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Sinc Methods and Chebyshev Cardinal Functions for Solving Singular Boundary Value Problems

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Abstract: *In this paper we consider boundary value problems with singularity in equation or solution. To solve these problems, we apply single exponential and double exponential transformations of sinc-Galerkin and Chebyshev cardinal functions. Numerical examples highlight efficiency of Chebyshev cardinal functions and sinc-Galerkin method in problems with singularity in equations. It is illustrated that in problems with singular solutions, Chebyshev cardinal functions is not applicable. However, sinc-Galerkin method overcomes to this difficultly.*
Keywords: *Sinc method, Singular points, Double and single exponential transformation, Chebyshev cardinal functions.*

1. Introduction

The sinc method is a highly efficient numerical method that has been developed by Frank Stenger, the pioneer of this field, and his colleagues [1, 2], it is widely used in various fields of numerical analysis, solution of integral, ordinary differential and partial differential equations [3-13]. Sinc-Galerkin is one of the sinc methods that used in this paper for solving boundary value problems with singular solutions. Despite most of the numerical methods, sinc-Galerkin method comprehends problems that have singular solutions. Conventional form of these methods is SE transformation. There are several advantages to using approximations based on sinc numerical methods. First, unlike most numerical techniques, it is now well-established that they are characterized by exponentially decaying errors [17]. Second, they are highly efficient and adaptable in handling problems with singularities [18]. Finally, due to their rapid convergence, sinc numerical methods do not suffer from the common instability problems associated with other numerical methods [19].

Takahasi and Mori [9] proposed the double exponential transformation for one dimensional numerical integration in 1974. The effectiveness of the DE transformation technique in numerical integration naturally suggests that the DE transformation technique could be useful in other numerical methods. In 1997, Sugihara [10] established the “meta-optimality” of the DE formula in a mathematically rigorous manner, and since then it has turned out that the DE transformation is also useful for other various kinds of numerical methods. Indeed, it has been demonstrated in [11-14] that, the use of the sinc method incorporated with the DE transformation gives highly efficient numerical methods for functions approximation, indefinite numerical integration, and the solution of differential equations.

Interpolate approximate base function have received considerable attention in dealing with various problems. The main characteristic behind the approach using this technique is that it reduces these problems to those of solving a system of algebraic equations thus greatly simplifying the problem. In this paper, a Chebyshev cardinal function is used for numerical solution of differential equations, with the goal of obtaining efficient computational solutions. Several papers have appeared in the literature concerned with the application of Chebyshev cardinal functions [24-29].

Some of important applicable problems have singularity in their solutions. Modeling of demotion of a rigid body around a fixed point redounds the Kowaleveski equation that has singularity in solution. The Lorenz model of atmospheric circulation is another example of differential equations with singular solution [20]. The Painlve equations that appear in several applications such as statistical mechanics, random matrix models, plasma physics, nonlinear waves, have singularity in their solutions [21]. In solving problems with singular solutions numerical methods often cannot pass singular point with successfully.

In this article we apply the DE and SE transformation sinc-Galerkin method and Chebyshev cardinal functions to solve boundary-value problems:

$$L(y)=p(x)y''+q(x)y'+u(x)y=f(x,y) \quad (1)$$

$$y(a)=y(b)=0$$

Where $p(x),q(x),u(x)$ and $f(x,y)$, are analytic functions. In (1), it is possible $p(x)$ has zeros in (a,b), or solution $y(x)$ has singularity on (a,b). It is shown that, proposed methods are applicable and in second case only SE and DE sinc-Galerkin methods overcomes on the singular points difficultly but DE transformation sinc-Galerkin method is more accurate and Chebyshev cardinal functions method is not suitable for solving our problems in the

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second case.

