International Journal of
Applied Mathematics,
Electronics and Computers

IJAMEC

ISSN:2147-8228 www.atscience.org/IJAMEC

Advanced Technology and Science

Original Research Paper

Fourier-type integral transforms in modeling of transversal oscillation[#]

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Accepted 15th August 2014

Abstract. The model of transversal oscillation for an elastic piecewise-homogeneous rod is constructed. In order to find a solution of this model a Fourier-type integral transforms method for the fourth-order differential equations is developed. The decomposition theorem is proved by Cauchy contour integration method.The conditions of existence for fundamental solutions of the initial - boundary value problem are established and explicit expressions of these fundamental solutions are found.

Keywords:*eigenfunction, fourth-order differential equation, fundamental solution, Green's function***.**

1. Introduction

An eigenfunction of a linear operator A, defined on some function space, is any non-zero function f in that space that returns from the operator exactly as is, except for a multiplicative scaling factor. Eigenfunctions play an important role in many branches of physics. Using eigenfunctions the laws of mathematical physics can be described, for example, the law of the transversal oscillations of semi-limited piecewise-homogeneous rod.

Laws of the transverse oscillations of rod are found theoretically by Euler. These laws on the experience are checked by Chladni, Lissajous, Mercadieretc. Transversal oscillations modeling of semi-limited piecewise-homogeneous rod

$$I_n^+=\left\{x:\ x\in\bigcup_{k=1}^{n+1}\left(l_{k-1},l_k\right),\ l_0>0,\ l_{n+1}=\infty\right\}.$$

leads to solve the system of fourth-order differential equations

$$\left(\frac{\partial^2}{\partial t^2}+A_j^4\frac{d^4}{dx^4}\right)y_j(t,x)=0,\ t>0,\ x\in\left(l_{j-1},l_j\right),\ \ j=1,\dots n+1. \tag{1}$$

Here function $y_j\left(x,t\right)$ is the ordinate of the deformed axis of j- layer at the point x in time t;

$$y_j\left(x,t\right),\ \ x\in\left(l_{j-1},l_j\right),\ j=1,\dots,n+1,\ \ t>0$$

E_j – an elastic modulus (or Young's modulus) of j- layer rod;

I_j – the moment of inertia of j- layer rod, a line perpendicular to the plane of thex,yand passing through the gravity center of areas S;

ρ_j -the density of j- layer rod;

$$A_j^4=\frac{E_jI_j}{\rho_jS}.$$

The boundary conditions have the form:

$$y=f\left(t\right),\frac{\partial^2y}{\partial x^2}=0 \tag{2}$$

when $x=0,t>0.$

Further formulate the conditions at the points of coupling intervals.

The conditions for continuity of the ordinates and for the tangent of the bending moment and clipping efforts must be true:

$$y_j\left(l_j,t\right)=y_{j+1}\left(l_j,t\right),\ y_j'\left(l_j,t\right)=y_{j+1}'\left(l_j,t\right), \tag{3}$$

$$E_jI_jy_j''\left(l_j,t\right)=E_{j+1}I_{j+1}y_{j+1}''\left(l_j,t\right),$$
$$E_jI_jy_j'''\left(l_j,t\right)=E_{j+1}I_{j+1}y_{j+1}'''\left(l_j,t\right).$$

Let's assume, the rod was in an equilibrium state before the hanging point was set into motion. The initial conditions have the form

$$y=0,\frac{\partial y}{\partial t}=0$$

when $t=0,x\geq 0.$

2. Problem Statement

We construct the integral transform by a method of delta-shaped sequences. Integrated transform is generated on the set I_n^+ by the fourth -order differential operator B:

$$B=\sum_{k=1}^{n}A_k^4\theta\left(x-l_{k-1}\right)\theta\left(l_k-x\right)\frac{d^4}{dx^4}+A_{n+1}^4\theta\left(x-l_n\right)\frac{d^4}{dx^4},$$

here A_k - matrix of size$\sigma\times\sigma$, all eigenvalues are the real positive number,

$\theta(x)$ - the Heaviside step function.

