ψ-SECONDARY SUBMODULES OF A MODULE

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Abstract. Let $R$ be a commutative ring with identity and $M$ be an $R$-module. Let $\psi: S(M) \rightarrow S(M) \cup \{\emptyset\}$ be a function, where $S(M)$ denote the set of all submodules of $M$. The main purpose of this paper is to introduce and investigate the notion of $\psi$-secondary submodules of an $R$-module $M$ as a generalization of secondary submodules of $M$.

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1. Introduction

Throughout this paper, $R$ will denote a commutative ring with identity, $\mathbb{Z}$ and $\mathbb{N}$ will denote the ring of integers and the set of positive integers, respectively. We will denote the set of ideals of $R$ by $S(R)$ and the set of all submodules of $M$ by $S(M)$, where $M$ is an $R$-module.

Let $M$ be an $R$-module. A proper submodule $P$ of $M$ is said to be prime if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [5]. A non-zero $R$-module $M$ is said to be secondary if for each $a \in R$ the endomorphism of $M$ given by multiplication by $a$ is either surjective or nilpotent [8]. A non-zero submodule $N$ of $M$ is said to be second if for each $a \in R$, the endomorphism of $N$ given by multiplication by $a$ is either surjective or zero [9].

Anderson and Bataineh in [1] defined the notion of $\phi$-prime ideals as follows: let $\phi: S(R) \rightarrow S(R) \cup \{\emptyset\}$ be a function. Then, a proper ideal $P$ of $R$ is $\phi$-prime if for $r, s \in R$, $rs \in P \setminus \phi(P)$ implies that $r \in P$ or $s \in P$ [1]. A proper ideal $I$ of $R$ is said to be $\phi$-primary if for $a, b \in R$ with $ab \in I \setminus \phi(I)$, then either $a \in I$ or $b \in \sqrt{I}$ [1].

Zamani in [10] extended this concept to prime submodule. For a function $\phi: S(M) \rightarrow S(M) \cup \{\emptyset\}$, a proper submodule $N$ of $M$ is called $\phi$-prime if whenever $r \in R$ and $x \in M$ with $rx \in N \setminus \phi(N)$, then $r \in (N :_R M)$ or $x \in N$. Bataineh and Kuhail in [4] generalized the concept of $\phi$-prime submodules to $\phi$-primary
submodules. For a function \( \phi : S(M) \to S(M) \cup \{\emptyset\} \), a proper submodule \( N \) of \( M \) is called \( \phi \)-primary if whenever \( r \in R \) and \( x \in M \) with \( rx \in N \setminus \phi(N) \), then \( x \in N \) or \( r^n \in (N :_R M) \) for some \( n \in \mathbb{N} \).

Let \( \psi : S(M) \to S(M) \cup \{\emptyset\} \) be a function. Farshadifar and Ansari-Toroghy in [6], defined the notation of \( \psi \)-second submodules of \( M \) as a dual notion of \( \phi \)-prime submodules of \( M \). A non-zero submodule \( N \) of \( M \) is said to be a \( \psi \)-second submodule of \( M \) if \( r \in R \), \( K \) a submodule of \( M \), \( rN \subseteq K \), and \( r\psi(N) \nsubseteq K \), then \( N \subseteq K \) or \( rN = 0 \).

The main purpose of this paper is to introduce and study the concept of \( \psi \)-secondary submodules of \( M \) as a generalization of the notion of secondary submodules of \( M \). Also, the notion of \( \psi \)-secondary submodules of \( M \) can be regarded as a generalization of the notion of \( \psi \)-second submodules of \( M \). We say that a non-zero submodule \( N \) of \( M \) is a \( \psi \)-secondary submodule of \( M \) if \( r \in R \), \( K \) a submodule of \( M \), \( rN \subseteq K \), and \( r\psi(N) \nsubseteq K \), then \( N \subseteq K \) or \( r^n N = 0 \) for some \( n \in \mathbb{N} \). In fact the notion of \( \psi \)-secondary submodules is a dual notion of \( \phi \)-primary submodules. There are some works about \( \phi \)-primary submodules. It is natural to ask the following question: To what extent does the dual of these results hold for \( \psi \)-secondary submodules of an \( R \)-module? The aim of this paper is to provide some information in this case. Among the other results, we have shown that if \( N \) is a \( \psi \)-secondary submodule of \( M \) such that \( Ann_R(N) \psi(N) \nsubseteq N \), then \( N \) is a secondary submodule of \( M \) (see Theorem 2.5). Also, we have proved that if \( H \) is a submodule of \( M \) such that for all ideals \( I \) and \( J \) of \( R \), \( (H :_M I) \subseteq (H :_M J) \) implies that \( J \subseteq I \), then \( H \) is a secondary submodule of \( M \) if and only if \( H \) is a \( \psi_1 \)-secondary submodule of \( M \) (see Corollary 2.9). In Theorem 2.10, it is shown that for a submodule \( S \) of \( M \), we have

(a) If \( S \) is a \( \psi \)-secondary submodule of \( M \) such that \( Ann_R(\psi(S)) \subseteq \phi(Ann_R(S)) \), then \( Ann_R(S) \) is a \( \phi \)-primary ideal of \( R \).

