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Araştırma Makalesi – Research Article

3 Boyutlu Lorentz Uzayında Null Cartan Helisler: Bir Yaklaşım

Zehra ÖZDEMİR[1*]

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ÖZ

Bu çalışmada, Lorentzian 3-uzayında ekseni Killing vektör alanı olan null Cartan helisler araştırılmıştır. Bu uzayda sabit Killing ekseninin spacelike, timelike ve null (lightlike) olma durumları göz önünde bulundurularak helis eğrileri türetilmiştir. Daha sonra, bu eğrilerin Bishop eğrilikleri ve parametrik denklemleri elde edilmiştir. Son olarak, çeşitli örnekler verilmiş ve bu örnekler Mathematica programı yardımıyla görselleştirilmiştir.

***Anahtar Kelimeler*- Minkowski Uzayı, Ligthtlike Eğriler ve Yüzeyler, Cartan Eğrilikleri, Çatı Alanları**

[1]*Sorumlu yazar iletişim: zehra.ozdemir@amasya.edu.tr (https://orcid.org/0000-0001-9750-507X)
Matematik Bölümüı, Amasya Üniversitesi, Amasya, Türkiye

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Null Cartan Helices in Lorentzian 3-Space: An Approximation

ABSTRACT

In this work, we investigate the null Cartan helices in Lorentzian 3-space. We derive the helices with the constant timelike, spacelike and lightlike Killing axis in Lorentzian 3-space. Then, we calculate the Bishop curvatures of the null Cartan helix and obtain the explicit parametric equations of these curves by using the Bishop curvatures. Finally, we present various examples and draw their images using the Mathematica.

Keywords- **Minkowski Space, Ligthtlike Curves And Surfaces, Cartan Curvatures, Frame Fields.**

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I.INTRODUCTION

In Lorentzian 3-space, we have three types of curves, namely spacelike, timelike and lightlike(null) curve. Since the induced metric on a null curve is degenerate, the null curves different from the timelike and spacelike curves. Therefore, null curves are usually more appropriate to explain some physical phenomena. For instance, the solution of the 2-dimensional wave equation showed that strings are equal to a single null curve or pairs of null curves (see for details [16-19, 24]). Besides, the solution of the variational problem of a null curve is a null elastic curve evolving by rigid motions in the rotational Killing vector field direction (see [12-14]).

On the other hand, a helix defined as a curve whose tangent vector makes a constant angle with a fixed direction. The helix has various applications to natural scientists, mathematics, fractal geometry, computer-aided design, computer graphics, physics, etc. Moreover, DNA, carbon nanotube, screws, springs, etc. have the helical shapes. The authors, in [1-3], described the helical structures in nature using the variational approach and characterized by the constancy of the ratio between torsion and curvature. On the other hand, null curves have been studied by various researchers:

In [4, 6, 7, 9, 10, 22], the authors give various basic characterizations of null curves. Ferrandez et al. examined the Lancret-type theorem for null generalized helices in a Lorentzian 3-manifold and gave various characterizations for these curves [11]. In [8], Çöken et al. reproduced the Cartan frame equations of the null curves in 4- dimensional Minkowski space E_1^4 and characterized some special curves by using these equations. In [21], the authors introduced various characterizations of null helices and illustrated some examples in E_1^3 .

In the present paper, we introduce three types of null Cartan helices in the Lorentzian 3-space. The helices defined in one of the following equivalent ways:

i. T_1 makes a constant angle with a unit fixed constant Killing vector field ;
ii. N_1 makes a constant angle with a unit fixed constant Killing vector field ;

iii. N_2 makes an angle with a unit fixed constant Killing vector field ;

iv. The ratio of the curvature κ and torsion τ is constant, that is,

$\frac{\tau}{\kappa} = const.$

Similar characterizations are given by Yampolsky et.al. [25].

