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Research Article

Ulam-Hyers stability for a nonlinear Volterra integro-differential equation

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Abstract

In this work, the Ulam-Hyers stability and the Ulam-Hyers-Rassias stability for the nonlinear Volterra integro-differential equations are established by employing the method of successive approximation. Some simple examples are given to illustrate the main results.

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1. Introduction

Integro-differential equations are a significant research topic from the theoretical point of view as well as of their applications. The reader is referred to the monograph of Volterra [16], Lakshmikantham [11], Medlock [12], the papers [4, 6, 7] and the references therein. Especially, studying the stability of Ulam-Hyers and Ulam-Hyers-Rassias for differential equations and integro-differential equations has been of great interest (see [5, 8, 14, 15, 17]) in recent years. In [5, 15], by utilizing method of the fixed point theorem, authors presented some kinds of the Hyers-Ulam-Rassias stability to the Volterra integro-differential equations.

In [1, 2], M. Gachpazan and O. Baghani have established the Ulam-Hyers stability for nonlinear integral equation and linear integral equation of the second kind by using successive approximation method. After that with the above idea, Huang et al. [3] proved the Ulam-Hyers stability for delay differential equations of the first order by using successive approximation method. Kucche et al. [9, 10] also applied to prove the Ulam-Hyers stability and E_α–Ulam-Hyers stability results for nonlinear implicit fractional differential equations.

Motivatived by M. Gachpazan and O. Baghani [1, 2], and Huang et al. [3], we investigate the Ulam-Hyers stability and Ulam-Hyers-Rassias stability for nonlinear Volterra integro-differential equations (NVIDE) of the form

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$$\begin{cases} u'(t) = f(t,u(t)) + \int_a^t g(t,s,u(s))ds, & t \in J \\ u(a) \ = \ u_0, \end{cases} \tag{1.1}$$

where $J = [a,b]$, f and g are continuous functions.

The rest of this paper is presented as follows: the Ulam-Hyers stability of (1.1) is presented in Section 2, and the Ulam-Hyers-Rassias stability of (1.1) is given in Section 3.

Now, we present the definition of some types of the Ulam-Hyers stability which will be used throughout this paper. First of all, let $\varepsilon > 0$ and $\psi \in C(J, \mathbb{R}_+)$. We consider the following inequalities:

$$\left| v'(t) - \mathcal{P}(t) \right| \leq \varepsilon, \quad t \in J, \tag{1.2}$$

and

$$\left| v'(t) - \mathcal{P}(t) \right| \leq \varepsilon\psi(t), \quad t \in J, \tag{1.3}$$

where

$$\mathcal{P}(t) := f(t,v(t)) + \int_a^t g(t,s,v(s))ds.$$

Definition 1.1. The problem (1.1) is Ulam-Hyers stable if there is a constant $K_f > 0$ such that for each $\varepsilon > 0$ and for each solution $v \in C^1(J,\mathbb{R})$ of (1.2) there is a solution u of (1.1) satisfying

$$\left| v(t) - u(t) \right| \leq K_f \varepsilon.$$

Definition 1.2. The problem (1.1) is Ulam-Hyers-Rassias stable concerning $\psi \in C(J, \mathbb{R}_+)$ if there is a constant $C_f > 0$ such that for each $\varepsilon > 0$ and for each solution $v \in C^1(J,\mathbb{R})$ of (1.3) there is a solution u of (1.1) satisfying

$$\left| v(t) - u(t) \right| \leq C_f \varepsilon \psi(t).$$

2. Ulam-Hyers stability for NVIDE (1.1)

In this section, by employing the successive approximation method, we shall present the Ulam-Hyers stability for NVIDE (1.1).