2. Chebyshev Cardinal Functions

In this section, we first present a brief review of the Chebyshev cardinal functions for solving differential equations. Chebyshev cardinal functions of order N in [-1,1] are defined as [29]:

$$\phi_j(x) = \frac{T_{N+1}(x)}{T_{N+1}(x_j)(x - x_j)}, \; j = 1,2,\ldots,N+1 \qquad (2)$$

where $T_{N+1}(x)$ is the first kind Chebyshev function of order $N+1$ in $[-1,1]$ defined by $T_{N+1}(x) = cos((N+1)arccos(x))$, subscript x denotes x-differentiation and x_j, $j = 1,2,\ldots,N+1$, are the zeros of $T_{N+1}(x)$ defined by $cos((2j-1)/(2N+2)), j = 1,2,\ldots,N+1$. We change the variable $t = (x+1)L/2$ to use these functions on $[0,L]$. Now any function $f(t)$ on $[0,L]$ can be approximated as

$$f(t) = \sum_{j=1}^{N+1} f(t_j)\phi_j(t) = F^T \Phi_N(t), \qquad (3)$$

where $t_j, j = 1,2,\ldots,N+1$, are the shifted points of $x_j, j = 1,2,\ldots,N+1$, by transforming $t = (x+1)L/2$ here we choose t_j so that, $t_1 < t_2 < \ldots < t_{N+1}$,

$$F = [f(t_1), f(t_2), \ldots, f(t_{N+1})]^T,$$

$$\Phi_N(t) = [\phi_1(t), \phi_2(t), \ldots, \phi_{N+1}(t)]^T. \qquad (4)$$

2.1 The Operational Matrix of Derivative

The differentiation of vector $\Phi_N(t)$ in (4) can be expressed as

$$D\Phi_N = \mathbf{D}\Phi_N, \qquad (5)$$

where $\mathbf{D}$ is $(N+1)\times(N+1)$ operational matrix of derivative for Chebyshev cardinal functions. The matrix $\mathbf{D}$ can be obtained by the following process. Let

$$D\Phi_N(t) = [\phi_1'(t), \phi_2'(t), \ldots, \phi_{N+1}'(t)]^T.$$

It is shown [24] that the matrix $\mathbf{D}$ is the form

$$\mathbf{D} = \begin{pmatrix} \phi'_1(t_1) & \cdots & \phi'_1(t_{N+1}) \\ \vdots & \ddots & \vdots \\ \phi'_{N+1}(t_1) & \ldots & \phi'_{N+1}(t_{N+1}) \end{pmatrix},$$

where $\phi_j'(t_j) = \sum_{\substack{l=1 \\ i\neq j}}^{N+1} \frac{1}{t_i - t_j}, \quad j = 1,\ldots,N+1,$

$$\phi_k'(t_j) = \frac{\beta}{T_{N+1}(t_j)} \prod_{\substack{l=1 \\ l\neq k,j}}^{N+1} (t_k - t_l),$$

$$j,k = 1,\ldots,N+1, j \neq k, \qquad (6)$$

and $\beta = \frac{2\times(2N+1)}{L(N+1)}$. Note that

$$\frac{T_{N+1}(t)}{(t-t_j)} = \beta \prod_{\substack{k=1 \\ k\neq j}}^{N+1} (t - t_k). \qquad (7)$$

3. Sinc Bases, SE and DE Transformations

Sinc function is demonstrated on $-\infty < x < \infty$ by

$$sinc(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases} \qquad (8)$$

This function is translated with evenly spaced nodes are given as

$$S(k,h)(x) = sinc\left(\frac{x-kh}{h}\right), \; k = 0, \pm1, \pm2, \ldots, \; h > 0 \qquad (9)$$

If $f(z)$ is analytic on a strip domain

$$|Imz| < d, \qquad (10)$$

in the z-plane and $|f(z)| \to 0$ as $z \to \pm\infty$ then, the series

$$C(f,h) = \sum_{k=-\infty}^{\infty} f(kh)\, sinc\left(\frac{z-kh}{h}\right), \qquad (11)$$

converges, we call it whittaker cardinal expansion.

