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A New Generalization of Injective Modules

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Abstract

In this paper, as a generalization of injective modules, we define two different modules: modules that have the property $(\delta - SE)$ and modules that have the property $(\delta - SSE)$, and we investigate basic properties of them. Namely, modules that have a δ-supplement that is a direct summand in its every extension and modules that have a strong δ-supplement in its every extension are tackled here. Particularly, it is proved that a ring whose modules have the property $(\delta - SE)$ is δ-semiperfect. Let R be a ring, M be an R-module with $IM = 0$ for an ideal of R. It is shown that if the R-module M has the property $(\delta - SE)$, then so does $\overline{R}$-module M, under a special condition, where $\overline{R} = \frac{R}{I}$. Finally we supply an example showing that a module that has the property $(\delta - SE)$ may not have the property $(\delta - SSE)$.

Keywords: δ-supplement, $\bigoplus$- δ-supplement, module extension, δ-semiperfect ring.

İnjektif Modüllerin Yeni Bir Genelleştirmesi

Öz

Bu çalışmada, injektif modüllerin yeni bir genelleştirmesi olarak iki farklı modül tanımlanmakta ve bunların temel özellikleri incelenmektedir. Bunlardan birincisi $(\delta - SE)$ özelliğine sahip modüller, yani her genişlemesinde direkt toplam terimi olacak şekilde bir δ-tümleyene sahip olan modüller; ikincisi ise $(\delta - SSE)$ özelliğine sahip modüller, yani her genişlemesinde güçlü $\delta -$tümleyene sahip olan modüllerdir. Özel olarak, tüm modülleri $(\delta - SE)$ özelliğine sahip olan halkaların δ-yarı mükemmel olduğu ispatlanmıştır. Ayrıca R bir halka, M bir R-modül ve R nin $IM = 0$ koşulunu sağlayan I ideali için $\overline{R} = \frac{R}{I}$ olmak üzere, özel bir şart altında R-modül olarak $(\delta - SE)$ özelliğine sahip M modülünün $\overline{R}$-modül olarak da $(\delta - SE)$ özelliğine sahip olduğu gösterilmiştir. Çalışmanın sonunda $(\delta - SE)$ özelliğine sahip fakat $(\delta - SSE)$ özelliğine sahip olmayan modüle bir örnek verilmiştir.

Anahtar kelimeler: δ-tümleyen, $\bigoplus$- δ-tümleyen, modül genişlemesi, δ-yarı mükemmel halka.

Introduction

In this paper, all rings are associative with identity and all modules are unital left modules, unless otherwise specified. The notation $(N < M)$ $N \leq M$ means that N is a (proper) submodule of M or M is an extension of N. A submodule $N \leq M$ is called essential in M if $N \cap X \neq 0$ for every nonzero submodule X of M and denoted by $N \trianglelefteq M$. Dually, a proper

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submodule N of M is called small in M if $N+X\neq M$ for every proper submodule X of M and denoted by $N\ll M$. Let N and X be submodules of M . X is called a supplement of N in M if it is minimal with respect to the property $M=N+X$, equivalently, $M=N+X$ and $N\cap X\ll X$. If every submodule of M has a supplement in M , M is called supplemented. The module M is called $\oplus$-supplemented if every submodule of M has a supplement in M that is a direct summand of M. A module M is called strongly supplemented or lifting if every submodule U of M has a strong supplement V in M, i.e., $M = U+V, U\cap V \ll V$ and $(U\cap V) \oplus U' = U$ for some submodule U' of U. Note that a module M is called lifting if and only if every submodule N of M includes a direct summand K of M such that $\frac{N}{K}\ll\frac{M}{K}$. For detailed information related to supplemented, $\oplus$-supplemented and lifting modules and other concepts given here we refer to [1], [2] and [3] respectively.