II. PRELIMINARIES

Let M be a three-dimensional Lorentzian manifold then the non-degenerate metric tensor g on M has the form

$$g(x,y) = -x_1y_1 + x_2y_2 + x_3y_3$$

for all $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in \chi(M)$. Then we denote (M, g) of the Lorentzian manifold M with the metric g . We say that g is positive (negative) definite on M if $g(x,x) > 0$ $(g(x,x) < 0)$ for any non-zero $x \in \chi(M)$. Moreover, if $g(x,x) \geq 0$ $(g(x,x) \leq 0)$ for any $x \in \chi(M)$ and there exist a non-zero $x \in \chi(M)$ with $g(x,x) = 0$, we say that g is positive (negative) semi-definite on M. Then (M, g) is called as Lorentzian manifold. The Lorentzian curvature tensor R of M is a $(1,3)$ tensor and denoted by the following equation

$$R(X,Y)Z = -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z.$$

Let π be a non-degenerate tangent plane to M at p then the sectional curvature, denoted by K, of

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π presented as

$$K(u,v) = \frac{g(R(u,v)u,v)}{g(u,u)g(v,v)-g(u,v)^2}.$$

If the sectional curvature of the Lorentzian manifold (M,g) is a constant, then it is called a Lorentzian space form. Then, the curvature tensor R on the Lorentzian space form satisfies

$$R(X,Y)Z = C\{g(Z,X)Y - g(Z,Y)X\},$$

where C is the constant sectional curvature.

A non-zero vector $x \in \chi(M)$ is said to be space-like if $g(x,x) > 0$, time-like if $g(x,x) < 0$ and lightlike (null) if $g(x,x) = 0$. Any two vectors $x, y \in \chi(M)$ are called orthogonal if $g(x,y) = 0$. Two null vectors are orthogonal if and only if they are linearly dependent.

Let $\gamma : I \to M;\ t \to \gamma(t)$ be a smooth curve in Lorentzian manifold (M,g) . Then, the smooth curve γ is said to be a null (light-like or isotropic) curve if the tangent vector $T = \gamma'$ of γ at any point is a null vector. If a null curve parameterized by the pseudo-arc function s then γ is called a null Cartan curve, namely,

$$s(t) = \int_0^t g(\gamma''(t), \gamma''(t))dt.$$

The null Cartan frame $\{T,N,B\}$ along a non-geodesic null Cartan curve γ satisfies

$$T' = \kappa N,$$

$$N' = -\kappa T + \tau B,$$

$$B' = -\tau B,$$

where the first Cartan curvature $\kappa(s) = 1$ and the torsion $\tau(s)$ is an arbitrary function. If $\tau(s) = 0$, the null Cartan curve is said to be a *null Cartan cubic*. A null Cartan frame $\{T,N,B\}$ along the curve γ satisfies

$$g(T,T) = g(B,B) = 0, g(N,N) = 1,$$

$$g(T,N) = g(N,B) = 0, g(T,B) = -1,$$

and

$$T \times N = -T, N \times B = -B, B \times T = N. \quad [9].$$

The Frenet frame does not define when the second derivative of the curve vanishes at some points. Therefore, we need an alternative frame at these points. In [5], the Bishop frame in 3-Euclidean space is derived by Bishop. It consists of the velocity vector field T_1 and two normal vector fields N_1 and N_2 . This frame is obtained by rotating the normal and the binormal Frenet vectors N and B in the normal plane. Besides, it is well defined at points that the curve has zero second derivative. The Bishop frame is established of non-null Frenet vector fields in Minkowski space by Özdemir et al. [23]. Then, the Bishop frames of pseudo null and null Cartan curves are presented and practiced by Grbović et al. [15].

Theorem 2.1. *Let γ be a null Cartan curve in 3-dimensional Lorentzian space parameterized by pseudo-arc s with the Cartan curvatures $\kappa(s) = 1$ and the torsion $\tau(s)$. Then the Bishop frame $\{T_1, N_1, N_2\}$ and the Cartan frame $\{T,N,B\}$ of γ have the following relation:*

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$$\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -k_2 & 1 & 0 \\ \frac{k_2^2}{2} & -k_2 & 1 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

and the null Cartan frame equations are given as follows

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} k_2 & k_1 & 0 \\ 0 & 0 & k_1 \\ 0 & 0 & -k_2 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix},$$

here the first Bishop curvature $k_1(s) = 1$ *, the second Bishop curvature satisfies the following first order non-linear differential equation*

$$k_2'(s) = -\frac{1}{2}k_2^2(s) - \tau(s).$$

The Bishop frame $\{T_1, N_1, N_2\}$ *satisfies the conditions*

$$g(T_1, T_1) = g(N_2, N_2) = 0, g(N_1, N_1) = 1,$$

$$g(T_1, N_1) = g(N_1, N_2) = 0, g(T_1, N_2) = -1, \qquad [15]$$

By using the Theorem 2.1. it is obtained that among all null Cartan curves in Lorentzian 3-space only the null Cartan cubics have two Bishop frames, which are given in the following corollary [15].