Let's consider solving problem of separate matrix system for (n+1) iterated parabolic equations

$$\left(A_m^{-4}\frac{\partial^2}{\partial t^2}+\frac{d^4}{dx^4}\right)v_m\left(t,x\right)=0,\ \ t>0,\ \ x\in I_n^+,\ \ m=\overline{1,n+1}, \tag{4}$$

limited on the set D,

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 # This paper has been presented at the International Conference on Advanced Technology&Sciences (ICAT'14) held in Antalya (Turkey), August 12-15, 2014.

$$D=(0,\infty)\times I_n^+=\left\{(t,r):\ t>0,\ \ x\in\bigcup_{k=1}^{n+1}(l_{k-1},l_k),\ l_0>0,\ l_{n+1}=\infty\right\}$$

on the initial conditions

$$\begin{aligned} &v_m(t,x)\big|_{t=0}=g_m(x),x\in I_n^+,\\ &\left.\frac{\partial v_m(t,r)}{\partial t}\right|_{t=0}=0,\ x\in I_n^+,\ m=\overline{1,n+1}, \end{aligned} \tag{5}$$

on the boundary conditions

$$\begin{aligned} &\left.\sum_{i=0}^{3}\alpha_{e1,i}^{0}\frac{\partial^i}{\partial x^i}v_1\right|_{x=l_0}=0,\ e=\overline{1,2},\\ &\left.\frac{\partial^i}{\partial x^i}v_{n+1}\right|_{x=\infty}=0,\ i=\overline{0,3} \end{aligned} \tag{6}$$

and the coupling conditions

$$\begin{aligned} &\sum_{i=0}^{3}\alpha_{m1,i}^{k}\frac{\partial^i}{\partial x^i}v_k=\sum_{i=0}^{3}\alpha_{m2,i}^{k}\frac{\partial^i}{\partial x^i}v_{k+1},\\ &x=l_k,\ k=\overline{1,n};\ m=\overline{1,4}. \end{aligned} \tag{7}$$

Here $v_m=v_m(t,x)$ – is the unknown vector-valued function of size σ×1, $g_m(x)$ –is the set of vector-valued functions of size σ×1, $\alpha_{j,e,i}^{k};\ j=\overline{1,4},\ e=\overline{1,2},\ i=\overline{0,3}$ - matrixes of size σ×σ.

The special solution $H_{j,s}(t,x,\xi)$, meeting conditions

$$H_{j,s}(t,x,\xi)\big|_{t=0}=\delta(x-\xi),\ x,\xi\in I_n^+,\ j=\overline{1,n+1}$$

$$\left.\sum_{i=0}^{3}\alpha_{e1,i}^{0}\frac{\partial^i}{\partial x^i}H_{1,s}\right|_{x=l_0}=0,\ e=\overline{1,2},$$

$$\left.\frac{\partial^i}{\partial x^i}H_{n+1,s}\right|_{x=\infty}=0,\ i=\overline{0,3}$$

$$\left[\alpha_{m1}^{k}\frac{d}{dx}+\beta_{m1}^{k}\right]H_{k,s}=\left[\alpha_{m2}^{k}\frac{d}{dx}+\beta_{m2}^{k}\right]H_{k+1,s},x=l_k,$$
$$k=\overline{1,n},\ m=\overline{1,2},$$

is called as the matrix fundamental solution or the Green's function.We can write the solution of the General boundary value problem (4)-(7), if we know influence function $H_{j,s}=H_{j,s}(t,x,\xi)$. Our next goal is to clarify the conditions of existence of the influence functions $H_{j,s}$ and finding explicit expressions for these functions.