The singular submodule of a module M is $Z(M)$ containing the elements of M whose annihilators are essential in R. A module M is called singular if $Z(M)=M$ [4]. In [5], Zhou introduced the concept of δ-small submodules as a generalization of small submodules. A submodule N of M is called δ-small in M if for any submodule X of M provided that $\frac{M}{X}$ singular, $M=N+X$ implies that $X=M$ and denoted by the notation $N\ll_\delta M$. $\delta(M)$ indicates the sum of all δ-small submodules of M . A submodule X of M is said to be a δ-supplement of a submodule N in M if $N+X=M$ and $N\cap X\ll_\delta X$ [6]. A module M is called δ-supplemented if every submodule of M has a δ-supplement in M . Of course, every supplemented module is δ-supplemented but the converse is not true in general. In [6], Koşan defined and investigated δ-lifting (or strongly δ-supplemented) modules besides δ-supplemented modules as a generalization of lifting modules. A module M is called δ-lifting (strongly δ-supplemented) if every submodule N of M has a strong δ-supplement K' in M, i.e., for any $N\leq M$, there exists a decomposition $M=K\oplus K'$ such that $K\leq N$ and $N\cap K'\ll_\delta K'$. Also M is called $\oplus$-δ-supplemented if every submodule of M has a δ-supplement which is a direct summand of M [8]. Clearly δ-lifting modules are $\oplus$-δ-supplemented.

A module M is called injective if it is a direct summand in its every extension [9]. Since every direct summand of a module M is also a supplement submodule in M, in [10] Zöschinger introduced the module with the property (E), namely it has a supplement in its every extension, as a generalization of injective modules. Recently, several authors have studied remarkable generalizations of these modules (see in [11], [12]). In [13] and [14], modules that have the properties $(SE), (SSE)$ and $(\delta - E)$ were defined respectively.

In this study we give a new generalization of injective modules using the notion of δ-supplement. By following the terminology and notation as in [13], we call a module M has the property $(\delta - SE)$ if in any extension N of M, M has a δ-supplement that is a direct summand of N. We call a module M has the property $(\delta - SSE)$ as a proper generalization of modules with the property (SSE) if it has a strong δ-supplement in its every extension. As an example we prove simple modules having the property $(\delta - SSE)$ and even we prove that semisimple modules with $\delta(M) \ll_{\delta} M$ and $\frac{M}{\delta(M)}$ singular over δ-semilocal rings also have the property $(\delta - SSE)$. Moreover being injective and having the properties $(\delta - SE)$ and $(\delta - SSE)$ for a module M coincides over a left δ-V-ring. We answer when all submodules of a module M has the property $(\delta - SE)$. Zöschinger proved in [10] that if an R-module M with $IM = 0$ for an ideal I of R has the property (E) then so does $\overline{R}$-module M, where $\overline{R}$ is the ring $\frac{R}{I}$. We give the similar fact for modules that have the property $(\delta - SE)$ under a specific condition in Proposition 9. In addition we show that in Proposition 10 modules that have the property $(\delta - SE)$ also have the property $(\delta - SSE)$ if they are fully invariant submodules in every extension. At the end we give the fact that if every left R-module has the property $(\delta - SE)$, then the ring R is δ-semiperfect.

Clearly we have the following hierarchy:

Injective module ⇨ module that has the property $(\delta - SSE)$ ⇨ module that has the property $(\delta - SE)$ ⇨ module that has the property $(\delta - E)$.

Results

Lemma 1: Let M be a module. Then $Soc(\delta(M)) \ll_{\delta} M$.

Proof: In aspect of brevity, let say $Soc(\delta(M)) = X$ and assume $M = X + Y$ for any submodule Y of M with $\frac{M}{Y}$ singular. Setting $Z = X \cap Y$, we have $X = Z \oplus Z'$ for some $Z' \leq X$ and $M = X + Y = (Z \oplus Z') + Y = Z' \oplus Y$. It is a known fact from [4, Prop. 1.24] that any simple right R-module is either singular or projective. So in particular there exist two cases for any simple submodule S of Z'.

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Case 1: Assume that S is singular. Since it is a direct summand and δ-small (=small, because of singularity) submodule in M, Z' is δ-small in M as a direct summand. So $Z' = M$ since $\frac{M}{Y}$ is singular.

Case 2: Assume that this submodule is projective. Since it is also semisimple we can say it is δ-small in Z'. By the same way in case (1), again $Z' \ll_\delta M$. Hence $X \ll_\delta M$.

Recall that a submodule K of a module M is the weak δ-supplement of a submodule L in M if $M = K + L$ and $K \cap L \ll_\delta M$.

Proposition 2: Let M be a semisimple module. Then the following statements are equivalent.

1. M has the property $(\delta - E)$;
2. M has the property $(\delta - SE)$;
3. M has a weak δ-supplement in every extension N;
4. For every module N with $M \leq N$, there exists a submodule K of N such that $N = M + K$ and $M \cap K \leq \delta(N)$;
5. M has the property $(\delta - SSE)$.