If $f(x)$ be a real function, sinc expansion (11) is defined on $-\infty < x < \infty$, while the equation that we want to solve is defined $a < x < b$, and hence we need some transformation which the given interval transform on to

$-\infty < x < \infty$. In many of applications of the sinc method transformation

$$\phi(z) = \log\left(\frac{z-a}{b-z}\right), \qquad (12)$$

has been used. The map ϕ carries the eye-shaped region

$$D_E = \left\{z = x + iy: \left|\arg\left(\frac{z-a}{b-z}\right)\right| < d < \frac{\pi}{2}\right\}, \qquad (13)$$

on to

$$D_d = \{\zeta = \xi + i\eta: |\eta| < d < \pi/2\}. \qquad (14)$$

Define h by

$$h = \sqrt{\frac{\pi d}{\alpha N}}, \quad 0 < \alpha \leq 1. \qquad (15)$$

The h is the mesh size in D_d for the uniform grids $\{kh\}, -\infty < k < \infty$. The base functions on (a,b) are given by

$$S(j,h)o\phi(x) = sinc\left(\frac{\phi(x) - jh}{h}\right). \qquad (16)$$

The sinc grid points $z \in (a,b)$ in D_E will be denoted by x because they are real. The inverse images of the equispaced grids in SE transformation are

$$x = \phi^{-1}(t) = \psi(t) = \frac{a + be^t}{1 + e^t} \qquad (17)$$

or

$$x = \phi^{-1}(t) = \psi(t) = \frac{b-a}{2} \operatorname{tgh}\left(\frac{t}{2}\right) + \frac{b+a}{2}. \qquad (18)$$

In DE transformation, we can use

$$t = \phi(x) = \log\left(\frac{1}{\pi}\log\left(\frac{x-a}{b-x}\right) + \sqrt{\left(\frac{1}{\pi}\log\left(\frac{x-a}{b-x}\right)\right)^2 + 1}\right) \qquad (19)$$

$$x = \phi^{-1}(t) = \psi(t) = \frac{b-a}{2} \operatorname{tgh}\left(\frac{\pi}{2}\sinh(t)\right) + \frac{b+a}{2}, \qquad (20)$$

which Takahasi and Mori proposed for numerical integration [9]. One of the best reasons for using (20) is optimality of this transformation. It usually gives significantly faster convergence than (18) [9, 10 and 15].

Definition 1 Let D be a simply-connected domain which satisfies $(a,b) \in D$, and let β, γ be positive constants. Then $L_{\beta,\gamma}(D)$ denotes the family of all functions f that satisfy the following conditions: (i) f is analytic in D; (ii) there exists a constant C such that

$$|f(z)| \leq C|z-a|^\beta |z-b|^\gamma, \qquad (21)$$

holds for all z in D. For later convenience, let us denote $L_\beta(D) = L_{\beta,\beta}(D)$ and introduce a function

$Q(z) = (z-a)(z-b)$. If

$$\phi_{a,b}^{SE}(D_d) = \left\{z \in \mathbb{C} \middle| \arg\left(\frac{z-a}{b-z}\right) < d\right\}, \qquad (22)$$

when $f \in L_\alpha\left(\phi_{a,b}^{SE}(D_d)\right)$ for some positive constants d and α, the next theorem guarantees the exponential convergence of the SE-sinc approximation.

Theorem 1 Let $f \in L_\alpha\left(\phi_{a,b}^{SE}(D_d)\right)$ for d with $0 < d < \pi/2$, Let also N be a positive integer, and h be given by (15), Then there exists a constant C independent of N, such that

$$max_{a\le t\le b}\left|f(t) - \sum_{j=-N}^{N} f\left(\phi_{a,b}^{SE}(jh)\right) S(j,h)\left(\left(\phi_{a,b}^{SE}\right)^{-1}(t)\right)\right|$$

$$\le C\sqrt{N}exp(-\sqrt{\pi d\alpha N}). \tag{23}$$

Proof: Ref [1].

In DE transformation, if

$$\phi_{a,b}^{DE}(D_d) = \left\{z \in \mathbb{C} \middle| \arg\left(\frac{1}{\pi}\log\left(\frac{x-a}{b-x}\right) + \sqrt{(\frac{1}{\pi}\log\left(\frac{x-a}{b-x}\right))^2 + 1}\right) < d\right\} \tag{24}$$

and $f \in L_\alpha\left(\phi_{a,b}^{DE}(D_d)\right)$ we have:

Theorem 2 Let $f \in L_\alpha\left(\phi_{a,b}^{DE}(D_d)\right)$ for d with $0 < d < \pi/2$, let N be a positive integer, and let h be selected by the formula

$$h = \frac{\log(\frac{2dN}{\alpha})}{N}. \tag{25}$$

Then there exists a constant C which is independent of N, such that

$$max_{a\le t\le b}\left|f(t) - \sum_{j=-N}^{N} f\left(\phi_{a,b}^{DE}(jh)\right) S(j,h)\left(\left(\phi_{a,b}^{DE}\right)^{-1}(t)\right)\right|$$

$$\le C\exp\left(\frac{-\pi dN}{\log(\frac{2dN}{\alpha})}\right). \tag{26}$$

Proof: Ref [22, 23].