Remark 2.1. We observe that if the function v is a solution of (1.2), then there is a continuous function $\delta(t)$ on J such that $\left|\delta(t)\right| \leq \varepsilon$ and

$$v'(t) = \mathcal{P}(t) + \delta(t).$$

Let $f : J \times \mathbb{R} \to \mathbb{R}$ and $g : J \times J \times \mathbb{R} \to \mathbb{R}$ are continuous functions. We consider the following hypotheses:

(H1) There exist positive constants L_1, L_2 such that for each $(t,s) \in J \times J$ and $w_1, w_2 \in \mathbb{R}$ one has

$$|f(t,w_1) - f(t,w_2)| \leq L_1|w_1 - w_2|,$$
$$|g(t,s,w_1) - g(t,s,w_2)| \leq L_2|w_1 - w_2|.$$

(H2) Let $\psi \in C(J, \mathbb{R}_+)$ in the inequality (1.3). Assume that there exists a constant $C > 0$ such that $kC^k \leq (b-a)C^{k-1}$, $\forall k \geq 1$, and $0 < CL < 1$, and the following hypothesis is satisfied, for $t \in J$,

$$\int_a^t \psi(s)ds \leq C\psi(t).$$

Theorem 2.2. *Assume that f and g satisfy the hypothesis (H1). Then, for each $\varepsilon > 0$ if the function v satisfies (1.2), there exists a unique solution u of (1.1) provided $u_0 = v_0$ and u satisfies the following estimate*

$$|u(t) - v(t)| \leq \varepsilon b \exp((b-a)(1+L)). \tag{2.1}$$

Proof. For each $\varepsilon > 0$ and let the function v satisfy (1.2), then basing on Remark 2.1 one has that then there is a continuous function $\delta(t)$ on J such that $|\delta(t)| \leq \varepsilon$ and $v'(t) = \mathcal{P}(t) + \delta(t)$. This yields that the function v satisfies the integral equation

$$v(t) = v_0 + \int_a^t \mathcal{P}(s)ds + \int_a^t \delta(s)ds, \tag{2.2}$$

where

$$\int_a^t \mathcal{P}(s)ds = \int_a^t \Big[f(s, v(s)) + \int_a^s g(s,\tau,v(\tau))d\tau\Big]ds.$$

We consider the sequence $(u_n)_{n\geq 0}$ defined as follows: $u_0(t) = v(t)$ and for $n = 1, 2, \ldots$

$$u_n(t) = v_0 + \int_a^t \mathcal{P}_{n-1}(s)ds, \tag{2.3}$$

where

$$\int_a^t \mathcal{P}_{n-1}(s)ds = \int_a^t \Big[f(s, u_{n-1}(s)) + \int_a^s g(s,\tau,u_{n-1}(\tau))d\tau\Big]ds.$$

By (2.2) and (2.3), for $n = 1$ one has

$$|u_1(t) - u_0(t)| = \left|v_0 + \int_a^t \mathcal{P}_0(s)ds - v(t)\right| = \left|\int_a^t \delta(s)ds\right| \leq \varepsilon(t-a), \ \forall t \in J. \tag{2.4}$$

For $n = 1, 2, \ldots$, from the hypothesis (H1) one has

$$\begin{aligned} |u_{n+1}(t) - u_n(t)| &= \left|\int_a^t \mathcal{P}_n(s)ds - \int_a^t \mathcal{P}_{n-1}(s)ds\right| \\ &\leq L\int_a^t |u_n(s) - u_{n-1}(s)|ds + L\int_a^t\int_a^s |u_n(r) - u_{n-1}(r)|drds, \end{aligned}$$

where $L = \max\{L_1, L_2\}$. In particular, for $n = 1$ and by (2.4) one gets

$$\begin{aligned} |u_2(t) - u_1(t)| &\leq \varepsilon L\int_a^t (s-a)ds + \varepsilon L\int_a^t\int_a^s (r-a)drds \\ &= \varepsilon L\Big(\frac{(t-a)^2}{2!} + \frac{(t-a)^3}{3!}\Big) \end{aligned}$$

and so, for $n = 2$, one also obtains

$$\begin{aligned} |u_3(t) - u_2(t)| &\leq \varepsilon L^2\int_a^t \left(\frac{(s-a)^2}{2!} + \frac{(s-a)^3}{3!}\right)ds + \varepsilon L^2\int_a^t\int_a^s \left(\frac{(r-a)^2}{2!} + \frac{(r-a)^3}{3!}\right)drds \\ &= \varepsilon L^2\Big(\frac{(t-a)^3}{3!} + 2\frac{(t-a)^4}{4!} + \frac{(t-a)^5}{5!}\Big) \\ &\leq 3\varepsilon L^2\Big(\frac{(t-a)^3}{3!} + \frac{(t-a)^4}{4!} + \frac{(t-a)^5}{5!}\Big) \end{aligned}$$