Proof: (1) ⇒ (2): Let N be any extension of M. By (1), we have $N = M + K$ and $M \cap K \ll_\delta K$ for some submodule $K \leq M$. Since M is semisimple, there exists a submodule X of M such that $M = (M \cap K) \oplus X$. So $(M \cap K) \cap X = K \cap X = 0$. Therefore, $N = M + K = [(M \cap K) + X] + K = K + X$. This means that $N = K \oplus X$ and so M has the property $(\delta - SE)$.

(2) ⇒ (3) and (3) ⇒ (4) are clear.

(4) ⇒ (5): Let $M \leq N$. Then there exists a submodule K of N such that $N = M + K$ and $M \cap K \leq \delta(N)$. By Lemma 1, we obtain that $Soc(\delta(N)) \ll_\delta N$. Since M is semisimple we can write the decomposition $M = (M \cap K) \oplus X$ for some submodule $X \leq M$. It follows that $M \cap K = Soc(M \cap K) \leq Soc(\delta(N)) \ll_\delta N$. Applying [5, Lemma 1.4], $M \cap K$ is a δ-small submodule of N. Since $N = M + K = [(M \cap K) \oplus X] + K = X \oplus K$, we obtain that $M \cap K \ll_\delta K$ by [15]. Hence K is a strong δ-supplement of M in N.

(5) ⇒ (1) is clear.

Recall that a ring R is called δ-semilocal if $\frac{R}{\delta(R)}$ is semisimple, where $\delta(R) = \delta({}_R R)$ denotes the sum of all δ-small left ideals of R [16].

Lemma 3: The following statements for a module M are equivalent :

a) M is δ-semilocal;
b) Any $N \in Gen(M)$ is δ-semilocal.

Proof: For every $N \in Gen(M)$ there exists a set Λ and an epimorphism $f: M^{(\Lambda)} \longrightarrow N$. Since $f(\delta(M)) \leq \delta(N)$ and $\frac{M^{(\Lambda)}}{\delta(M^{(\Lambda)})} \cong (\frac{M}{\delta(M)})^{(\Lambda)}$ always holds, we get an epimorphism $\overline{f}: (\frac{M}{\delta(M)})^{(\Lambda)} \longrightarrow \frac{N}{\delta(N)}$. Hence N is δ-semilocal.

Corollary 4: Let M be a semisimple R-module over a δ-semilocal ring R with $\delta(M) \ll_\delta M$

and $\frac{M}{\delta(M)}$ singular. Then M has the property $(\delta - SSE)$.

Proof: Let N be an R-module with $M \leq N$. Since R is δ-semilocal, we obtain that N is semilocal by Lemma 3. Therefore, there exists a submodule K of N such that $N = M + K$ and $M \cap K \leq \delta(N)$. Applying Proposition 2, we derive that M has the property $(\delta - SSE)$.

It is clear that every injective module has the property $(\delta - SSE)$. The following example shows that a module that has the property $(\delta - SSE)$ need not to be injective. Firstly, we need the following lemma.

Lemma 5: Every simple module has the property $(\delta - SSE)$.

Proof: It is similar to the proof of Lemma 2.1 in [13].

A uniserial module M is an R-module over a ring R whose submodules are totally ordered by inclusion. Dually, a serial module is a direct sum of uniserial modules. Any simple module is trivially uniserial, and likewise semisimple modules are serial modules. A ring R is left serial if the R module R is serial [17].

Example: $\frac{\mathbb{Z}}{n\mathbb{Z}}$ is uniserial when n is a prime power but not injective as a $\mathbb{Z}$-module. So the $\mathbb{Z}$-module $\frac{\mathbb{Z}}{p\mathbb{Z}}$, where p is prime, has the property $(\delta - SSE)$ but it is not injective.

Lemma 6: Let M be a module that has the property $(\delta - SE)$. Suppose that N is an extension of M such that $\delta(N) = 0$. Then M is a direct summand of N.

Proof: Let N be any extension of M. Since M has the property $(\delta - SE)$ there exist submodules K and K' of N such that $N = M + K$, $M \cap K \ll_{\delta} K$ and $N = K \oplus K'$. By the hypothesis, $M \cap K \leq \delta(N) = 0$. It follows that $N = M \oplus K$.

Recall from [8] that a ring R is a left δ-V-ring if and only if $\delta(M) = 0$ for every left R-module M.

Proposition 7: For a module M over a left δ-V-ring, the following statements are equivalent.