Corollary 2.2. *Let* γ *be a null Cartan cubic in Lorentzian 3-space parameterized by pseudo-arc* s *. Then the Bishop frame* $\{T_1, N_1, N_2\}$ *and the Cartan frame* $\{T, N, B\}$ *have the following relation:*

(i)

$$\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -k_2 & 1 & 0 \\ \frac{k_2^2}{2} & -k_2 & 1 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

and the null Cartan frame equations are given by

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} k_2 & k_1 & 0 \\ 0 & 0 & k_1 \\ 0 & 0 & -k_2 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix}.$$

where the Bishop curvatures satisfy $k_1(s) = 1$ *and* $k_2(s) = \frac{2}{s}$;

(ii)

$$\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

and the null Cartan frame equations are given by

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} k_2 & k_1 & 0 \\ 0 & 0 & k_1 \\ 0 & 0 & -k_2 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix},$$

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here the Bishop frame curvatures satisfy $k_1(s) = 1$ *and* $k_2(s) = 0$ *[15]* .

The cross products of the Bishop frame vectors satisfy

$$T_1 \times N_1 = -T_1, N_1 \times N_2 = -N_2, N_2 \times T_1 = N_1. \quad [15].$$

Lemma 2.3. Let $\gamma \; : \; I \subset R \rightarrow M$ be a null Cartan curve in Lorentzian space form $(M(C), g)$ and V be a vector field along the curve γ then the variation of γ defined by $\Gamma \; : \; I \times (-\varepsilon, \varepsilon) \rightarrow M(C)$ with $\gamma(s,0)$ the initial null Cartan curve satisfy $\Gamma(s,0) = \gamma(s)$. The variational vector field has the notion $V(s) = \frac{\partial \Gamma(s,t)}{\partial s}$. In this setting, the variations of the speed function $v(s,t) = \left\| \frac{\partial \Gamma(s,t)}{\partial s} \right\|$, and the Bishop curvature functions $k_1(s,t)$ and $k_2(s,t)$ at $t = 0$ are calculated as follows:

(a)

$$V(v) = \left(\frac{\partial v}{\partial t}(s,t) \right) \Big|_{t=0} = -v\rho,$$
$$V(k_1) = \left(\frac{\partial k_1}{\partial t}(s,t) \right) \Big|_{t=0} = g\big(R(V,T_1)T_1 + \nabla_{T_1}^2 V - k_2 \nabla_{T_1} V, N_1\big) + 2\rho k_1,$$
$$V(k_2) = \left(\frac{\partial k_2}{\partial t}(s,t) \right) \Big|_{t=0} = g((k_1 - 1)R(V,T_1)T_1 + (1 - k_1)\nabla_{T_1}^2 V - k_2 \nabla_{T_1} V,$$
$$+k_1 k_2 \nabla_{T_1} V, N_2) - 2\rho' - 2\rho k_2 - k_1 k_2' + 2\rho k_1 k_2,$$

where $\rho = g(\nabla_{T_1} V, T_1)$ *[20]* .

Proposition 2.4. *Let* $V(s)$ *be the restriction to* $\gamma(s)$ *of a Killing vector field* V *of* M *then the variations of the Bishop curvature functions and speed function of* γ *satisfy:*

$$V(v) = V(k_1) = V(k_2) = 0, \quad [20].$$

Corollary 2.5. V *is a Killing vector field along the null Cartan curve* γ *if and only if it satisfies the following conditions:*

i. $g(\nabla_{T_1} V, T_1) = 0,$

ii. $g(\nabla_{T_1}^2 V, N_1) - k_2 g(\nabla_{T_1} V, N_1) = 0,$

iii. $g((k_1 - 1)\nabla_{T_1}^2 V - k_2 \nabla_{T_1} V + k_1 k_2 \nabla_{T_1} V, N_2) + C(k_1 - 1) = 0$ *[20].*

The helix called degenerate if the Killing vector field V is a null vector field. The helix is said to be non-degenerate if the Killing vector field V is a non-null vector field [1].