(b) If \( \psi(S) = (0 :_M \phi(Ann_R(S))) \), \( M \) is a comultiplication \( R \)-module and \( Ann_R(S) \) is a \( \phi \)-primary ideal of \( R \), then \( S \) is a \( \psi \)-secondary submodule of \( M \).

The Example 2.11 shows that the condition “\( M \) is a comultiplication \( R \)-module” in Theorem 2.10 (b) can not be omitted. Also, it is shown that if \( a \) is an element of \( R \) such that \( (0 :_M a) \subseteq a(0 :_M aAnn_R((0 :_M a))) \) and \( (0 :_M a) \) is a \( \psi_1 \)-secondary submodule of \( M \), then \( (0 :_M a) \) is a secondary submodule of \( M \) (see Theorem 2.17).

Finally, in Theorem 2.18, we characterize \( \psi \)-secondary submodules of \( M \).
2. Main results

**Definition 2.1.** Let $M$ be an $R$-module. We say that a non-zero submodule $N$ of $M$ is a *weak secondary submodule of $M$* if $r \in R$, $K$ a submodule of $M$, $rN \subseteq K$, and $rM \not\subseteq K$, then $N \subseteq K$ or $r^n N = 0$ for some $n \in \mathbb{N}$.

Clearly, every secondary submodule of an $R$-module $M$ is a weak secondary submodule of $M$. But the converse is not true in general, as we see in the following example.

**Example 2.2.** Due to the fact that in logic if $P$ is false, then $P \Rightarrow Q$ is true, every $R$-module is a weak secondary submodule of itself but not every $R$-module is a secondary $R$-module. For example, the $\mathbb{Z}$-module $\mathbb{Z}$ is weak secondary which is not secondary.

**Definition 2.3.** Let $M$ be an $R$-module, $S(M)$ be the set of all submodules of $M$, and let $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. We say that a non-zero submodule $N$ of $M$ is a $\psi$-secondary submodule of $M$ if $r \in R$, $K$ a submodule of $M$, $rN \subseteq K$, and $r\psi(N) \not\subseteq K$, then $N \subseteq K$ or $r^n N = 0$ for some $n \in \mathbb{N}$.

In Definition 2.3, since $r\psi(N) \not\subseteq K$ implies that $r(\psi(N)+N) \not\subseteq K$, there is no loss of generality in assuming that $N \subseteq \psi(N)$ in the rest of this paper. Let $M$ be an $R$-module. We use the following functions $\psi : S(M) \to S(M) \cup \{\emptyset\}$.

\[
\psi_i(N) = (N :_M \text{Ann}^i_R(N)), \quad \forall N \in S(M), \quad \forall i \in \mathbb{N},
\]

\[
\psi_\sigma(N) = \sum_{i=1}^{\infty} \psi_i(N), \quad \forall N \in S(M).
\]

\[
\psi_M(N) = M, \quad \forall N \in S(M).
\]

Then it is clear that the set of all $\psi_M$-secondary submodules is exactly the set of all weakly secondary submodules. Clearly, for any submodule and every positive integer $n$, we have the following implications:

secondary $\Rightarrow \psi_{n-1}$ - secondary $\Rightarrow \psi_n$ - secondary $\Rightarrow \psi_\sigma$ - secondary.

For functions $\psi, \theta : S(M) \to S(M) \cup \{\emptyset\}$, we write $\psi \leq \theta$ if $\psi(N) \subseteq \theta(N)$ for each $N \in S(M)$. So whenever $\psi \leq \theta$, any $\psi$-secondary submodule is $\theta$-secondary.

**Theorem 2.4.** [3, 2.8]. For a submodule $S$ of an $R$-module $M$ the following statements are equivalent.

(a) $S$ is a secondary submodule of $M$. 


(b) $S \neq 0$ and $rS \subseteq K$, where $r \in R$ and $K$ is a submodule of $M$, implies either $r^nS = 0$ for some $n \in \mathbb{N}$ or $S \subseteq K$.

