} - DHX_nDHX_n^{\dagger_4} - DHX_{n-1}DHX_{n-1}^{\dagger_4}$$
$$+ 2(2p-q)\left(H_{2n+3}\varepsilon + H_{2n+2}j\varepsilon\right)$$
$$- 2e\left(F_{2n-1} + F_{2n}j + \left(F_{2n+3} + F_{2n+1}\right)\varepsilon + 2F_{2n+2}j\varepsilon\right)$$

$$DHX_n^2 + DHX_{n-1}^2 = 2(2p-q)DHX_{2n-1} - DHX_nDHX_n^{\dagger_5} - DHX_{n-1}DHX_{n-1}^{\dagger_5}$$
$$+ (2p-q)\left[H_{2n+4} + 2H_{2n+2}j + \left(2H_{2n+3} + 2H_{2n} + H_{2n+5}\right)\varepsilon + 2\left(2H_{2n+2} + H_{2n}\right)j\varepsilon\right]$$
$$- e\left[F_{2n+3} + 2F_{2n+1} + F_{2n-1} + 2\left(F_n + F_{2n+2}\right) + \left(4F_{2n+3} + F_{2n+5}\right)\varepsilon + \left(6F_{2n+2} + F_{2n}\right)j\varepsilon\right]$$

iv)

$$DHY_n \times DHX_n^{\dagger_2} - DHY_n^{\dagger_2} \times DHX_n = 4(-1)^n\left[2p^2\varepsilon + p^2j\varepsilon\right]$$

$$DHY_n \times DHX_n^{\dagger_3} - DHY_n^{\dagger_3} \times DHX_n = 4(-1)^n\left[\left(p^2 - 2pq + 2q^2\right)j - 2p^2\varepsilon\right]$$

$$DHY_n \times DHX_n^{\dagger_4} - DHY_n^{\dagger_4} \times DHX_n = \left[4(-1)^{n-1}\left(p^2 + 2pq - 2q^2\right)\right]j$$
$$+ \frac{8(-1)^np^2}{(V_n + V_{n+1}i)(H_n + H_{n+1}i)}\begin{bmatrix} -p^2F_{2n+1} + pq\left(-2F_{2n+2} - F_{2n}\right) \\ +q^2\left(F_{2n+1} + F_{2n}\right) \end{bmatrix}j\varepsilon$$

$$DHY_n \times DHX_n^{\dagger_5} - DHY_n^{\dagger_5} \times DHX_n = 2(-1)^{n-1}p^2j$$

The proofs of these identities are easily seen by following the similar ways in the proof of Theorem 3.4. Finally, the special values $p = 1$ and $q = 0$ provide the above equations.

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