Let $V(s)$ be an axis then its causal characters can be of three families of vectors, namely, spacelike, timelike and lightlike(null). Therefore, we can define the following representations for null Cartan helices in Lorentzian 3-space.

III. NULL CARTAN HELICES IN LORANTZIAN 3-SPACE

Definition 3.1. *Let* γ *be a curve with the Bishop frame* $\{T_1, N_1, N_2\}$ *in Lorentzian 3-space. If there exist a constant Killing vector field* V *along the curve* γ *such that* $g(V, T_1) = const.,$ $g(V, N_1) = const.,$ *or* $g(V, N_2) = const.,$ *respectively, then the curve* γ *is called the first kind of null Cartan helix, second kind of null Cartan helix, or third kind of null Cartan helix with the Killing axis* V.

Theorem 3.1. *Let* γ *be a null Cartan curve and* V *be a Killing vector field on a Lorentzian space form* $(M(C), g)$. *The curve* γ *is a first kind of null Cartan helix of* $(M(C), g; V)$ *if and only if* γ *has the following curvatures*

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$k_1 = 1,\ k_2 = \frac{A}{a}$ or $k_2 = 0$

and axis

$V = \frac{A^2-\varepsilon}{2}T_1 + AN_1 + aN_2,$

where $A = const.,$ *and* $\varepsilon = g(V,V)$.

Proof. Let γ be an arc-length parameterized first kind of null Cartan helix with the Killing vector field in a Lorentzian space form $(M(C),g)$ then we have $g(V,\mathrm{T}_1) = -a = const.$ Therefore, V can be written as follows

(1) $\mathrm{V} = \zeta\mathrm{T}_1 + \eta\mathrm{N}_1 + a\mathrm{N}_2$

Calculating the derivative of eq. (1), we have the following equation

(2) $\nabla_{T_1}V = (\zeta' + \zeta k_2)T_1 + (\zeta + \eta')N_1 + (\eta - ak_2)N_2,$

Then using the first equation in the Corollary 2.5. we obtain

$k_2 = \frac{\eta}{a}.$

If we differentiate the eq. (2), we get

(3) $\nabla^2_{T_1}V = (\zeta'' + 2\zeta' k_2 + \zeta k_2' + \zeta k_2^2)T_1 + (2\zeta' + \zeta k_2 + \eta'')N_1 + (\zeta + \eta')N_2,$

On the other hand, we have the following equation for simply connected space form $(M(C),g)$

(4) $R(V,T_1)T_1 = C(g(T_1,V)T_1 - g(T_1,T_1)V).$

Since γ is a null Cartan curve we have $g(T_1,T_1) = 0$ and we calculate $g(T_1,V) = -a.$ So, we obtain the Lorentzian curvature tensor of $M(C)$ equal to the following equation

(5) $R(V,T_1)T_1 = -aCT_1.$

Considering the eqs. (1)-(6) with the second equation in Corollary 2.5. we reach the following second order linear differential equation

(6) $2\zeta' - k_2\eta' + \eta'' = 0.$

In these cases, the last equation in Corollary 2.5. is provided automatically.

Furthermore, if we use the equation $g(V,V) = \varepsilon,$ we get

$\eta^2 - \zeta a = \varepsilon.$

By combining the eq. (7) and eq. (8) we obtain the following second order non-linear differential equation

(7) $3\eta\eta' + a\eta'' = 0.$

Then the solution of the eq. (9) calculated as follows

(9) $\eta = A$ or $\eta = \sqrt{\frac{2ac_1}{3}}tanh(\sqrt{\frac{3c_1}{2a}}(s + c_2)).$

This gives the second curvature of the helix as

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(10) $k_2 = \frac{A}{a}$ or $k_2 = \sqrt{\frac{2c_1}{3a}} tanh(\sqrt{\frac{3c_1}{2a}}(s + c_2))$.