Theorem 2.5. Let $M$ be an $R$-module and $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Let $N$ be a $\psi$-secondary submodule of $M$ such that $\text{Ann}_R(N)\psi(N) \not\subseteq N$. Then $N$ is a secondary submodule of $M$.

Proof. Let $a \in R$ and $K$ be a submodule of $M$ such that $aN \subseteq K$. If $a\psi(N) \not\subseteq K$, then we are done because $N$ is a $\psi$-secondary submodule of $M$. Thus suppose that $a\psi(N) \subseteq K$. If $a\psi(N) \not\subseteq N$, then $a\psi(N) \not\subseteq N \cap K$. Hence $aN \subseteq N \cap K \subseteq K$ or $a^nN = 0$ for some $n \in \mathbb{N}$, as required. So let $a\psi(N) \subseteq N$. If $\text{Ann}_R(N)\psi(N) \not\subseteq K$, then $(a + \text{Ann}_R(N))\psi(N) \not\subseteq K$. Hence, there exists $x \in \text{Ann}_R(N)$ such that $(a + x)\psi(N) \not\subseteq K$. Thus $(a + x)N \subseteq K$ implies that $N \subseteq K$ or $a^nN = (a^n + x^n)N \subseteq (a + x)^nN = 0$ for some $n \in \mathbb{N}$, since $N$ is a $\psi$-secondary submodule of $M$. So suppose that $\text{Ann}_R(N)\psi(N) \subseteq K$. Since by assumption, $\text{Ann}_R(N)\psi(N) \not\subseteq N$, there exists $b \in \text{Ann}_R(N)$ such that $b\psi(N) \not\subseteq N$. Hence $b\psi(N) \not\subseteq N \cap K$. This in turn implies that $(a + b)\psi(N) \not\subseteq N \cap K$. Thus $(a + b)N \subseteq N \cap K$ implies that $N \subseteq N \cap K \subseteq K$ or $a^nN = (a^n + b^n)N \subseteq (a + b)^nN = 0$ for some $n \in \mathbb{N}$, as desired. 

Corollary 2.6. Let $N$ be a weak secondary submodule of an $R$-module $M$ such that $\text{Ann}_R(N)M \not\subseteq N$. Then $N$ is a secondary submodule of $M$.

Proof. In Theorem 2.5 set $\psi = \psi_M$. 

Corollary 2.7. Let $M$ be an $R$-module and $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. If $N$ is a $\psi$-secondary submodule of $M$ such that $(N :_M \text{Ann}_R^2(N)) \subseteq \psi(N)$, then $N$ is a $\psi_\sigma$-secondary submodule of $M$.

Proof. If $N$ is a secondary submodule of $M$, then the result is clear. So suppose that $N$ is not a secondary submodule of $M$. Then by Theorem 2.5, we have $\text{Ann}_R(N)\psi(N) \subseteq N$. Therefore, by assumption, 

$$(N :_M \text{Ann}_R^2(N)) \subseteq \psi(N) \subseteq (N :_M \text{Ann}_R(N)).$$

This implies that $\psi(N) = (N :_M \text{Ann}_R^2(N)) = (N :_M \text{Ann}_R(N))$ because always $(N :_M \text{Ann}_R(N)) \subseteq (N :_M \text{Ann}_R^2(N))$. Now 

$$(N :_M \text{Ann}_R^3(N)) = ((N :_M \text{Ann}_R^2(N)) :_M \text{Ann}_R(N)) = ((N :_M \text{Ann}_R(N)) :_M \text{Ann}_R(N)) = (N :_M \text{Ann}_R^2(N)) = \psi(N).$$

By continuing, we get that $\psi(N) = (N :_M \text{Ann}_R^i(N))$ for all $i \geq 1$. Therefore, $\psi(N) = \psi_\sigma(N)$ as needed. 

Theorem 2.8. Let $M$ be an $R$-module and $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Let $H$ be a submodule of $M$ such that for all ideals $I$ and $J$ of $R$, $(H :_M I) \subseteq (H :_M J)$ implies that $J \subseteq I$. If $H$ is not a secondary submodule of $M$, then $H$ is not a $\psi_1$-secondary submodule of $M$.

