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Blow up of solutions for a parabolic equation of Kirchhoff-type with multiple nonlinearities

Erhan PİŞKİN[1], *Fatma EKİNCİ[2]

[1]Dicle University, Department of Mathematics, 21280 Diyarbakır, Turkey.
E-mail: episkin@dicle.edu.tr
[2]Dicle University, Department of Mathematics, Institute of Natural and Applied Sciences, 21280 Diyarbakır, Turkey.

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* Corresponding author.
E-mail address:
ekincifatma2017@gmail.com

ABSTRACT

In this work, we considered a class of nonlinear parabolic equation with Krichhoff-type. We study a blow up of solutions with negative initial energy.

1. Introduction

In this work, we are interested in the blow up of solutions of the following reaction-diffusion equations with multiple nonlinearities

$$\begin{cases} u_t - M(\|\nabla u\|^2)\Delta u + |u|^{q-2}u_t = |u|^{p-2}u, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t \geq 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where $M(s) = 1 + s^{\gamma}$ $(\gamma > 0,\ s \geq 0)$, Ω is a bounded domain in R^n $(n = 1,2,3)$ with smooth boundary $\partial\Omega$ and $p, q > 2$ are real numbers.

In the absence of $|u|^{q-2}u_t$ term, equation (1) becomes to the following equation

$$u_t - M(\|\nabla u\|^2)\Delta u = |u|^{p-2}u. \tag{2}$$

Han and Li [1] investigated the long time behaviours of solution for problem (2) by using potential well method and variational method. They studied global exitence and blow up of solution when initial energy was supercritical, critical and subcritical. Moreover, Han et al. [2] considered the upper and lower bounds for the blow-up time and gave a new blow-up criterion for problem (2) when the initial energy is positive.

Tuan et al. [3] considered the following equation

$$u_t - M(\|\nabla u\|^2)\Delta u = F\big(x, t, u(x,t)\big). \tag{3}$$

They gave the circumstances for the existence of the first time backward problem (3) and indicated that problem (3) is ill-posed in the sense of Hadamard. In [4, 5] the authors investigated problem (3) with initial-boundary conditions.

Kundu et al. [6] considered the following equation

$$u_t - (1 + \|\nabla u\|^2)\Delta u = F(x,t), \tag{4}$$

with the initial and boundary conditions of Dirichlet type.

In [7, 8] the authors investigated the following equation

$$u_t - M(\|\nabla u\|)\Delta u = F(x,t), \tag{5}$$

where M is nonlinear nonlocal form in u.

This work is organized as follows: In section 2, we present some notations and stament of assumptions. In section 3, the blow up of the solution is given.

2. Preliminaries

In this part, we shall give some assumptions which will be used along this paper. Let $\|.\|$, , $\|.\|_p$ and $(u,v) = \int_\Omega u(x)v(x)\,dx$ denote the usual $L^2\,(\Omega)$ norm, $L^p\,(\Omega)$ norm and inner product of $L^2\,(\Omega)$, respectively.

To state and prove our main result, we need the following assumptions.
For the numbers p and q, we assume that

$$\begin{cases} 2 < q < p < +\infty \ \ if\ n = 1,2, \\ 2 < q < p \le \frac{2(n-1)}{n-2} \ \ if\ n \ge 3. \end{cases} \tag{6}$$

Similar to [9], $u(x,t)$is a solution of problem (1) on $\Omega \times [0,T)$ if

$$\begin{cases} u \in L^\infty\left(0,T;H_0^1(\Omega)\right), \\ u_t \in L^2\left(0,T;L^2(\Omega)\right), \\ |u|^{q-2}u_t \in L^2\left(\Omega \times [0,T)\right) \end{cases} \tag{7}$$

satisfying the initial condition $u(x,0) = u_0(x)$ and

$$\int_\Omega [M(\|\nabla u\|^2)\nabla u \nabla v + u_t v + |u|^{q-2}u_t v - |u|^{p-2}uv]\,dx = 0, \tag{8}$$

for all $v \in\ C(0,T;H_0^1(\Omega))$.
The energy functional associated with problem (1) is

$$E(t) = -\frac{1}{p}\|u\|_p^p + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}, \tag{9}$$

where $u \in H_0^1(\Omega)$.