We introduce notations:

$$M_{m,k}\equiv\begin{pmatrix}\alpha_{1m,0}^{k}\dots\alpha_{1m,3}^{k}\\ \dots\dots\dots\dots\dots\dots\\ \alpha_{4m,0}^{k}\dots\alpha_{4m,3}^{k}\end{pmatrix},\ k=\overline{1,n};\ m=\overline{1,2},$$

and demand fulfillment of the following condition:

$$\det M_{mk}\equiv C_{mk}\neq 0,k=\overline{1,n};m=\overline{1,2}. \tag{8}$$

Let's assume that the desired vector-valued function $U_j(t,x)$ is Laplace's original on *t*. In the images of the Laplace we get a problem about a construction of limited on the set I_n^+ solutions of the separate matrix system of ordinary differential equations

$$\begin{aligned} &\left(\frac{d^4}{dx^4}-q_j^4\right)U_j^*(p,x)=p\bar{g}_j(x),\ \ j=\overline{1,n+1},\\ &q_j^4=-A_j^{-4}p^2,\ \bar{g}_j(x)=A_j^{-4}g_j(x) \end{aligned} \tag{9}$$

on the boundary conditions

$$\begin{aligned} &\left.\sum_{i=0}^{3}\alpha_{e1,i}^{0}\frac{d^i}{dx^i}v_1^*\right|_{x=l_0}=0,\ e=\overline{1,2},\\ &\left\|\left.\frac{d^i}{dx^i}v_{n+1}^*(p,x)\right\|\right|_{x=\infty}<\infty,\ i=\overline{0,3} \end{aligned} \tag{10}$$

and the contact conditions in the points of joint

$$\begin{aligned} &\sum_{i=0}^{3}\alpha_{m1,i}^{k}\frac{d^i}{dx^i}v_k^*=\sum_{i=0}^{3}\alpha_{m2,i}^{k}\frac{d^i}{dx^i}v_{k+1}^*,x=l_k,\\ &m=\overline{1,4},\ k=\overline{1,n}. \end{aligned} \tag{11}$$

Let's define images $H_{j,s}^*$ of matrix fundamental solution as solutions of the following boundary value problem for the separate matrix system of ordinary differential equations (9):

$$\left(\frac{d^4}{dx^4}-q_j^4\right)H_{j,s}^*(p,x,\xi)=A_j^{-4}\delta(x-\xi),\ \ j,s=\overline{1,n+1},$$
$$q_j^4=-A_j^{-4}p^2,$$

$$\left.\sum_{i=0}^{2q-1}\alpha_{e1,i}^{0}\frac{d^i}{dx^i}H_{1,s}^*\right|_{x=l_0}=0,\ e=\overline{1,2}$$

$$\left\|\left.\frac{d^i}{dx^i}H_{n+1,s}^*(p,x,\xi)\right\|\right|_{x=\infty}<\infty,\ i=\overline{0,3}$$

$$\sum_{i=0}^{3}\alpha_{m1,i}^{k}\frac{d^i}{dx^i}H_{k,s}^*=\sum_{i=0}^{3}\alpha_{m2,i}^{k}\frac{d^i}{dx^i}H_{k+1,s}^*,x=l_k,$$
$$m=\overline{1,4},\ k=\overline{1,n}.$$

For images of matrix influence functions $H_{j,s}^*$ formulas are fair:

if k< s

$$H_{k,s}^*=\left(\varphi_k(x)\overset{0}{\varphi}{}_1^{-1}(q_1)-\psi_k(x)\overset{0}{\psi}{}_1^{-1}(q_1)\right)\left(\overset{0}{\varphi}_1(q_1)\ \ \overset{0}{\psi}_1(q_1)\right)\Omega_s^{-1}(\xi)\begin{pmatrix}0\\ \mathrm{E}\end{pmatrix},$$
$$k=\overline{1,n+1},\ s=\overline{1,n+1},$$

if k> s

$$H_{k,s}^*=-\psi_k(x)\overset{0}{\psi}{}_1^{-1}(q_1)\left(\overset{0}{\varphi}_1(q_1)\ \ \overset{0}{\psi}_1(q_1)\right)\Omega_s^{-1}(\xi)\begin{pmatrix}0\\ \mathrm{E}\end{pmatrix},$$
$$l_{k-1}<x<l_k,\ l_{s-1}<\xi<l_s,\ k=\overline{1,n+1},\ s=\overline{1,n+1},$$