Proof. As $H$ is not a secondary submodule of $M$, there exists $r \in R$ and a submodule $K$ of $M$ such that $r^nH \neq 0$ for each $n \in \mathbb{N}$ and $H \nsubseteq K$, but $rH \subseteq K$ by Theorem 2.4. We have $H \nsubseteq K \cap H$ and $rH \subseteq K \cap H$. If $r(H :_M \text{Ann}_R(H)) \nsubseteq K \cap H$, then by our definition $H$ is not a $\psi_1$-secondary submodule of $M$. So let $r(H :_M \text{Ann}_R(H)) \subseteq K \cap H$. Then $r(H :_M \text{Ann}_R(H)) \subseteq K \cap H \subseteq H$. Thus $(H :_M \text{Ann}_R(H)) \subseteq (H :_M r)$ and so by assumption, $r \in \text{Ann}_R(H)$. This is a contradiction. \[\square\]

Corollary 2.9. Let $M$ be an $R$-module and $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Let $H$ be a submodule of $M$ such that for all ideals $I$ and $J$ of $R$, $(H :_M I) \subseteq (H :_M J)$ implies that $J \subseteq I$. Then $H$ is a secondary submodule of $M$ if and only if $H$ is a $\psi_1$-secondary submodule of $M$.

An $R$-module $M$ is said to be a comultiplication module if for every submodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $N = (0 :_M I)$ [2]. It is easy to see that $M$ is a comultiplication module if and only if $N = (0 :_M \text{Ann}_R(N))$ for each submodule $N$ of $M$.

Theorem 2.10. Let $M$ be an $R$-module, $\phi : S(R) \to S(R) \cup \{\emptyset\}$, and $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be functions.

(a) If $S$ is a $\psi$-secondary submodule of $M$ such that $\text{Ann}_R(\psi(S)) \subseteq \phi(\text{Ann}_R(S))$, then $\text{Ann}_R(S)$ is a $\phi$-primary ideal of $R$.

(b) If $\psi(S) = (0 :_M \phi(\text{Ann}_R(S))$, $M$ is a comultiplication $R$-module and $\text{Ann}_R(S)$ is a $\phi$-primary ideal of $R$, then $S$ is a $\psi$-secondary submodule of $M$.

Proof. (a) Let $ab \in \text{Ann}_R(S) \setminus \phi(\text{Ann}_R(S))$ for some $a, b \in R$. Then $ab\psi(S) \neq 0$ by assumption. If $a\psi(S) \subseteq (0 :_M b)$, then $ab\psi(S) = 0$, a contradiction. Thus $a\psi(S) \nsubseteq (0 :_M b)$. Therefore, $S \subseteq (0 :_M b)$ or $a^nS = 0$ for some $n \in \mathbb{N}$ because $S$ is a $\psi$-secondary submodule of $M$.

(b) Let $a \in R$ and $K$ be a submodule of $M$ such that $aS \subseteq K$ and $a\psi(S) \nsubseteq K$. As $aS \subseteq K$, we have $S \subseteq (K :_M a)$. It follows that

$$S \subseteq ((0 :_M \text{Ann}_R(K)) :_M a) = (0 :_M a\text{Ann}_R(K)).$$
This implies that $a\text{Ann}_R(K) \subseteq \text{Ann}_R((0 :_M a\text{Ann}_R(K))) \subseteq \text{Ann}_R(S)$. Hence, $a\text{Ann}_R(K) \subseteq \text{Ann}_R(S)$. If $a\text{Ann}_R(K) \subseteq \phi(\text{Ann}_R(S))$, then $\psi(S) = (0 :_M \phi(\text{Ann}_R(S))) \subseteq ((0 :_M \text{Ann}_R(K)) :_M a)$. As $M$ is a comultiplication $R$-module, we have $a\psi(S) \subseteq K$, a contradiction. Thus $a\text{Ann}_R(K) \not\subseteq \phi(\text{Ann}_R(S))$ and so as $\text{Ann}_R(S)$ is a $\phi$-primary ideal of $R$, we conclude that $a^nS = 0$ for some $n \in \mathbb{N}$ or

$$S = (0 :_M \text{Ann}_R(S)) \subseteq (0 :_M \text{Ann}_R(K)) = K,$$

as needed. \hfill \Box

The following example shows that the condition “$M$ is a comultiplication $R$-module” in Theorem 2.10 (b) can not be omitted.

**Example 2.11.** Let $R = \mathbb{Z}$, $M = \mathbb{Z} \oplus \mathbb{Z}$, and $S = 2\mathbb{Z} \oplus 2\mathbb{Z}$. Clearly, $M$ is not a comultiplication $R$-module. Suppose that $\phi : S(R) \to S(R) \cup \{\emptyset\}$ and $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be functions such that $\phi(I) = I$ for each ideal $I$ of $R$ and $\psi(S) = S$. Then clearly, $\text{Ann}_R(S) = 0$ is a $\phi$-primary ideal of $R$ and $\psi(S) = M = (0 :_M \phi(\text{Ann}_R(S)))$. But as $3S \subseteq 6\mathbb{Z} \oplus 6\mathbb{Z}$, $S \not\subseteq 6\mathbb{Z} \oplus 6\mathbb{Z}$, and $3^nS \neq 0$ for each $n \in \mathbb{N}$, we have that $S$ is not a $\psi$-secondary submodule of $M$.