However, the torsion of the curve γ is constant on the condition that $k_2 = \frac{c}{a}$ or $c_1 = 0$, that is,$k_2 = 0$.

As a consequently, we determine the Killing axis of the helix satisfy

(11) $V = \frac{A^2-\varepsilon}{2} T_1 + AN_1 + aN_2$.

Conversely, if γ satisfy the eq. (10) and eq. (11) then we can easily show that γ is a first kind of null Cartan helix with the Killing axis V.

Theorem 3.2. *Let γ be a null Cartan curve and V be a Killing vector field on a Lorentzian space form $(M(C), g)$. The curve γ is a second kind of null Cartan helix of $(M(C), g; V)$ if and only if γ satisfy the following constant curvatures*

$k_1 = 1, k_2 = \frac{b}{\upsilon}$,

and axis

$V = \frac{b^2-\varepsilon}{\upsilon} T_1 + bN_1 + \upsilon N_2$,

where $\varepsilon = g(V, V)$, and b, υ are some constants.

Proof. Let γ be an arc-length parameterized second kind of null Cartan helix with the Killing vector field in the Lorentzian space form $(M(C), g)$ then we have $g(V, N_1) = b = const$. Therefore, V may be written as follows

(12) $V = \xi T_1 + bN_1 + \upsilon N_2$.

By differentiating of the eq. (12) and the Bishop frame formulas we get

(13) $\nabla_{T_1} V = (\xi' + \xi k_2)T_1 + \xi N_1 + (b + \upsilon' - \upsilon k_2)N_2$.

Then the first equation in the Corollary 2.5. yields

(14) $k_2 = \frac{b+\upsilon'}{\upsilon}$.

By differentiating of the eq. (13), we obtain

(15) $\nabla^2_{T_1} V = (\xi'' + 2\xi' k_2 + \xi k_2' + \xi k_2^2)T_1 + (2\xi' + \xi k_2)N_1 + \xi N_2$.

On the other hand, we have the following equation for simply connected space form $(M(C), g)$

(16) $R(V, T_1)T_1 = C(g(T_1, V)T_1 - g(T_1, T_1)V)$.

Since γ is a null Cartan curve we have $g(T_1, T_1) = 0$ and we calculate $g(T_1, V) = -\upsilon$. Therefore we obtain the Lorentzian curvature tensor of $M(C)$ equal to the following equation

(17) $R(V, T_1)T_1 = \upsilon C T_1$.

In combination with the eqs. (12)-(17) and the second equation in Corollary 2.5., we get

(18) $\xi = const$.

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In these cases, the last equation in Corollary 2.5. is provided automatically.

Moreover, if we use the equation $g(V,V)=\varepsilon,$ we get

(19) $b^2-\xi\upsilon=\varepsilon.$

If we combine with the eq. (18) and eq. (19) we obtain

(20) $\upsilon=const.$

This yield

(21) $k_2=\frac{b}{\upsilon},$

As a consequently, we determine the Killing axis of the helix satisfy

(22) $V=\frac{b^2-\varepsilon}{\upsilon}T_1+bN_1+\upsilon N_2.$

Conversely, if γ satisfy the eq. (21) and eq. (22) then we can easily show then γ is a second kind of null Cartan helix with the Killing axis $V.$

Theorem 3.3. *Let γ be a null Cartan curve and V be a Killing vector field on a Lorentzian space form $(M(C),g)$. The curve γ is a third kind of null Cartan helix of $(M(C),g;V)$ if and only if the Bishop curvatures of γ satisfy*

$k_1=1,$

$k_2=-\sqrt{2c_1(2\varepsilon c_1+c)}\,tanh\frac{\sqrt{2c_1(2\varepsilon c_1+c)}(s+c_4)}{2},-c_1(2\varepsilon c_1+c)>0,$or

$k_2=-\sqrt{-2c_1(2\varepsilon c_1+c)}\,tan\frac{\sqrt{-2c_1(2\varepsilon c_1+c)}(s+c_4)}{2},-c_1(2\varepsilon c_1+c)<0,$

$V=cT_1+\varsigma N_1+\frac{\varsigma^2-\varepsilon}{c}N_2,$

Where for spacelike Killing vector field $\varepsilon=1,$ for timelike Killing vector field $\varepsilon=-1$ and $a, c_1,$ c_2 are some constants.