By comparing (23) and (26) it is observed that convergence by the DE transformation as N become large is much faster than that by SE transformation. In fact it is proved that DE sinc approximation is optimal in some sense in the approximation [14, 15].

For solving problem (1) with sinc methods, we need a lemma.

Lemma 1 Let ϕ be the conformal one-to-one mapping of the simply connected domain D_E to D_d Given by (13). Then

$$\delta_{jk}^{(0)} = [S(j,h)o\phi(x)\,]_{x=x_k} = \begin{cases}1, & j = k,\\ 0, & j \ne k,\end{cases} \tag{27}$$

$$\delta_{jk}^{(1)} = h\frac{d}{d\phi}[S(j,h)o\phi(x)]_{x=x_k}$$

$$= \begin{cases}0, & j = k,\\ \frac{(-1)^{k-j}}{k-j}, & j \ne k,\end{cases} \tag{28}$$

$$\delta_{jk}^{(2)} = h^2\frac{d^2}{d\phi^2}[S(j,h)o\phi(x)]_{x=x_k}$$

$$= \begin{cases}\frac{-\pi^2}{3}, & j = k,\\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & j \ne k.\end{cases} \tag{29}$$

Proof: Ref [1].

3.1 Sinc-Galerkin Method

In linear problem we have

$$\mathcal{L}(y) = p(x)y" + q(x)y' + u(x)y = f(x), \tag{30}$$

$$y(a) = y(b) = 0.$$

We consider (30) and its approximation solution by

$$y_m(x) = \sum_{j=-N}^{N}\alpha_j S_j(x), \quad m = 2N + 1, \tag{31}$$

where $S_j(x)$ is the function $S(j,h)o\phi(x)$ for some fixed step size h. The unknown coefficients α_j is determined that

$$\langle\mathcal{L}[y_m] - f, S_n\rangle = 0, \quad n = 1, \dots, N, \tag{32}$$

or

$$\langle p(x)y"(x), S_n\rangle + \langle q(x)y'(x), S_n\rangle +$$

$$\langle u(x)y(x), S_n\rangle = \langle f(x), S_n\rangle, \quad n = 1, \dots, N. \tag{33}$$

The used inner product is defined by

$$\langle f(x), g(x)\rangle = \int_a^b f(x)g(x)w(x)dx, \tag{34}$$

where $w(x) = \frac{1}{\phi'(x)}$ [1].

Theorem 3. The following relations hold

$$\langle p(x)y", S_k\rangle \approx h\sum_{j=-N}^{N}\sum_{i=0}^{2}\frac{y(x_j)}{\phi'(x_j)h^i}\delta_{kj}^{(i)}g_{2,i}, \tag{35}$$

$$\langle q(x)y', S_k\rangle \approx -h\sum_{j=-N}^{N}\sum_{i=0}^{1}\frac{y(x_j)}{\phi'(x_j)h^i}\delta_{kj}^{(i)}g_{1,i}, \tag{36}$$

and

$$\langle G, S_k\rangle \approx h\frac{G(x_k)w(x_k)}{\phi'(x_k)} \tag{37}$$

where

$g_{2,2} = (pw)(\phi')^2$, $g_{2,1} = (pw)\phi" + 2(pw)'\phi'$, $g_{2,0} = (pw)"$, $g_{1,1} = (qw)\phi'$, $g_{1,0} = (qw)'$ and

$G = u(x)y$ or $G = f(x)$.

Proof: Ref [16].