if k = s

$$H_{k,k}^*=\begin{cases}\left(\varphi_k(x)\overset{0}{\varphi}{}_1^{-1}(q_1)-\psi_k(x)\overset{0}{\psi}{}_1^{-1}(q_1)\right)\left(\overset{0}{\varphi}_1(q_1)\ \ \overset{0}{\psi}_1(q_1)\right)\Omega_s^{-1}(\xi)\begin{pmatrix}0\\ \mathrm{E}\end{pmatrix},\\ l_{k-1}<x<\xi<l_k;\\ -\psi_k(x)\overset{0}{\psi}{}_1^{-1}(q_1)\left(\overset{0}{\varphi}_1(q_1)\ \ \overset{0}{\psi}_1(q_1)\right)\Omega_s^{-1}(\xi)\begin{pmatrix}0\\ \mathrm{E}\end{pmatrix},\\ l_{k-1}<\xi<x<l_k,\ \ k=\overline{1,n+1}.\end{cases}$$

Matrix functions are present in the expressions for images of matrix influence functions $H_{j,s}^*$

$$\varphi(x,\lambda)=\sum_{k=1}^{n}\theta(x-l_{k-1})\theta(l_k-x)\varphi_k(x,\lambda)+\theta(x-l_n)\varphi_{n+1}(x,\lambda),$$

$$\psi(x,\lambda)=\sum_{k=1}^{n}\theta(x-l_{k-1})\theta(l_k-x)\psi_k(x,\lambda)+\theta(x-l_n)\psi_{n+1}(x,\lambda).$$

For $\varphi_{n+1}=\left(\varphi_{n+1,0}\quad\varphi_{n+1,1}\right);\ \psi_{n+1}=\left(\psi_{n+1,0}\quad\psi_{n+1,1}\right)$ we have values

$$\psi_{n+1,0}\left(x,{}_1\sqrt[4]{q_{n+1}^4}\right)=e^{{}_1\sqrt[4]{q_{n+1}^4}\,x},$$

$$\psi_{n+1,1}\left(x,{}_3\sqrt[4]{q_{n+1}^4}\right)=e^{{}_3\sqrt[4]{q_{n+1}^4}\,x},$$

$$\varphi_{n+1,0}\left(x,{}_0\sqrt[4]{q_{n+1}^4}\right)=e^{{}_0\sqrt[4]{q_{n+1}^4}\,x},$$

$$\varphi_{n+1,1}\left(x,{}_1\sqrt[4]{q_{n+1}^4}\right)=e^{{}_2\sqrt[4]{q_{n+1}^4}\,x},$$

where the *j* radical branch is designated by a symbol ${}_j\sqrt[4]{\ }$. Other pairs of functions φ_m, ψ_m uniquely defined by coupling conditions:

$$\sum_{i=0}^{3}\alpha_{j1,i}^{k}\frac{d^i}{dx^i}(\phi_k,\psi_k)=\sum_{i=0}^{3}\alpha_{j2,i}^{k}\frac{d^i}{dx^i}(\phi_{k+1},\psi_{k+1}),\ x=l_k,\ k=\overline{1,n},\ j=\overline{1,4}.$$

Furthermatrixes $\overset{0}{\varphi}_1,\overset{0}{\psi}_1$ and Ω_k are defined by relations:

$$\begin{pmatrix}\overset{0}{\varphi}_1 & \overset{0}{\psi}_1\end{pmatrix}=\begin{pmatrix}\overset{0}{\varphi}_{11} & \overset{0}{\psi}_{11}\\ \overset{0}{\varphi}_{21} & \overset{0}{\psi}_{21}\end{pmatrix},\ \begin{pmatrix}\overset{0}{\varphi}_{e1} & \overset{0}{\psi}_{e1}\end{pmatrix}=\sum_{i=0}^{3}\alpha_{e1,i}^{0}\frac{d^i}{dx^i}\begin{pmatrix}\varphi_1 & \psi_1\end{pmatrix}\Big|_{x=l_0},\ e=\overline{1,2},$$

$$\Omega_k=\begin{pmatrix}\varphi_k & \psi_k\\ \varphi_k^{/} & \psi_k^{/}\\ \varphi_k^{//} & \psi_k^{//}\\ \varphi_k^{///} & \psi_k^{///}\end{pmatrix}$$