Proof. Let γ be an arc-length parameterized third kind of null Cartan helix with the Killing vector field in Lorentzian space form $(M(C),g)$ then we have $g(V,N_2)=-c=const.$ Therefore, V is written

(23) $V=cT_1+\varsigma N_1+\sigma N_2,$

Using the derivative of the eq. (23) and the Bishop frame formulas in the Corollary 2.5 we obtain

(24) $k_2=\frac{\varsigma+\sigma'}{\sigma}.$

If we use the second derivation of the eq. (23) and combine the eq. (24) with the second equation in Corollary 2.5. we get

(25) $\varsigma''-k_2\varsigma'=0,$

In these cases, the last equation in Corollary 2.5. is provided automatically.

On the other hand, if we use the equation $g(V,V)=\varepsilon,$ we get

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(26) $\sigma = \frac{\varsigma^2-\varepsilon}{c}$.

If the eq. (24) and eq. (26) are fulfilled in the eq. (25) we obtain a second order non-linear differential equation

(27) $(\varepsilon - \varsigma^3)\varsigma'' + c\varsigma\varsigma' + 2\varsigma\varsigma'^2 = 0$.

The solution of the eq. (27) is obtained as

$$\varsigma(s) = -\sqrt{\frac{(2\varepsilon c_1+c)}{2c_1}}\, tanh\frac{\sqrt{2c_1(c+2\varepsilon c_1)}(s+c_2),-c_1(2\varepsilon c_1+c)>0}{2}$$

or

$$\varsigma(s) = -\sqrt{\frac{-(2\varepsilon c_1+c)}{2c_1}}\, tan\frac{\sqrt{-2c_1(c+2\varepsilon c_1)}(s+c_2)}{2}, -c_1(2\varepsilon c_1 + c) < 0.$$

Then the curvatures and the axis of the third kind null Cartan helices have the following second Bishop curvature

(28) $k_2 = -\sqrt{2c_1(2\varepsilon c_1 + c)}\, tanh\frac{\sqrt{2c_1(2\varepsilon c_1+c)}(s+c_4)}{2}, -c_1(2\varepsilon c_1 + c) > 0,$

or

$$k_2 = -\sqrt{-2c_1(2\varepsilon c_1 + c)}\, tan\frac{\sqrt{-2c_1(2\varepsilon c_1+c)}(s+c_4)}{2}, -c_1(2\varepsilon c_1 + c) < 0,$$

and the axis

(29) $$V = cT_1 + \varsigma N_1 + \frac{\varsigma^2-\varepsilon}{c}N_2.$$

Conversely, if γ satisfy the eq. (28) and eq. (29) then we can easily show that γ is a third kind of null Cartan helix with the Killing axis V.

Corollary 3.4. (**Main result**) Let γ be a null Cartan helix and V be a Killing vector field on a Lorentzian space form $(M(C), g)$. Then γ has one of the following equivalent characterizations:

i. T_1 makes a constant angle with a unit fixed constant Killing vector field ;

ii. N_1 makes a constant angle with a unit fixed constant Killing vector field ;

iii. N_2 makes an angle with a unit fixed constant Killing vector field ;

iv. The ratio of the curvature κ and torsion τ is constant, that is,

$\frac{\tau}{\kappa} = const.$

3.1. Parametric representations of null Cartan helices

From the above theorem we obtain that the null Cartan helices have the constant torsion and the position vector of the helices satisfy the following higher-order linear ordinary differential equation:

$$\gamma^{(4)} + 2\tau\gamma'' = 0.$$

The solution of the differential equation gives the parametric representation of all null Cartan

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helices following three cases:

If $\tau = -\lambda < 0$, then $\gamma(s) = \Psi_1 + \Psi_2 s + \frac{\Psi_3}{2\lambda} exp(\sqrt{2\lambda} s) + \frac{\Psi_4}{2\lambda} exp(-\sqrt{2\lambda} s)$,