Direct differentiation of (9), using (1), leads to

$$E^{'}(t) = -\|u_t\|^2 - \int_\Omega |u|^{q-2}{u_t}^2 dx < 0, \tag{10}$$

and then

$$E(t) \le E(0). \tag{11}$$

3. Blow up of solutions

In this section, we deal with the blow up result of the solution for problem (1).

Theorem 3.1 Suppose that (6) hold and $p > 2(\gamma+1)$, $u_0 \in H_0^1(\Omega)$ and u is a local solution of problem (1) and $E(0) < 0$. Then, the solution of the problem (1) blows up in finite time.

Proof. We set

$$H(t) = -E(t). \tag{12}$$

Thanks to (10) and (12), we get

$$H^{'}(t) = -E^{'}(t) \ge 0. \tag{13}$$

Since $E(0) < 0$, we have

$$H(0) = -E(0) > 0. \tag{14}$$

Then, by the integrate (13), we obtain

$$0 < H(0) \le H(t). \tag{15}$$

From (9) and (12), we have

$$H(t) - \frac{1}{p}\|u\|_p^p = -\frac{1}{2}\|\nabla u\|^2 - \frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} < 0. \tag{16}$$

Therefore,

$$0 < H(0) \le H(t) \le \frac{1}{p}\|u\|_p^p. \tag{17}$$

Let us define the functional

$$G(t) = H(t)^{1-\sigma} + \frac{\varepsilon}{2}\|u\|^2, \tag{18}$$

where $\varepsilon > 0$ small be chosen later and $0 < \sigma < (p-q)/p$.

By derivativing (18) to time and by (1), we get

$$\begin{aligned} G'(t) &= (1-\sigma)H(t)^{-\sigma}H'(t) + \varepsilon\int_\Omega uu_t dx \\ &= (1-\sigma)H(t)^{-\sigma}H'(t) + 2(\gamma+1)\varepsilon H(t) + 2(\gamma+1)\varepsilon E(t) \\ &\quad -\varepsilon\|\nabla u\|^2 - \varepsilon\|\nabla u\|^{2(\gamma+1)} + \varepsilon\|u\|_p^p - \varepsilon\int_\Omega |u|^{q-2}uu_t dx \\ &= (1-\sigma)H(t)^{-\sigma}H'(t) + 2(\gamma+1)\varepsilon H(t) \\ &\quad +\varepsilon\left(1 - \frac{2(\gamma+1)}{p}\right)\|u\|_p^p + \varepsilon\gamma\|\nabla u\|^2 - \varepsilon\int_\Omega |u|^{q-2}uu_t dx. \end{aligned} \tag{19}$$

In order to estimate the last terms in the right-hand side of (19), we use the following Young's inequality,

$$ab \le \delta^{-1}a^2 + \delta b^2, \quad \delta > 0,$$

so we have

$$\begin{aligned} \int_\Omega |u|^{q-2}uu_t dx &\le \int_\Omega |u|^{\frac{q-2}{2}}u_t|u|^{\frac{q-2}{2}}u dx \\ &\le \delta^{-1}\int_\Omega |u|^{q-2}{u_t}^2 dx + \delta\int_\Omega |u|^q\, dx. \end{aligned}$$

Hence, (19) becomes the following

$$\begin{aligned} G'(t) &\ge (1-\sigma)H(t)^{-\sigma}H'(t) + 2(\gamma+1)\varepsilon H(t) + \varepsilon\left(1 - \frac{2(\gamma+1)}{p}\right)\|u\|_p^p \\ &\quad +\varepsilon\gamma\|\nabla u\|^2 - \varepsilon\delta\|u\|_q^q - \varepsilon\delta^{-1}\int_\Omega |u|^{q-2}{u_t}^2 dx. \end{aligned} \tag{20}$$

We choose δ such that $\delta^{-1} = MH^{-\sigma}(t)$ for M enough large constants to be fixed later. Therefore, (20) becomes

$$\begin{aligned} G'(t) &\ge (1-\sigma-M\varepsilon)H(t)^{-\sigma}H'(t) + 2(\gamma+1)\varepsilon H(t) + \varepsilon\left(1 - \frac{2(\gamma+1)}{p}\right)\|u\|_p^p \\ &\quad +\varepsilon\gamma\|\nabla u\|^2 - \varepsilon M^{-1}H^\sigma(t)\|u\|_q^q. \end{aligned} \tag{21}$$