In the future, the condition will be considered fulfilled:

$$\det\begin{pmatrix}E_{\sigma 2} & 0_{\sigma 2}\\ \alpha_{11,0}^{0} & \alpha_{11,1}^{0}\\ \alpha_{21,0}^{0} & \alpha_{21,1}^{0}\end{pmatrix}\neq 0, \qquad (12)$$

$E_{\sigma 2}$ - identity matrix of size $2\sigma\times 2\sigma$, $0_{\sigma 2}$ - zero matrixof size $2\sigma\times 2\sigma$.

The solution of the matrix boundary problem (9)-(11) is based on the following statements.

Lemma 1. The determinant $\det(\Omega_k)$ doesn't depend on the variable ξ. If the condition (12) is satisfied, all matrixes Ω_k are invertible (or nondegenerate matrix)

$$\det(\Omega_k)\neq 0.$$

Proof. Verification of the first statement of the lemma is trivial. It is enough to establish that $\det(\Omega_1)\neq 0$. The last requirement immediately follows from the identity

$$\begin{pmatrix}\varphi_1 & \psi_1\\ \varphi_1^{/} & \psi_1^{/}\\ \overset{0}{\varphi}_1 & \overset{0}{\psi}_1\end{pmatrix}=\begin{pmatrix}E_{\sigma 2} & 0_{\sigma 2}\\ \alpha_{11,0}^{0} & \alpha_{11,1}^{0}\\ \alpha_{21,0}^{0} & \alpha_{21,1}^{0}\end{pmatrix}\Omega_1(\xi).$$

Lemma 2. If the condition (12) is satisfied, then conditions of unlimited resolvability of the problem (9)-(11) are satisfied

for $p=\sigma+i\tau$ with $\mathrm{Re}\,p\geq 0$, where

σ_0 - abscissa of convergence of Laplace integral,

and $Jmp=\tau\in(-\infty,+\infty)$

matrixes $\Omega_s(\xi),\ s=\overline{1,n+1}$ - nondegenerate:

$$\det\ \Omega_s(\xi)\neq 0. \qquad (13)$$

Lemma 3. If conditions of unlimited resolvability of the problem are satisfied, then limited on the set I_n^+ the solution of separate system (9) has the form:

$$v_j^*(p,x)=\sum_{m=0}^{n}\int_{l_m}^{l_{m+1}}H_{j,m+1}^{*}(p,x,\xi)\,p\bar{g}_{m+1}(\xi)\,d\xi,\ \ j=\overline{1,n+1}. \qquad (14)$$

Proof. On the basis of the Lemma 2, the right part of the formula (14) has the meaning. Therefore direct validate of each conditions (9)-(11) is possible.

Let's return to originals in the formula (14). We apply the inverse Laplace transform formula, we have

$$v_j(t,x)=\frac{1}{2\pi i}\int_0^{\infty i}e^{pt}\sum_{m=0}^{n}\int_{l_m}^{l_{m+1}}H_{j,m+1}^{*}(p,x,\xi)\,p\bar{g}_{m+1}(\xi)\,d\xi dp+$$

$$+\int_{-\infty i}^{0}e^{pt}\sum_{m=0}^{n}\int_{l_m}^{l_{m+1}}H_{j,m+1}^{*}(p,x,\xi)\,p\bar{g}_{m+1}(\xi)\,d\xi dp,\ j=\overline{1,n+1}.$$