and for $n \geq 4$ we have

$$|u_n(t) - u_{n-1}(t)| \leq \varepsilon n L^{n-1}\Big(\frac{(t-a)^n}{n!} + \frac{(t-a)^{n+1}}{(n+1)!} + \ldots + \frac{(t-a)^{2n}}{(2n)!} + \frac{(t-a)^{2n+1}}{(2n+1)!}\Big). \tag{2.5}$$

Then, the estimation (2.5) can be rewritten by:

$$\begin{aligned}\big|u_n(t)-u_{n-1}(t)\big| &\leq \frac{\varepsilon(t-a)\big(L(t-a)\big)^{n-1}}{(n-1)!}\bigg(1+\frac{(t-a)}{n+1}+\frac{(t-a)^2}{(n+1)(n+2)}+\dots\\ &\qquad\qquad +\frac{(t-a)^n}{(n+1)(n+2)\dots 2n}+\frac{(t-a)^{n+1}}{(n+1)(n+2)\dots 2n(2n+1)}\bigg)\\ &\leq \varepsilon b\frac{\big(L(t-a)\big)^{n-1}}{(n-1)!}\bigg(1+\frac{(t-a)}{1!}+\frac{(t-a)^2}{2!}+\dots+\frac{(t-a)^n}{n!}+\frac{(t-a)^{n+1}}{(n+1)!}\bigg)\\ &\leq \varepsilon b\frac{\big(L(t-a)\big)^{n-1}}{(n-1)!}\exp(t-a).\end{aligned}$$

Furthermore, if we assume that

$$\big|u_n(t)-u_{n-1}(t)\big| \leq \varepsilon b\frac{\big(L(t-a)\big)^{n-1}}{(n-1)!}\exp(t-a), \tag{2.6}$$

then one also gets

$$\big|u_{n+1}(t)-u_n(t)\big| \leq \varepsilon b\frac{\big(L(t-a)\big)^{n}}{n!}\exp(t-a), \quad \forall t\in J.$$

This yields that

$$\sum_{n=0}^{\infty}\big|u_{n+1}(t)-u_n(t)\big| \leq \varepsilon b\exp(b-a)\sum_{n=0}^{\infty}\frac{\big(L(t-a)\big)^{n}}{n!}. \tag{2.7}$$

Since the right-hand series is convergent to the function $\exp(L(t-a))$, for each $\varepsilon>0$ we deduce the series $u_0(t)+\sum_{n=0}^{\infty}[u_{n+1}(t)-u_n(t)]$ is uniformly convergent concerning the norm $|\cdot|$ and

$$\sum_{n=0}^{\infty}\big|u_{n+1}(t)-u_n(t)\big| \leq \varepsilon b\exp((b-a)(1+L)). \tag{2.8}$$

Assume that

$$\tilde{u}(t)=u_0(t)+\sum_{n=0}^{\infty}[u_{n+1}(t)-u_n(t)]. \tag{2.9}$$

Then,

$$u_j(t)=u_0(t)+\sum_{n=0}^{j}[u_{n+1}(t)-u_n(t)] \tag{2.10}$$

is the j^{th} partial of the series (2.9). From (2.9) and (2.10), we obtain

$$\lim_{j\to\infty}\big|\tilde{u}(t)-u_j(t)\big|=0, \quad \forall t\in J.$$

Define $u(t)=\tilde{u}(t), \forall t\in J$. We observe that the limit of the above sequence is a solution to the following equation:

$$u(t)=v_0+\int_a^t \mathcal{P}(s)ds, \quad \forall t\in J, \tag{2.11}$$

where

$$\mathcal{P}(t):=f(t,u(t))+\int_a^t g(t,s,u(s))ds.$$

By (2.3), (2.11) and the hypothesis (H1), one has that

$$\begin{aligned}\left|u(t) - v_0 - \int_a^t \mathcal{P}(s)ds\right| &= \left|\tilde{u}(t) - \Big(u_j(t) - \int_a^t \mathcal{P}_{j-1}(s)ds\Big) - \int_a^t \mathcal{P}(s)ds\right| \\ &\leq \left|\tilde{u}(t) - u_j(t)\right| + \int_a^t \left|\mathcal{P}_{j-1}(s) - \mathcal{P}(s)\right|ds \\ &\leq \left|\tilde{u}(t) - u_j(t)\right| + L\int_a^t \left|u_{j-1}(s) - u(s)\right|ds \\ &+ L\int_a^t\int_a^s \left|u_{j-1}(r) - u(r)\right|drds. \end{aligned} \tag{2.12}$$

Combining (2.9) and (2.10), we get

$$\left|\tilde{u}(t) - u_j(t)\right| \leq \sum_{n=j+1}^{\infty} \left|u_{n+1}(t) - u_n(t)\right|$$

and by the estimation (2.7), one has

$$\left|u(t) - u_j(t)\right| \leq \varepsilon b \exp(b-a) \sum_{n=j+1}^{\infty} \frac{\big(L(t-a)\big)^n}{n!}, \quad \forall t \in J. \tag{2.13}$$

Hence, it follows from the inequalities (2.12) and (2.13) that

$$\begin{aligned}\left|u(t) - v_0 - \int_a^t \mathcal{P}(s)ds\right| &\leq \varepsilon b e^{b-a} \sum_{n=j+1}^{\infty} \frac{\big(L(t-a)\big)^n}{n!} \\ &+ \varepsilon L b e^{b-a}\Big(\int_a^t \sum_{n=j+1}^{\infty} \frac{\big(L(s-a)\big)^n}{n!}ds + \int_a^t\int_a^s \sum_{n=j+1}^{\infty} \frac{\big(L(r-a)\big)^n}{n!}drds\Big) \\ &\leq \varepsilon b e^{b-a}\left[\sum_{n=j+1}^{\infty} \frac{\big(L(t-a)\big)^n}{n!} + \sum_{n=j+1}^{\infty} L^{n+1}\Big(\frac{(t-a)^{n+1}}{(n+1)!} + \frac{(t-a)^{n+2}}{(n+2)!}\Big)\right]. \end{aligned} \tag{2.14}$$

Taking limit as $n \to \infty$, we see that the right-hand series of (2.14) is convergent. Therefore, one deduces that

$$\left|u(t) - v_0 - \int_a^t \mathcal{P}(s)ds\right| \leq 0, \quad \forall t \in J.$$

This means that

$$u(t) = v_0 + \int_a^t \mathcal{P}(s)ds, \quad \forall t \in J, \tag{2.15}$$

which is a solution of (1.1). In addition, from the estimation (2.8), we have the estimate as follows:

$$\left|u(t) - v(t)\right| \leq \varepsilon b \exp((b-a)(1+L)).$$

To show the uniqueness of solution to the problem (1.1), we assume that $\widehat{u}(t)$ is another solution of (1.1), which has the form

$$\widehat{u}(t) = v_0 + \int_a^t \widehat{\mathcal{P}}(s)ds, \quad \forall t \in J, \tag{2.16}$$

where

$$\widehat{\mathcal{P}}(t) := f(t, \widehat{u}(t)) + \int_a^t g(t, s, \widehat{u}(s))ds.$$

By using the hypothesis (H1), one obtains

$$\gamma(t) \leq L\int_a^t \gamma(s)ds + L\int_a^t\int_a^t \gamma(r)drds, \quad \forall t \in J.$$

where $\gamma(t) := |u(t) - \widehat{u}(t)|$. Then by applying Gronwall's lemma (see Theorem 2.1 in [13]), we infer that $\gamma(t) = 0$ on J. So, $u(t) = \widehat{u}(t)$. This completes the proof. □

3. Ulam-Hyers-Rassias stability for NVIDE (1.1)

In this section, with the same manner as in Section 2, by using the successive approximation method, we also present the Ulam-Hyers-Rassias stability for NVIDE (1.1).