If use theorem 3 for replacing in inner product (32) we obtain following theorem:

Theorem 4 If the assumed approximate solution of the boundary-value problem (30) is (31), then the discrete sinc-Galerkin system for the determination of the unknown coefficients α_j is given by

$$\sum_{j=-N}^{N}\left(\sum_{i=0}^{2}\frac{1}{h^i}\delta_{kj}^{(i)}\frac{g_{2,i}(x_j)}{\phi'(x_j)}\alpha_j - \sum_{i=0}^{1}\frac{1}{h^i}\delta_{kj}^{(i)}\frac{g_{1,i}(x_j)}{\phi'(x_j)}\alpha_j\right) + \frac{u(x)w(x_k)}{\phi'(x_k)}\alpha_k = \frac{f(x_k)w(x_k)}{\phi'(x_k)}, -N \le k \le N. \tag{38}$$

We can rewrite (38) in following system:

$$A\alpha = D\left(\frac{wf}{\phi'}\right)1, \tag{39}$$

$A = \sum_{i=0}^{2}\frac{1}{h^i}I^{(i)}D(a_i)$, $D(g(x))_{ij} = \begin{cases}g(x_i) & i = j\\ 0 & i \ne j\end{cases}$, $I^{(i)}, 0 \le i \le 2$, the $m \times m$ matrices whose jk-th entry is given by (27)-(29), α be the m-vector with i-th component given by α_i, 1 be the m-vector each of whose components is 1 and the functions $a_i(x), 0 \le i \le 2$ are given by:

$$a_0 = \frac{g_{2,0}-g_{1,0}+uw}{\phi'},\ a_1 = \frac{g_{2,1}-g_{1,1}+uw}{\phi'},\ a_2 = \frac{g_{2,2}}{\phi'}.$$

Proof: Ref [3, 16].

By solving the obtained algebraic linear system, the vector α and so the approximation solution is determined.

Now we consider nonlinear boundary value problem

$$\mathcal{L}(y) = p(x)y" + q(x)y' + u(x)y + r(x)y^n = f(x), \tag{40}$$

$$y(a) = y(b) = 0.$$

If (31) be a approximation solution of (40), the unknown coefficients α_j is determined that

$$\langle p(x)y", S_n\rangle + \langle q(x)y', S_n\rangle + \langle u(x)y, S_n\rangle + \langle r(x)y^n, S_n\rangle = \langle f(x), S_n\rangle, \quad n = 1, \dots, N. \tag{41}$$

Lemma 2 we have

$$\langle \mathrm{r}y^n, S_n \rangle \approx \frac{hw(x_k)y^n(x_k)r(x_k)}{\phi'(x_k)}$$

Proof: Reff [3,16].

If use theorem 3 and lemma 2 for replacing in inner product (41) we obtain following theorem:

Theorem 5 If the assumed approximate solution of the boundary-value problem (40) be (31), then the discrete sinc-Galerkin system for the determination of the unknown coefficients α_j is given by

$$\sum_{j=-N}^{N}\left(\sum_{i=0}^{2}\frac{1}{h^i}\delta_{kj}^{(i)}\frac{g_{2,i}(x_j)}{\phi'(x_j)}\alpha_j - \sum_{i=0}^{1}\frac{1}{h^i}\delta_{kj}^{(i)}\frac{g_{1,i}(x_j)}{\phi'(x_j)}\alpha_j\right)$$

$$+\frac{u(x)w(x_k)}{\phi'(x_k)}\alpha_k + \frac{w(x_k)r(x_k)}{\phi'(x_k)}\alpha_k^n = \frac{f(x_k)w(x_k)}{\phi'(x_k)},$$

$$-N \le k \le N. \qquad (42)$$

We can rewrite (42) in following system:

$$A\alpha + E\alpha^n = F \qquad (43)$$

where $E = D\left(\frac{rw}{\phi'}\right)$, $F = D\left(\frac{wf}{\phi'}\right)1$, and A is given by (39). For solving nonlinear system (40), we can use Newton s method.

4. Numerical Examples

Here we present some examples that are solved by DE and SE sinc-Galerkin methods and Chebyshev cardinal functions. These examples have singular point in equations or solutions. Comparisons show that Chebyshev cardinal functions is a good method for solving problems with singular equations and better than sinc methods. It is not appropriate for solving problems with singular solutions. The DE sinc-Galerkin method gives better results than SE sinc-Galerkin method. The problems are solved with Matlab on a personal computer.