We consider that the function

$$\begin{pmatrix}\overset{0}{\phi}_1,\overset{0}{\psi}_1\end{pmatrix}\Omega_s^{-1}(\rho,\lambda)\begin{pmatrix}0\\ \mathrm{E}\end{pmatrix}$$

is analytic in the half-plane $\mathrm{Re}\,p\geq 0$ according to condition (12). We make the change $p=i\lambda$ in the first integral, and in the second integral $p=-i\lambda$ we will present the problem solution (4)-(7) in the form:

$$v_j(t,x)=-\frac{1}{2\pi i}\int_0^{\infty}\sum_{m=0}^{n}\int_{l_m}^{l_{m+1}}\left(e^{-i\lambda t}\varphi_k(x)\overset{0}{\varphi}{}_1^{-1}(q_1)-e^{i\lambda t}\psi_k(x)\overset{0}{\psi}{}_1^{-1}(q_1)\right)\cdot$$

$$\cdot\begin{pmatrix}\overset{0}{\varphi}_1(q_1) & \overset{0}{\psi}_1(q_1)\end{pmatrix}\Omega_s^{-1}(\xi)\begin{pmatrix}0\\ \mathrm{E}\end{pmatrix}\bar{g}_{m+1}(\xi)\,d\xi\lambda d\lambda,\ j=\overline{1,n+1}.$$

In the found representation for the mixed boundary problem (4)-(7) we will pass to the limit at $t\to 0$. We obtain the integral representation for initial conditions (5):

$$g_j(x)=-\frac{1}{2\pi i}\int_0^{\infty}\sum_{m=0}^{n}\int_{l_m}^{l_{m+1}}\left(\varphi_k(x)\overset{0}{\varphi}{}_1^{-1}(q_1)-\psi_k(x)\overset{0}{\psi}{}_1^{-1}(q_1)\right)\cdot \qquad (15)$$

$$\cdot\begin{pmatrix}\overset{0}{\varphi}_1(q_1) & \overset{0}{\psi}_1(q_1)\end{pmatrix}\Omega_s^{-1}(\xi)\begin{pmatrix}0\\ \mathrm{E}\end{pmatrix}\bar{g}_{m+1}(\xi)\,d\xi\lambda d\lambda,\ j=\overline{1,n+1}.$$

If to put

$$u(x,\lambda)=\sum_{k=1}^{n}\theta(x-l_{k-1})\theta(l_k-x)u_k(x,\lambda)+\theta(x-l_n)u_{n+1}(x,\lambda),$$

where

$$u_k=\varphi_k\,\overset{0}{\varphi}{}_1^{-1}-\psi_k\,\overset{0}{\psi}{}_1^{-1},$$

and

$$u^*(\xi,\lambda)=\sum_{k=1}^{n}\theta(\xi-l_{k-1})\theta(l_k-\xi)u_k^*(\xi,\lambda)+\theta(\xi-l_n)u_{n+1}^*(\xi,\lambda),$$

where

$$u_k^*=\begin{pmatrix}\overset{0}{\varphi}_1(\lambda),\overset{0}{\psi}_1(\lambda)\end{pmatrix}\Omega_k^{-1}(\xi,\lambda)\begin{pmatrix}0\\ \mathrm{E}\end{pmatrix}A_k^{-4},\ \ k=\overline{1,n+1},$$

then integral representation (15) leads to an integral representation of Dirac measure [18]

$$\delta(x-\xi)=-\frac{1}{2\pi i}\int_0^{\infty}u(x,\lambda)\,u^*(\xi,\lambda)\,\lambda d\lambda. \qquad (16)$$

The integral representation of Dirac measure generates direct F_{n+} and inverse F_{n+}^{-1} Fourier transformation of fourth order on the Cartesian axis with n points division by the rules:

$$F_{n+}[g](\lambda)=\sum_{m=0}^{n}\int_{l_m}^{l_{m+1}}u_{m+1}^{*}(\xi,\lambda)\,g_{m+1}(\xi)\,d\xi\equiv\tilde{g}(\lambda), \qquad (17)$$

$$F_{n+}^{-1}[\hat{g}](x)=-\frac{1}{2\pi i}\int_0^{\infty}u(x,\lambda)\,\tilde{g}(\lambda)\,\lambda d\lambda\equiv g(x), \qquad (18)$$

$$g(x)=\sum_{k=1}^{n}\theta(x-l_{k-1})\theta(l_k-x)g_k(x)+\theta(x-l_n)g_{n+1}(x).$$

Following the decomposition theorem can be proved by the Cauchy's method of contour integration [14].