Remark 3.1. We observe that if the function v is a solution of (1.3), then there is a continuous function $\xi(t)$ on J such that $|\xi(t)| \le \varepsilon\psi(t)$ and

$$v'(t) = \mathcal{P}(t) + \xi(t).$$

Theorem 3.2. *Assume that the hypotheses (H1) and (H2) are held. Then, for each $\varepsilon > 0$, if the function v satisfies (1.3), there is a unique solution u of (1.1) with $u_0 = v_0$, and u satisfies the following estimate, for $t \in J$,*

$$|v(t) - u(t)| \le \varepsilon \frac{(b-a)}{(1-C)(1-CL)}\psi(t). \tag{3.1}$$

Proof. For each $\varepsilon > 0$ let the function v satisfy (1.3), then by basing on Remark 3.1, one has that there is a continuous function $\xi(t)$ on J such that $|\xi(t)| \le \varepsilon\psi(t)$ and $v'(t) = \mathcal{P}(t) + \xi(t)$. This yields that the function v satisfies the integral equation as follows:

$$v(t) = v_0 + \int_a^t \mathcal{P}(s)ds + \int_a^t \xi(s)ds, \tag{3.2}$$

where

$$\int_a^t \mathcal{P}(s)ds = \int_a^t \Big[f(s, v(s)) + \int_a^s g(s,\tau,v(\tau))d\tau\Big]ds.$$

Similar to the proof of Theorem 2.2, we also reconsider the sequence $(u_n)_{n\ge 0}$ defined as in (2.3) with $u_0(t) = v(t)$, $\forall t \in J$. Now, by (2.3), the hypothesis (H3) and (3.2), for $n = 1$ one has

$$|u_1(t) - u_0(t)| = \left|v_0 + \int_a^t \mathcal{P}_0(s)ds - v(t)\right| \le \varepsilon \int_a^t \psi(s)ds \le \varepsilon C\psi(t), \ \forall t \in J.$$

For $n = 1, 2, \dots$ and from the hypothesis (H1), one has

$$|u_{n+1}(t) - u_n(t)| \le L\int_a^t \left(|u_n(s) - u_{n-1}(s)|ds + \int_a^s |u_n(r) - u_{n-1}(r)|dr\right) ds,$$

where $L = \max\{L_1, L_2\}$. In particular, for $n = 1$ one has

$$|u_2(t) - u_1(t)| \le \varepsilon LC\int_a^t \psi(s)ds + \varepsilon LC\int_a^t\int_a^s \psi(r)drds = \varepsilon L(C^2 + C^3)\psi(t), \quad \forall t \in J$$

and so, for $n = 2$, we also obtain

$$\begin{aligned}|u_3(t) - u_2(t)| &\le L\int_a^t |u_2(s) - u_1(s)|ds + L\int_a^t\int_a^s |u_2(r) - u_1(r)|drds\\ &\le 3\varepsilon L^2(C^3 + C^4 + C^5)\psi(t).\end{aligned}$$

and for $n \ge 4$ we have

$$|u_n(t) - u_{n-1}(t)| \le n\varepsilon(C^n + C^{n+1} + \ldots + C^{2n} + C^{2n+1})L^{n-1}\psi(t). \tag{3.3}$$

Then, by the hypothesis (H3), the estimation (2.5) is rewritten by:

$$\begin{aligned}\left|u_n(t)-u_{n-1}(t)\right| &\leq \varepsilon(b-a)(CL)^{n-1}(1+C^1+\ldots+C^{n+1})\psi(t)\\ &\leq \varepsilon(b-a)\Big(\frac{1-C^{n+1}}{1-C}\Big)(CL)^{n-1}\psi(t), \quad \forall t\in J.\end{aligned}$$

In addition, if the assumption

$$\left|u_n(t)-u_{n-1}(t)\right| \leq \varepsilon(b-a)\Big(\frac{1-C^{n+1}}{1-C}\Big)(CL)^{n-1}\psi(t), \quad \forall t\in J, \tag{3.4}$$

is satisfied, then by using the mathematical induction we also get

$$\left|u_{n+1}(t)-u_n(t)\right| \leq \varepsilon(b-a)\Big(\frac{1-C^{n+2}}{1-C}\Big)(CL)^{n}\psi(t), \quad \forall t\in J.$$