The following lemma is known, but we write it here for the sake of reference.

**Lemma 2.12.** Let $M$ be an $R$-module, $S$ a multiplicatively closed subset of $R$, and $N$ be a finitely generated submodule of $M$. If $S^{-1}N \subseteq S^{-1}K$ for a submodule $K$ of $M$, then there exists an $s \in S$ such that $sN \subseteq K$.

**Proof.** This is straightforward. \hfill \Box

**Proposition 2.13.** Let $M$ be an $R$-module, $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function, and $N$ be a $\psi$-secondary submodule of $M$. Then we have the following statements.

(a) If $K$ is a submodule of $M$ with $K \subseteq N$ and $\psi_K : S(M/K) \to S(M/K) \cup \{\emptyset\}$ is a function such that $\psi_K(N/K) = \psi(N)/K$, then $N/K$ is a $\psi_K$-secondary submodule of $M/K$.

(b) If $N$ is a finitely generated submodule of $M$, $S$ is a multiplicatively closed subset of $R$ with $\text{Ann}_R(N) \cap S = \emptyset$, and $S^{-1}\psi : S(S^{-1}M) \to S(S^{-1}M) \cup \{\emptyset\}$ is a function such that $(S^{-1}\psi)(S^{-1}N) = S^{-1}\psi(N)$, then $S^{-1}N$ is a $S^{-1}\psi$-secondary submodule of $S^{-1}M$.

**Proof.** (a) This is straightforward.

(b) As $N$ is a $\psi$-secondary submodule of $M$, we have $N \neq 0$. This implies that $S^{-1}N \neq 0$ since $N$ is finitely generated and $\text{Ann}_R(N) \cap S = \emptyset$ by using Lemma 2.12.
Let $a/s \in S^{-1}R$ and $S^{-1}K$ be a submodule of $S^{-1}M$ such that $(a/s)S^{-1}N \subseteq S^{-1}K$ and $(a/s)S^{-1}(\psi(S^{-1}N)) \not\subseteq S^{-1}K$. It follows that $(a/s)S^{-1}(\psi(N)) \not\subseteq S^{-1}K$. Now the result follows from the fact that $N$ is a $\psi$-secondary submodule of $M$ and Lemma 2.12.

\begin{proof}
\end{proof}

\textbf{Proposition 2.14.} Let $M$ and $\hat{M}$ be $R$-modules and $f : M \to \hat{M}$ be an $R$-monomorphism. Let $\psi : S(M) \to S(M) \cup \{\emptyset\}$ and $\hat{\psi} : S(\hat{M}) \to S(\hat{M}) \cup \{\emptyset\}$ be functions such that $\psi(f^{-1}(\hat{N})) = f^{-1}(\hat{\psi}(\hat{N}))$, for each submodule $\hat{N}$ of $\hat{M}$. If $\hat{N}$ is a $\hat{\psi}$-secondary submodule of $\hat{M}$ such that $\hat{N} \subseteq \text{Im}(f)$, then $f^{-1}(\hat{N})$ is a $\psi$-secondary submodule of $M$.

\textbf{Proof.} As $\hat{N} \neq \emptyset$ and $\hat{N} \subseteq \text{Im}(f)$, we have $f^{-1}(\hat{N}) \neq \emptyset$. Let $a \in R$ and $K$ be a submodule of $M$ such that $af^{-1}(\hat{N}) \subseteq K$ and $a\hat{\psi}(f^{-1}(\hat{N})) \not\subseteq K$. Then by using assumptions, $a\hat{N} \subseteq f(K)$ and $a\hat{\psi}(\hat{N}) \not\subseteq f(K)$. Thus $a^n\hat{N} = 0$ for some $n \in \mathbb{N}$ or $\hat{N} \subseteq f(K)$ since $\hat{N}$ is a $\hat{\psi}$-secondary submodule of $\hat{M}$. This implies that $a^n f^{-1}(\hat{N}) = 0$ or $f^{-1}(\hat{N}) \subseteq K$, as needed.

A proper submodule $N$ of an $R$-module $M$ is said to be \textit{completely irreducible} if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of submodules of $M$, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of $M$ is an intersection of completely irreducible submodules of $M$ [7].