1. M is injective;
2. M has the property $(\delta - SSE)$;
3. M has the property $(\delta - SE)$.

Proof: (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear. (3) $\Rightarrow$ (1) follows from Lemma 6.

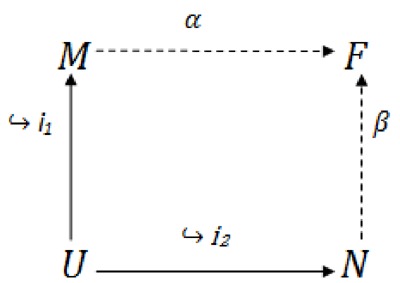
Theorem 8: For a module M the following statements are equivalent:

1. Every submodule of M has the property $(\delta - SE)$;
2. For any extension N of M and any submodule K such that $N = M +$

K, K contains a δ-supplement of M in N that is a direct summand of K.

Proof: $(1) \Rightarrow (2)$: Let N be any extension of M . Suppose that a submodule K of N satisfies $N = M + K$. By the hypothesis $M \cap K$ has the property $(\delta - SE)$, so $M \cap K$ has a δ-supplement L in K such that L is a direct summand of K , that is, $K = (M \cap K) + L$, $(M \cap K) \cap L = M \cap L \ll_\delta L$ and there exists a submodule L' of K such that $K = L \oplus L'$. Then $N = M + K = M + (M \cap K) + L = M + L$, $M \cap L \ll_\delta L$ and L is also a direct summand of K.

$(2) \Rightarrow (1)$: Let U be any submodule of M and N be any extension of U. We heve the following pushout diagram by the inclusion homomorphisms i_1 and i_2.



It follows that $F = Im(\alpha) + Im(\beta)$ and α is a monomorphism by the properties of pushout, and so $M \cong Im(\alpha)$. By assumption $Im(\alpha)$ has a δ-supplement V in F with $V \leq Im(\beta)$, i.e., $F = Im(\alpha) + V$, $Im(\alpha) \cap V \ll_\delta V$ and there exists a submodule K of $Im(\beta)$ such that $Im(\beta) = V \oplus K$. Then we get $N = \beta^{-1}(Im(\alpha)) + \beta^{-1}(V) = U + \beta^{-1}(V)$ and $U \cap \beta^{-1}(V) \ll_\delta \beta^{-1}(V)$. Since β is a monomorphism, we have that $N = \beta^{-1}(Im(\beta)) = \beta^{-1}(V) \oplus \beta^{-1}(K)$, which means that $\beta^{-1}(V)$ is a direct summand of N. Therefore U has the property $(\delta - SE)$.

Proposition 9: Let R be a ring, I be an ideal of R with $\overline{R} = \frac{R}{I}$ and M be an R-module that has the property $(\delta - SE)$ with $IM = 0$. If for any module K the submodule IK is fully invariant in K, then $\overline{R}$-module M also has the property $(\delta - SE)$.

Proof: It can be proved similar to Prop. 2.3 in [13].

Proposition 10: Let M be a module that has the property $(\delta - SE)$. If M is a fully invariant submodule in every extension, then $\overline{R}$-module M also has the property $(\delta - SSE)$.

Proof: It can be proved similar to Prop. 2.4 in [13].

Theorem 11: Let R be a ring. If every left R-module has the property $(\delta - SE)$, then R is δ-semiperfect.

Proof: Let assume that every left R-module has the property $(\delta - SE)$. Accordingly, every left ideal of R has a δ-supplement in R as a left R-module. Thus the module ${}_RR$ is $\oplus - \delta$-supplemented. Therefore R is a δ-semiperfect ring.

Generally it is obvious that modules that have the property $(\delta - SSE)$ also have the property $(\delta - SE)$ but the converse may not be true in specific cases. In [13], the author proved that every left R-module over a commutative artinian serial ring R has the property (SE). Now by using this fact, we give an example of a module that has the property $(\delta - SE)$ but does not have the property $(\delta - SSE)$.

Example 12: Let R be a local Dedekind domain. For an integer $n \geq 3$, it is a known fact that $\frac{R}{Rad(R)^n}$ is artinian serial as an R-module. Hence $\frac{R}{Rad(R)^n}$ has the property (SE) and so $(\delta - SE)$. On the other hand $\frac{R}{Rad(R)^n}$ does not have the property $(\delta - SSE)$ as an R-module as R is local (see in [13] and [18]).

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