This yields that

$$\sum_{n=0}^{\infty}\left|u_{n+1}(t)-u_n(t)\right| \leq \varepsilon(b-a)\Big(\frac{1}{1-C}\Big)\sum_{n=0}^{\infty}(CL)^{n}\psi(t). \tag{3.5}$$

By the hypothesis (H3), we observe that $\sum_{n=0}^{\infty}(CL)^n \to \frac{1}{1-CL}$ as $n\to\infty$. Hence, for every $\varepsilon>0$ we infer that the series $u_0(t)+\sum_{n=0}^{\infty}[u_{n+1}(t)-u_n(t)]$ is uniformly convergent on J and

$$\sum_{n=0}^{\infty}\left|u_{n+1}(t)-u_n(t)\right| \leq \varepsilon\frac{(b-a)}{(1-C)(1-CL)}\psi(t), \quad \forall t\in J. \tag{3.6}$$

With the same manner as in the proof of Theorem 2.2, we also can show that $u(t)$ is a solution of (1.1) which has form

$$u(t)=v_0+\int_a^t \mathcal{P}(s)ds, \quad \forall t\in J,$$

where

$$\mathcal{P}(t):=f(t,u(t))+\int_a^t g(t,s,u(s))ds.$$

In addition, the following estimate is also satisfied

$$\left|u(t)-v(t)\right| \leq \varepsilon\frac{(b-a)}{(1-C)(1-CL)}\psi(t), \quad \forall t\in J.$$

Therefore, (1.1) is Ulam-Hyers-Rassias stable. □

4. Examples

In this section, two simple examples are presented to illustrate our results.

Example 4.1. Consider the following problem

$$\begin{cases} u'(t)=1+\displaystyle\int_0^t u(s)ds, \quad \forall t\in[0,1],\\ u(0)=1, \end{cases} \tag{4.1}$$

We see that $v(t)=1$, $\forall t\in[0,1]$ complies with the following inequality

$$\left|v'(t)-1-\int_0^t v(s)ds\right| \leq 2.$$

Now, we can choose $v_0(t) = u(0) = 1$. By using the successive approximation method as in Theorem 2.2, we obtain the following successive solution to (4.1) as

$$\begin{aligned}
&v_0(t) = 1,\\
&u_1(t) = v(0) + \int_0^t \Big(1 + \int_0^s u(r)dr\Big)ds = 1 + t + \frac{t^2}{2!}.
\end{aligned}$$

Then it is no difficult to see that $u(t) = 1 + t + \frac{t^2}{2!}$ forms a solution (4.1) and one gets the estimate

$$\big|v(t) - u(t)\big| = \left|1 - \Big(1 + t + \frac{t^2}{2!}\Big)\right| \leq \frac{3}{2}.$$

Next, we define the function $u^*(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!}$ is also a solution of (4.1) and we also have

$$\big|v(t) - u^*(t)\big| = \left|1 - \Big(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!}\Big)\right| \leq \frac{17}{24}.$$

Therefore, it shows the function $u^*(t)$ is better approximate solution than the function $u(t)$.

Example 4.2. Consider the following problem

$$u'(t) = u(t) + \int_0^t \frac{u(s)}{1 + u(s)}ds, \tag{4.2}$$

where $t \in [0, 1]$. We set

$$f(t, u(t)) = u(t) \quad \text{and} \quad g(t, s, u(s)) = \frac{u(s)}{1 + u(s)}.$$

Then, we see that

$$\big|f(t, w_1) - f(t, w_2)\big| = |w_1 - w_2|$$

and

$$\begin{aligned}
\big|g(t, s, w_1(s)) - g(t, s, w_2(s))\big| &= \left|\frac{w_1(s)}{1 + w_1(s)} - \frac{w_2(s)}{1 + w_2(s)}\right|\\
&\leq \frac{|w_1 - w_2|}{(1 + w_1)(1 + w_2)}\\
&\leq |w_1 - w_2|.
\end{aligned}$$