If $\tau > 0$, then $\gamma(s) = \Phi_1 + \Phi_2 s + \frac{\Phi 3}{2\tau} sin(\sqrt{2\tau} s) + \Phi_4 \, cos(\sqrt{2\tau} s)$,

if $\tau = 0$, then $\gamma(s) = \Omega_1 + \Omega_2 s + \Omega_3 s^2 + \Omega_4 s^3$,

where Ψ_i, Φ_i, $\Omega_i \in E_1^3$, $i = 1,2,3,4$. Namely, the solution for first kind null Cartan helix can be given following two cases:

Case 1. Provided that γ is a first kind of null Cartan helix then we obtain following two characterizations:

i. if $k_2 = \frac{A}{a}$ then $\tau = -\frac{A^2}{a^2} < 0$. We deduce that

$$\gamma(s) = A_1 + A_2 s + \frac{a^2 A_3}{2A^2} exp(\sqrt{\frac{2A^2}{a^2}} s) + \frac{a^2 A_4}{2A^2} exp(-\sqrt{\frac{2A^2}{a^2}} s),$$

ii. if $k_2 = 0$ then $\tau = 0$ then we have

$$\gamma(s) = B_1 + B_2 s + B_3 s^2 + B_4 s^3,$$

where A_i, $B_i \in E_1^3$, $i = 1,2,3,4$.

Case 2. Suppose that γ is a second kind of null Cartan helix then the curve γ has the curvature $k_2 = \frac{b}{v}$. From here we obtain following characterization:

If $\tau = -\frac{b^2}{2v^2}$ then $\tau < 0$ along the curve γ . Thus, the solution of the second kind null Cartan helix

$$\gamma(s) = C_1 + C_2 s + \frac{v^2 C_3}{b^2} exp(\sqrt{\frac{b^2}{v^2}} s) + \frac{v^2 C_4}{b^2} exp(-\sqrt{\frac{b^2}{v^2}} s),$$

where $C_i \in E_1^3$, $i = 1,2,3,4$.

Case 3. Suppose that γ is a third kind of null Cartan helix then the curve γ has the curvature

$k_2 = -\sqrt{2c_1(2\varepsilon c_1 + c)} \, tanh \frac{\sqrt{2c_1(2\varepsilon c_1 + c)}(s + c_4)}{2}$. From here we obtain following two characterizations:

i. if $\tau = -c_1(2\varepsilon c_1 + c) < 0$ then we have

$$\gamma(s) = D_1 + D_2 s + \frac{D_3}{2(-c_1(2\varepsilon c_1 + c))} exp(\sqrt{2(-c_1(2\varepsilon c_1 + c))} s) + \frac{D_4}{2(-c_1(2\varepsilon c_1 + c))} exp(-\sqrt{2(-c_1(2\varepsilon c_1 + c))} s),$$

ii. if $\tau = -c_1(2\varepsilon c_1 + c) > 0$ then we have

$$\gamma(s) = E_1 + E_2 s + \frac{E_3}{2(-c_1(2\varepsilon c_1 + c))} sin(\sqrt{2(-c_1(2\varepsilon c_1 + c))} s) + \frac{E_4}{2(-c_1(2\varepsilon c_1 + c))} cos(\sqrt{2(-c_1(2\varepsilon c_1 + c))} s),$$

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where $D_i, E_i \in E_1^3, \quad i = 1,2,3,4.$

Example 3.1. *If we take* $A = 0, \quad a = -1, \quad c_2 = 0, \quad B_1 = (0,0,0), \quad B_2 = (\frac{1}{6}, 0, -\frac{1}{6}), \quad B_3 = (0, \frac{1}{2}, 0)$ *and* $B_4 = (\frac{1}{2}, 0, -\frac{1}{2})$, *we get following first kind of null Cartan helix*

$$\gamma(s) = (\frac{s^3}{2} + \frac{s}{6}, \frac{s^2}{2}, \frac{s^3}{2} - \frac{s}{6}).$$

The curve has the following Bishop curvatures:

$$k_1(s) = 1, k_2(s) = 0.$$

"The axis of the helix calculated as

$$V = \frac{-\varepsilon}{2} T_1 + a N_2.$$

The image of the first kind of null Cartan helix illustrated in Figure 1.