Theorem 1. If the function $g(x)$ is defined, piecewise continuous, absolutely summarized and has a limited variation on I_n^+, then for $x\in I_n^+$ the integral representation is true:

$$\frac{1}{2}\left[g(x-0)+g(x+0)\right]=-\frac{1}{2\pi i}\int_0^{\infty}u(x,\lambda)\left(\int_{l_0}^{\infty}u^*(\xi,\lambda)g(\xi)d\xi\right)\lambda d\lambda. \tag{19}$$

We will receive the main integrated transformation identity of the differential operator for the application of the integral transformations for the problems solution of mathematical physics

$$B=\sum_{k=1}^{n}A_k^4\theta(x-l_{k-1})\theta(l_k-x)\frac{d^4}{dx^4}+A_{n+1}^4\theta(x-l_n)\frac{d^4}{dx^4}.$$

Theorem 2. If the function $g(x)$ is defined, four times differentiable on the set I_n^+, $g(x)$, then the main identity for the function satisfying to the coupling conditions (7) and vanishing at infinity together with the derivatives to the third order holds:

$$F_{n+}\left[B(g)\right](\lambda)=\lambda^2F_{n+}[g](\lambda)-\begin{pmatrix}\alpha_{11,0}^0 & \alpha_{11,1}^0 & \alpha_{11,2}^0 & \alpha_{11,3}^0\\ \alpha_{21,0}^0 & \alpha_{21,1}^0 & \alpha_{21,2}^0 & \alpha_{21,3}^0\end{pmatrix}\begin{pmatrix}g_1(l_0)\\ g_1^{/}(l_0)\\ g_1^{//}(l_0)\\ g_1^{///}(l_0)\end{pmatrix}. \tag{20}$$

Applies the formula of integration by parts in the form

$$\int_0^{\infty}u(x)v^{(IV)}(x)dx=\begin{pmatrix}u & u^{/} & u^{//} & u^{///}\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}\begin{pmatrix}v\\ v^{/}\\ v^{//}\\ v^{///}\end{pmatrix}\Bigg|_{x=0}^{x=\infty}+$$

$$+\int_0^{\infty}v(x)u^{(IV)}(x)dx.$$

Let's prove that all members outside the integral except the first member in the right part of the written-out formula will disappear. Let's use coupling conditions

$$M_{k1}\Omega_k=M_{k2}\Omega_{k+1},x=l_k$$

and its consequence

$$\Omega_k^{-1}M_{k1}^{-1}=\Omega_{k+1}^{-1}M_{k2}^{-1},x=l_k.$$

Not the zero member in the formula of integration by parts in the form

$$-\left(\overset{0}{\varphi}_1(\lambda),\overset{0}{\psi}_1(\lambda)\right)\Omega_k^{-1}(l_0,\lambda)\begin{pmatrix}g_1(l_0)\\ g_1^{/}(l_0)\\ g_1^{//}(l_0)\\ g_1^{///}(l_0)\end{pmatrix}.$$

When it is considered that

$$\left(\overset{0}{\varphi}_1(\lambda),\overset{0}{\psi}_1(\lambda)\right)=\begin{pmatrix}\alpha_{11,0}^0 & \alpha_{11,1}^0 & \alpha_{11,2}^0 & \alpha_{11,3}^0\\ \alpha_{21,0}^0 & \alpha_{21,1}^0 & \alpha_{21,2}^0 & \alpha_{21,3}^0\end{pmatrix}\Omega_1(l_0,\lambda),$$

we obtain the formula for member outside the integral in the form

$$\begin{pmatrix}\alpha_{11,0}^0 & \alpha_{11,1}^0 & \alpha_{11,2}^0 & \alpha_{11,3}^0\\ \alpha_{21,0}^0 & \alpha_{21,1}^0 & \alpha_{21,2}^0 & \alpha_{21,3}^0\end{pmatrix}\Omega_1(l_0,\lambda)\Omega_k^{-1}(l_0,\lambda)\begin{pmatrix}g_1(l_0)\\ g_1^{/}(l_0)\\ g_1^{//}(l_0)\\ g_1^{///}(l_0)\end{pmatrix}.$$

The theorem is proved.