This yields that the hypotheses of Theorem (2.2) is satisfied. That means Eq. (4.2) has unique solution on $[0, 1]$. Furthermore, if the function v satisfies

$$\left|v'(t) - v(t) - \int_0^t \frac{v(s)}{1 + v(s)}ds\right| \leq \varepsilon$$

then by basing on the result of Theorem 2.2, there exists a solution u of Eq. (4.2) satisfying

$$\big|u(t) - v(t)\big| \leq \varepsilon \exp(2), \quad \forall t \in [0, 1].$$

This means that the problem (4.2) is Ulam-Hyers stable.

5. Conclusion

The results of Ulam-Hyers stability to the nonlinear Volterra integro-differential equation have been investigated by Janfada et al. [5] and Sevgin et al. [15] by employing the fixed point theorem. In our paper, we present the Ulam-Hyers stability and the Ulam-Hyers-Rassias stability to NVIDE (1.1) by employing the method of successive approximation.

References

[1] M. Gachpazan and O. Baghani, *Hyers–Ulam stability of nonlinear integral equation*, Fixed Point Theory Appl. **2010**, 6 pages, 2010.

[2] M. Gachpazan and O. Baghani, *Hyers–Ulam stability of volterra integral equation*, Inter. J. Nonlinear Anal. Appl. **1** (2), 19–25, 2010.

[3] J. Huang and Y. Li, *Hyers–Ulam stability of delay differential equations of first order*, Math. Nachr. **289** (1), 60–66, 2016.

[4] H.V. Jain and H.M. Byrne, *Qualitative analysis of an integro–differential equation model of periodic chemotherapy*, Appl. Math. Lett. **25** (12), 2132–2136, 2012.

[5] M. Janfada and G. Sadeghi, *Stability of the Volterra integrodifferential equation*, Folia Math. **18** (1), 11–20, 2013.

[6] C. Jin and J. Luo, *Stability of an integro-differential equation*, Comput. Math. Appl. **57** (7), 1080–1088, 2009.

[7] M. Joshi, *An existence theorem for an integro-differential equation*, J. Math. Anal. Appl. **62** (1), 114–124, 1978.

[8] S.-M. Jung and J. Brzdek, *Hyers–Ulam stability of the delay equation* $y'(t) = \lambda y(t-\tau)$, Abstr. Appl. Anal. **2010**, Art. Id. 372176, 2010.

[9] K.D. Kucche and S.T. Sutar, *Stability via successive approximation for nonlinear implicit fractional differential equations*, Moroccan J. Pure Appl. Anal. **3** (1), 36–55, 2017.

[10] K.D. Kucche and S.T. Sutar, *On existence and stability results for nonlinear fractional delay differential equations*, Bol. Soc. Parana. Mat. **36** (4), 55–75, 2018.

[11] V. Lakshmikantham, *Theory of Integro-Differential Equations*, CRC Press, Boca Raton, Florida, NY, 1995.

[12] J.P. Medlock, *Integro-differential equation models in ecology and epidemiology*, University of Washington, PhD Thesis, 2004.

[13] J.A. Oguntuase, *On an inequality of Gronwall*, J. Inequal. Pure Appl. Math. **2** (1), Art. No. 9, 2001.

[14] A.Z. Rahim Shah, *A fixed point approach to the stability of a nonlinear volterra integrodifferential equation with delay*, Hacet. J. Math. Stat. **47** (3), 615–623, 2018.

[15] S. Sevgin and H. Sevli, *Stability of a nonlinear volterra integro-differential equation via a fixed point approach*, J. Nonlinear Sci. Appl. **9**, 200–207, 2016.

[16] V. Volterra, *Theory of Functionals and of Integral and Integro-Differential Equations*, Dover, New York, NY, 1959.

[17] A. Zada and S.O. Shah, *Hyers–Ulam stability of first-order non-linear delay diffierential equations with fractional integrable impulses*, Hacet. J. Math. Stat. **47** (5), 1196–1205, 2018.