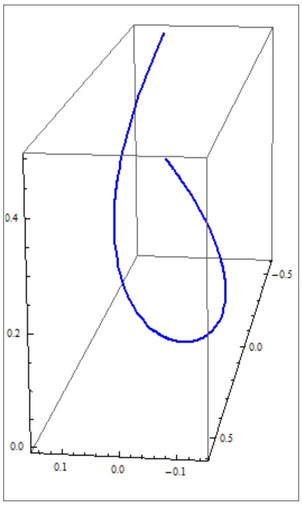

Figure 1. A first kind of null Cartan helix

Example 3.2. *If we choose* $b = \sqrt{2}, \quad v = 1, \quad C_1 = (0,0,0), \quad C_2 = (0, \frac{1}{\sqrt{2}}, 0), \quad C_3 = (\frac{1}{2}, 0, \frac{1}{2})$ *, and* $C_4 = (-\frac{1}{2}, 0, \frac{1}{2})$ *we obtain second kind of null Cartan helix parameterized as follows:*

$$\gamma(s) = (\frac{e^{\sqrt{2}s}}{4} - \frac{e^{-\sqrt{2}s}}{4}, \frac{s}{\sqrt{2}}, \frac{e^{\sqrt{2}s}}{4} + \frac{e^{-\sqrt{2}s}}{4}).$$

The curve has the following Bishop curvatures:

$$k_1(s) = 1, k_2(s) = \sqrt{2}.$$

Then the axis of the second kind of null Cartan helix given as

$$V = (2 - \varepsilon) T_1 + \sqrt{2} N_1 + N_2.$$

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The image of the helix plotted in Figure 2.

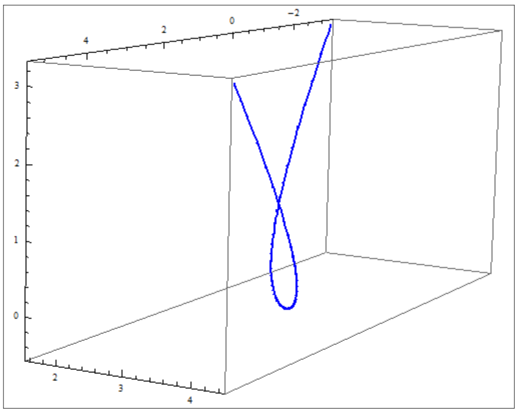

Figure 2. A second kind of null Cartan helix

Example 3.3. *If we take* $E_1 = (0,0,0)$, $E_2 = (\sqrt{2}, 0, 1)$, $E_3 = (0,1,0)$, $E_4 = (1, 0, \sqrt{2})$, $c_1 = -1$, $c = -\frac{1}{2}$ *and* $\varepsilon = 0$, *Then we obtain third kind of null Cartan helix parameterizaed as*

$\gamma(s) = (\cos s + \sqrt{2}s, \sin s, \sqrt{2}\cos s + s)$.

The curve has the following Bishop curvatures:

$k_1(s) = 1, k_2(s) = -\frac{1}{2}\tan\frac{s}{4}$.

The axis of the helix

$V = \frac{1}{2}T_1 - \frac{1}{2}\tan\frac{s}{4}N_1 - \frac{1}{4}\tan\frac{s}{4}N_2$.

The image of the helix illustrated in Figure 3.

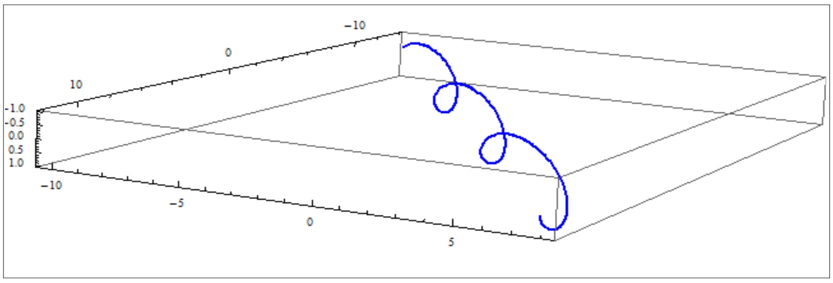

Figure 3. A third kind of null Cartan helix

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