\textbf{Remark 2.15.} Let $N$ and $K$ be two submodules of an $R$-module $M$. To prove $N \subseteq K$, it is enough to show that if $L$ is a completely irreducible submodule of $M$ such that $K \subseteq L$, then $N \subseteq L$.

\textbf{Proposition 2.16.} Let $M$ be an $R$-module, $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function, and let $N$ be a $\psi_1$-secondary submodule of $M$. Then we have the following statements.

\begin{enumerate}
\item[(a)] If $a \in R$, $aN \neq N$, then $(N :_M \sqrt{\text{Ann}_R(N)}) \subseteq (N :_M a)$.
\item[(b)] If $J$ is an ideal of $R$ such that $\sqrt{\text{Ann}_R(N)} \subseteq J$ and $JN \neq N$, then $(N :_M \sqrt{\text{Ann}_R(N)}) = (N :_M J)$.
\end{enumerate}

\textbf{Proof.} (a) Let $a \in R$ such that $aN \neq N$. If $a^nN = 0$ for some $n \in \mathbb{N}$, then clearly $(N :_M \sqrt{\text{Ann}_R(N)}) \subseteq (N :_M a)$. So let $a^nN \neq 0$ for each $n \in \mathbb{N}$. Now let $\hat{L}$ be a completely irreducible submodule of $M$ such that $N \subseteq \hat{L}$. Then $N \not\subseteq \hat{L} \cap aN$ and $aN \subseteq \hat{L} \cap aN$. Hence as $N$ is a $\psi_1$-secondary submodule of $M$, we have $a(N :_M \text{Ann}_R(N)) \subseteq \hat{L} \cap aN \subseteq \hat{L}$. Therefore, $a(N :_M \text{Ann}_R(N)) \subseteq N$ by Remark 2.15. Hence, $a(N :_M \sqrt{\text{Ann}_R(N)}) \subseteq a(N :_M \text{Ann}_R(N)) \subseteq N$. Thus $(N :_M \sqrt{\text{Ann}_R(N)}) \subseteq (N :_M a)$. 

(b) As $JN \neq N$, we have $aN \neq N$ for each $a \in J$. Thus by part (a), for each $a \in J$, $(N :_{M} \sqrt{Ann_{R}(N)}) \subseteq (N :_{M} a)$. This implies that

$$(N :_{M} J) = \cap_{a \in J}(N :_{M} a) \supseteq (N :_{M} \sqrt{Ann_{R}(N)}).$$

The inverse inclusion follows from the fact that $\sqrt{Ann_{R}(N)} \subseteq J$. \hfill \qed

**Theorem 2.17.** Let $M$ be an $R$-module, $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function, and let $a$ be an element of $R$ such that $(0 :_{M} a) \subseteq a(0 :_{M} Ann_{R}((0 :_{M} a)))$. If $(0 :_{M} a)$ is a $\psi$-secondary submodule of $M$, then $(0 :_{M} a)$ is a secondary submodule of $M$.

**Proof.** Let $N := (0 :_{M} a)$ be a $\psi$-secondary submodule of $M$. Then $(0 :_{M} a) \neq 0$. Now let $t \in R$ and $K$ be a submodule of $M$ such that $t(0 :_{M} a) \subseteq K$. If $t(N :_{M} Ann_{R}(N)) \subseteq K$, then $t^{n}(0 :_{M} a) = 0$ for some $n \in \mathbb{N}$ or $(0 :_{M} a) \subseteq K$ since $(0 :_{M} a)$ is a $\psi$-secondary submodule of $M$. So suppose that $t(N :_{M} Ann_{R}(N)) \subseteq K$. Now we have $(t + a)(0 :_{M} a) \subseteq K$. If $(t + a)(N :_{M} Ann_{R}(N)) \subseteq K$, then as $(0 :_{M} a)$ is a $\psi$-secondary submodule of $M$,

$$t^{n}(0 :_{M} a) = (t^{n} + a^{n})(0 :_{M} a) \subseteq (t + a)^{n}(0 :_{M} a) = 0$$

for some $n \in \mathbb{N}$ or $(0 :_{M} a) \subseteq K$, and we are done. So assume that $(t + a)(N :_{M} Ann_{R}(N)) \subseteq K$. Then $t(N :_{M} Ann_{R}(N)) \subseteq K$ gives that $a(N :_{M} Ann_{R}(N)) \subseteq K$. Hence by assumption, $(0 :_{M} a) \subseteq K$ and the result follows from Theorem 2.4. \hfill \qed

**Theorem 2.18.** Let $N$ be a non-zero submodule of an $R$-module $M$ and let $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Then the following are equivalent:

(a) $N$ is a $\psi$-secondary submodule of $M$;

(b) for a submodule $K$ of $M$ with $N \nsubseteq K$, we have

$$\sqrt{(K :_{R} N)} = \sqrt{Ann_{R}(N)} \cup \sqrt{(K :_{R} \psi(N))};$$

(c) for a submodule $K$ of $M$ with $N \nsubseteq K$, we have $\sqrt{(K :_{R} N)} = \sqrt{Ann_{R}(N)}$

\text{or} $\sqrt{(K :_{R} N)} = \sqrt{(K :_{R} \psi(N))};$

(d) for any ideal $I$ of $R$ and any submodule $K$ of $M$, if $IN \subseteq K$ and $I \nsubseteq \sqrt{(K :_{R} \psi(N))}$, then $IN = 0$ or $N \nsubseteq K$;

(e) for each $a \in R$ with $a\psi(N) \nsubseteq aN$, we have $aN = N$ or $a^{n}N = 0$ for some $n \in \mathbb{N}$.

**Proof.** (a) $\Rightarrow$ (b) Let for a submodule $K$ of $M$ with $N \nsubseteq K$, we have $a \in \sqrt{(K :_{R} N)} \setminus \sqrt{(K :_{R} \psi(N))}$. Then $a^{n}N \subseteq K$ for some $n \in \mathbb{N}$ and $a^{n}\psi(N) \nsubseteq K$. Since $N$ is a $\psi$-secondary submodule of $M$, we have $a \in \sqrt{Ann_{R}(N)}$. As we may assume that $N \subseteq \psi(N)$, the other inclusion always holds.
(b) ⇒ (c) This follows from the fact that if a subgroup is a union of two subgroups, it is equal to one of them.

(c) ⇒ (d) Let \( I \) be an ideal of \( R \) and \( K \) be a submodule of \( M \) such that \( IN \subseteq K \) and \( I \nsubseteq \sqrt{(K :_R \psi(N))}. \) Suppose \( I \nsubseteq \sqrt{Ann_R(N)} \) and \( N \nsubseteq K. \) We show that \( I \subseteq \sqrt{(K :_R \psi(N))}. \) Let \( a \in I \) and first let \( a \nsubseteq \sqrt{Ann_R(N)}. \) Then, since \( aN \subseteq K, \) we have \( \sqrt{(K :_R N)} \neq \sqrt{Ann_R(N)}. \) Hence by our assumption \( \sqrt{(K :_R N)} = \sqrt{(K :_R \psi(N))}. \) So \( a \in \sqrt{(K :_R \psi(N))}. \) Now assume that \( a \in I \cap \sqrt{Ann_R(N)}. \) Let \( u \in I \setminus \sqrt{Ann_R(N)}. \) Then \( a + u \in I \setminus \sqrt{Ann_R(N)}. \) So by the first case, we have \( u \in \sqrt{(K :_R \psi(N))} \) and \( u + a \in \sqrt{(K :_R \psi(N))}. \) This gives that \( a \in \sqrt{(K :_R \psi(N))}. \) Thus in any case \( a \in \sqrt{(K :_R \psi(N))}. \) Therefore, \( I \subseteq \sqrt{(K :_R \psi(N))}, \) as desired.

(d) ⇒ (a) This is clear.

(a) ⇒ (c) Let \( a \in R \) such that \( a\psi(N) \nsubseteq aN. \) Then \( aN \subseteq aN \) implies that \( N \subseteq aN \) or \( a^nN = 0 \) for some \( n \in \mathbb{N} \) by part (a). Thus \( N = aN \) or \( a^nN = 0 \) for some \( n \in \mathbb{N}, \) as requested.

Proposition 2.21. Let \( M \) be an \( R \)-module and \( S : S(M) \to S(M) \cup \{\emptyset\} \) be a function. If \( \psi(N) = N, \) then \( N \) is a \( \psi \)-secondary submodule of \( M \) by Theorem 2.18 (e) ⇒ (a).

Let \( N \) be a non-zero submodule of an \( R \)-module \( M \) and let \( \psi : S(M) \to S(M) \cup \{\emptyset\} \) be a function. If \( \psi(N) = N, \) then \( N \) is a \( \psi \)-secondary submodule of \( M \) by Theorem 2.18 (e) ⇒ (a).

Example 2.19. Let \( N \) be a non-zero submodule of an \( R \)-module \( M \) and let \( \psi : S(M) \to S(M) \cup \{\emptyset\} \) be a function. If \( \psi(N) = N, \) then \( N \) is a \( \psi \)-secondary submodule of \( M \) by Theorem 2.18 (e) ⇒ (a).