By the embedding $L^p \hookrightarrow L^q \hookrightarrow L^2$, $(p > q > 2)$, from (17) and thanks to Young's inequality, we obtain

$$H^\sigma(t)\|u\|_q^q \le c_1\|u\|_p^{p\sigma}\|u\|_q^q \le c_2\|u\|_p^{p\sigma+q}, \tag{22}$$

where c_1 and c_2 are positive constants. Since $0 < \frac{q}{p} < 1$, now applying the following inequality

$$x^l \le (x+1) \le \left(1 + \frac{1}{z}\right)(x+z), \ \forall x \ge 0, 0 \le l \le 1, z > 0,$$

especially, by the selection of σ; taking $x = \|u\|_p^p$, $l = \frac{(p\sigma+q)}{p}$, $z = H(0)$, and by using (17), we have

$$\begin{aligned} \|u\|_p^{p\sigma+q} &\le \left(1 + \frac{1}{H(0)}\right)\left(\|u\|_p^p + H(0)\right) \\ &\le c_3\|u\|_p^p. \end{aligned} \tag{23}$$

Thanks to (21) and (23), we get

$$\begin{aligned} G'(t) &\ge (1-\sigma-M\varepsilon)H(t)^{-\sigma}H'(t) + 2(\gamma+1)\varepsilon H(t) \\ &\quad +\varepsilon\left(1 - \frac{2(\gamma+1)}{p} - c_3M^{-1}\right)\|u\|_p^p + \varepsilon\gamma\|\nabla u\|^2 \end{aligned} \tag{24}$$

where we pick M such that $c_4 = 1 - \frac{2(\gamma+1)}{p} - c_3M^{-1} > 0$, once M is fixed, we choose ε small such that $1 - \sigma - M\varepsilon > 0$, (24) becomes

$$G'(t) \geq c_5\left(H(t) + \|\nabla u\|^2 + \|u\|_p^p\right), \tag{25}$$

where $c_5 > 0$. Thus

$$G(t) \geq G(0) > 0, \ \forall t \geq 0. \tag{26}$$

We now need to estimate $G(t)^{\frac{1}{1-\sigma}}$. Otherwise, the expression of $G(t)$ and applying Poincare's inequality, we obtain

$$\begin{aligned} G(t) &= H^{1-\sigma}(t) + \frac{\varepsilon}{2}\|u\|^2 \\ &\leq C(H^{1-\sigma}(t) + \|\nabla u\|^2). \end{aligned} \tag{27}$$

By using the above algebraic inequality for $x = \|\nabla u\|_2^{2/(1-\sigma)}$, $l = 1 - \sigma < 1$, $z = H^{1/(1-\sigma)}(0)$, we have

$$\begin{aligned} \|\nabla u\|_2^2 &\leq \left(1 + \frac{1}{H^{1/(1-\sigma)}(0)}\right)\left(\|\nabla u\|_2^{2/(1-\sigma)} + H^{1/(1-\sigma)}(0)\right) \\ &\leq C\|\nabla u\|_2^{2/(1-\sigma)}. \end{aligned}$$

Then, we have

$$G(t)^{\frac{1}{1-\sigma}} \leq C\left[H(t) + \|\nabla u\|^2 + \|u\|_p^p\right]. \tag{28}$$

Combining with (28) and (25), we arrive to

$$G'(t) \geq \beth G(t)^{\frac{1}{1-\sigma}}, \tag{29}$$

where $\beth$ is a positive constant.

Integration of (29) over $(0, t)$ gives us

$$G^{\frac{1}{1-\sigma}}(t) \geq \frac{1}{G^{-\frac{1}{1-\sigma}}(0) - \frac{\beth\sigma t}{1-\sigma}}. \tag{30}$$

The estimate (30) shows that G(t) blows up in time

$$T^* \leq \frac{1-\sigma}{\beth\sigma G^{\frac{1}{1-\sigma}}(0)}.$$

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