3.Modeling of transversal oscillation for an elastic piecewise-homogeneous rod

Let's apply the method of integral Fourier transforms from item 2 to the problem solving (1)-(3). Let's choose the parameters in conditions (9),(10),(11) :

$$A_m^4=\frac{E_mI_m}{\rho_mS},$$

$$\begin{pmatrix}\alpha_{11,0}^0 & \alpha_{11,1}^0 & \alpha_{11,2}^0 & \alpha_{11,3}^0\\ \alpha_{21,0}^0 & \alpha_{21,1}^0 & \alpha_{21,2}^0 & \alpha_{21,3}^0\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&0&1&0\end{pmatrix},$$

$$\begin{pmatrix}\alpha_{11,0}^m & \alpha_{11,1}^m & \alpha_{11,2}^m & \alpha_{11,3}^m\\ \alpha_{21,0}^m & \alpha_{21,1}^m & \alpha_{21,2}^m & \alpha_{21,3}^m\\ \alpha_{31,0}^m & \alpha_{31,1}^m & \alpha_{31,2}^m & \alpha_{31,3}^m\\ \alpha_{41,0}^m & \alpha_{41,1}^m & \alpha_{41,2}^m & \alpha_{41,3}^m\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&E_mI_m&0\\0&0&0&E_mI_m\end{pmatrix},$$

$$\begin{pmatrix}\alpha_{12,0}^m & \alpha_{12,1}^m & \alpha_{12,2}^m & \alpha_{12,3}^m\\ \alpha_{22,0}^m & \alpha_{22,1}^m & \alpha_{22,2}^m & \alpha_{22,3}^m\\ \alpha_{32,0}^m & \alpha_{32,1}^m & \alpha_{32,2}^m & \alpha_{32,3}^m\\ \alpha_{42,0}^m & \alpha_{42,1}^m & \alpha_{42,2}^m & \alpha_{42,3}^m\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&E_{m+1}I_{m+1}&0\\0&0&0&E_{m+1}I_{m+1}\end{pmatrix}.$$

In Fourier's images considering integral identity (20) problem (1)-(3) takes the form:

$$\left(\frac{d^2}{\partial t^2}+\lambda^2\right)\tilde{y}(t,\lambda)=\begin{pmatrix}f(t)\\0\end{pmatrix},\ t>0,$$

$\tilde{y}=0,\dfrac{\partial\tilde{y}}{\partial t}=0$ at $t=0.$

Its solution in Fourier's images has the form

$$\tilde{y}(t,\lambda)=\int_0^t\frac{\sin\lambda(t-\tau)}{\lambda}\begin{pmatrix}f(\tau)\\0\end{pmatrix}d\tau.$$

According to the theorem 1 we will find a formula for displacement $y_m(x,t)$

$$y_m(x,t)=-\frac{1}{2\pi i}\int_0^{\infty}u_m(x,\lambda)\int_0^t\frac{\sin\lambda(t-\tau)}{\lambda}\begin{pmatrix}f(\tau)\\0\end{pmatrix}d\tau\lambda d\lambda.$$

After change of the order of integration the formula takes the form:

$$y_m(x,t)=\int_0^tW_m(x,t-\tau)\begin{pmatrix}f(\tau)\\0\end{pmatrix}d\tau,$$

where

$$W_m(x,t-\tau) = -\frac{1}{2\pi i}\int_0^{\infty} u_m(x,\lambda)\frac{\sin\lambda(t-\tau)}{\lambda}\lambda d\lambda.$$

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