In these examples, the maximum absolute error at sinc points is taken as

$\|\mathrm{E_{SE}}\| = \max_{-\mathrm{N}\le i\le \mathrm{N}}\left|y_{\mathrm{exact}}(x_i) - y_{\mathrm{N,SE\ sinc-Galerkin}}(x_i)\right|$,

where $x_k = \frac{a+be^{kh}}{1+e^{kh}}$, with $d = \frac{\pi}{2}, \alpha = 1/2$ and

$\|\mathrm{E_{DE}}\| = \max_{-\mathrm{N}\le i\le \mathrm{N}}\left|y_{\mathrm{exact}}(x_i) - y_{\mathrm{N,DE\ sinc-Galerkin}}(x_i)\right|$,

where $x_k = \frac{\mathrm{b-a}}{2}\mathrm{tgh}\left(\frac{\pi}{2}\sinh(\mathrm{kh})\right) + \frac{\mathrm{b+a}}{2}$.

In tables 1, 4 and 7 we give the absolute errors in some points using proposed methods. In tables 2, 5 and 8 we present maximum error in SE sinc-Galerkin method and DE sinc-Galerkin method in all sinc points. In tables 3, 6 and 9 we give absolute errors in (a,b) with chebyshev cardinal functions. We use $N = 125$ in SE sinc-Galerkin method, $N = 50$ in DE sinc-Galerkin method and $N = 15$ In Chebyshev cardinal functions. It is observed that although N in DE transformation is smaller but it's better than SE transformation. In Chebyshev cardinal functions method our bases functions are interpolate functions of Chebyshev functions and by using low values of base functions, we have high accuracy of solutions of regular and singular equations, but the nonlinearity of these problems don't let that we can choose a high value of base functions. Thus in our examples, we choose low values of N.

Example 1. Consider the problem

$$xy'' - 2y' + y = \frac{-(-x^3 + x^2 + 3x + 3)}{e^x},$$

$y(-1) = y(1) = 0,$

with exact solution

$$y = \frac{(x^2 - 1)}{e^x}.$$

In this problem the singular point in equation is $x = 0$.

Table 1.The error of solving example 1

x_i	*Error in SE sinc-Galerkin*	*Error in Chebyshev cardinal functions*
-0.999995130475963	2.85 e-011	. 96 e-015
-0.687383065290916	1.32 e-006	2.23 e-010
-0.273823495469893	2.11 e-006	3.45 e-010
0	2.43 e-006	4.01 e-010
0.273823495469893	2.71 e-006	4.48 e-010
0.605940504455271	2.37 e-006	3.39 e-010
0.961616491413769	3.48 e-007	4.49 e-011
0.999858140267242	1.34 e-009	2.85 e-013

x_i *in DE transformation*	*Error in DE sinc-Galerkin*
-0.999994677382579	8.94 E-016
-0.672078625326399	1.40 e-011
-0.286634259852092	2.03 e-011
0	2.32 e-011
0.286634259852092	2.60 e-011
0.608126245913561	2.34 e-011
0.963240393303272	3.34 e-012
0.999814417834297	1.70 e-014

Table 2.Maximum error in SE sinc-Galerkin method and DE sine-Galerkin method

N	$\|E_{SE}\|$	$\|E_{DE}\|$
25	0.4737	1.41 e-006
50	1.88 e-002	2.72 e=011
75	6.77 e-004	9.55 e-011
100	4.23 e-005	2.46 e-010
125	2.76 e-006	3.55 e-010

Table 3. Max error in Chebyshev cardinal functions

N	*error*
10	1.4 e-005
15	4.0e-010
20	1.0e-019

Example 2. Consider the problem

$(10\mathrm{x}-9)^3\mathrm{y}'' - 2(10\mathrm{x}-9)^3\mathrm{y}' - (10\mathrm{x}-9)^3\mathrm{y}$

$= (1000\mathrm{x}^4 + 200\mathrm{x}^3 - 5590\mathrm{x}^2 + 7040\mathrm{x} - 223),$

$\mathrm{y}(-1) = \mathrm{y}(1) = 0$

with exact solution

$\mathrm{y} = -(\mathrm{x}^2 - 1)/(\mathrm{x} - 0.9)$

In this problem the singular point in solution is $\mathrm{x} = 0.9$.

Table 1. The error of solving example

x_i	*Error in SE sinc-Galerkin method*	*Error in Chebyshev cardinal functions*
-0.999938937287810	7.83 e-11	2.75 E-4
-0.886419699493427	1.65 e-7	1.21 E-1
-0.605940504455271	7.95 e-7	7.13 E-1
-0.398183994844089	1.60 e-6	1.15 E-0
0	4.80 e-5	1.92 E-0
0.273823495469893	9.62 e-6	2.21 E-0
0.605940504455271	2.18 e-5	6.64 E-0
0.913033166231486	4.28 e-5	4.49 E+1
0.999995130475963	3.33 e-9	2.08 E-3

x_i *in DE trasformation*	*Error in DE sinc-Galerkin method*
-0.999949158093507	1.31 e-16
-0.888259468294954	6.28 e-14
-0.608126245913561	3.05 e-13
-0.375635017328184	6.57 -13
0	1.91 e-12
0.286634259852092	3.76 e-12
0.608126245913561	8.61 e-12
0.913440889188212	1.76 e-11
0.999994677382579	3.32 e-15

Table 5. Maximum error in SE sinc-Galerkin method and DE sinc-Galerkin method

N	$\|E_{SE}\|$	$\|E_{DE}\|$
25	5.2519	2.27 e-006
50	0.65403	1.88 e-011
75	7.02 e-003	6.08 e-011
100	7.28 e-004	4.08 e-010
125	4.29 e-005	4.39 e-010

Table 2. Max error in Chebyshev cardinal functions

error	*N*
300 e001	10
400 e001	20

Example 3. Consider the nonlinear problem

$(3x-2)^4 y'' - (3x-2)^4 y - 3(3x-2)^4 y^4 = (-3x^8 + 12x^6 - 27x^5 + 36x^4 - 9 - 34 + 6x + 9)$

$y(-1) = y(1) = 0$

with exact solution

$y = (x^2 - 1)/(3x - 2)$

In this problem the singular point in solution is $x = 2/3$.

Table 3. The error of solving example 3

x_i	*Error in SE sinc-Galerkin method*	*Chebyshev cardinal functions*
-0.999999982347064	1.05 e -14	2.97 E-7
-0.398183994844089	7.96 e -08	2.11 E-0
-0.139579107197183	1.22 e -08	9.86 E-1
0.273823495469893	2.36 e -07	4.40 E-0
0.509448868030365	4.45 e -07	4.96 E-0
0.886419699493427	2.41 e -07	4.39 E-0
0.999858140267242	3.75 e -10	5.39 E-2

x_i *in DE trasformation*	*Error in DE sinc-Galerkin method*
-0.999999986180038	6.27 e-17
-0.375635017328184	8.71 e-14
-0.193494262754776	2.13 e-13
0.286634259852092	2.52 e-13
0.537116659907527	5.29 e-13
0.888259468294954	1.50 e-13
0.999814417834297	2.73 e-16

Table 8. Maximum error in SE sinc-Galerkin method and DE sinc-Galerkin method

N	$\|E_{SE}\|$	$\|E_{DE}\|$
25	1.1645	6.60 e-007
50	0.0209	3.06 e-012
75	2.7825e-004	2.05 e-008
100	8.9257e-006	7.99 e-008
125	8.8722e-007	8.60 e-008

Table 9. Max error in Chebyshev cardinal functions

N	*error*
3	-1500 e-0
6	-1500 e-0
10	-1500 e-0
13	-1500 e-0
15	-1500 e-0

5. Conclusion

In this paper we compared the DE sinc-Galerkin and SE sinc-Galerkin and Chebyshev cardinal functions for solving boundary value problems with singular point in equation or solutions. Chebyshev cardinal functions is a good method for solving problems with singular equations and better than sinc methods. The DE sinc-Galerkin and SE sinc-Galerkin are appropriate and the Chebyshev cardinal functions is not suitable for problems with singular solutions. It was observed that DE sinc-Galerkin with small N gives better results than SE sinc-Galerkin with bigger N. These results highlight the accuracy and potency of DE transformation.

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