Example 2.20. (a) Let \( p, q \) be two prime numbers, \( N = \langle 1/p + \mathbb{Z} \rangle, \) and \( K = \langle 1/q + \mathbb{Z} \rangle. \) Then clearly, \( N \oplus 0 = 0 \oplus K \) are weak secondary submodules of the \( \mathbb{Z} \)-module \( \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{q^\infty} \) but as \( p(N + K) \nsubseteq K, p(\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{q^\infty}) \nsubseteq K, N + K \nsubseteq K, \) and \( p^n(N + K) \neq 0 \) for each \( n \in \mathbb{N} \) we have that \( N + K \) is not a weak secondary submodule of the \( \mathbb{Z} \)-module \( \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{q^\infty}. \)

(b) Clearly, the submodules \( 2\mathbb{Z}_6 \) and \( 3\mathbb{Z}_6 \) are \( \psi \)-secondary submodules of \( 
\psi : S(\mathbb{Z}_6) \to S(\mathbb{Z}_6) \cup \{\emptyset\} \) is a function. But \( 2\mathbb{Z}_6 \cap 3\mathbb{Z}_6 = 0 \) is not a \( \psi \)-secondary submodule of \( \mathbb{Z}_6. \)
Proof. Let $a \in R$ with $aM \nsubseteq a(N_1 \cap N_2)$. If $aM \subseteq aN_1$ and $aM \subseteq aN_2$, then $aM \subseteq a(N_1 \cap N_2)$, a contradiction. If $aM \nsubseteq aN_1$ and $aM \nsubseteq aN_2$, then by Theorem 2.18 (a) $aN_1 = N_1$ or $a^mN_1 = 0$ for some $n \in \mathbb{N}$ and $aN_2 = N_2$ or $a^mN_2 = 0$ for some $m \in \mathbb{N}$. If $a^mN_2 = 0$ or $a^mN_1 = 0$, then $a^t(N_1 \cap N_2) = 0$ for some $t \in \mathbb{N}$ and we are done. So suppose that $aN_1 = N_1$ and $aN_2 = N_2$. Then $a(N_1 \cap N_2) = N_1 \cap N_2$. Finally if $aM \nsubseteq aN_1$, $aM \subseteq aN_2$, and $aN_1 = N_1$, then $aN_1 \nsubseteq aM \subseteq aN_2$. Hence, $N_1 \cap N_2 \subseteq N_1 = aN_1 \cap aN_2 = a(N_1 \cap N_2)$. It follows that $a(N_1 \cap N_2) = N_1 \cap N_2$, as needed.

Let $R_1$ and $R_2$ be two commutative rings with identity. Let $M_1$ and $M_2$ be $R_1$ and $R_2$-module, respectively and put $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an $R$-module and each submodule of $M$ is of the form $N = N_1 \times N_2$ for some submodules $N_1$ of $M_1$ and $N_2$ of $M_2$. Suppose that $\psi^i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ be a function for $i = 1, 2$. One can see that the $R_1 \times R_2$-module $S_1 \times 0$ and $0 \times S_2$, where $S_1$ is a secondary submodule of $M_1$ and $S_2$ is a secondary submodule of $M_2$, are secondary submodules of $M$. The following example, shows that this is not true for correspondence $\psi^1 \times \psi^2$-secondary submodules in general.

Example 2.22. Let $R_1 = R_2 = M_1 = M_2 = S_1 = \mathbb{Z}_6$. Then clearly, $S_1$ is a weak secondary submodule of $M_1$. However, $(2,1)(\mathbb{Z}_6 \times 0) \nsubseteq 2\mathbb{Z}_6 \times 3\mathbb{Z}_6$ and $(2,1)(\mathbb{Z}_6 \times \mathbb{Z}_6) \nsubseteq 2\mathbb{Z}_6 \times 3\mathbb{Z}_6$. But $(2,1)^n(\mathbb{Z}_6 \times 0) = 2\mathbb{Z}_6 \times 0 \neq 0 \times 0$ for each $n \in \mathbb{N}$, and $\mathbb{Z}_6 \times 0 \nsubseteq 2\mathbb{Z}_6 \times 3\mathbb{Z}_6$. Therefore, $S_1 \times 0$ is not a weak secondary submodule of $M_1 \times M_2$.

Theorem 2.23. Let $R = R_1 \times R_2$ be a ring and $M = M_1 \times M_2$ be an $R$-module, where $M_1$ is an $R_1$-module and $M_2$ is an $R_2$-module. Suppose that $\psi^i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ be a function for $i = 1, 2$. Then $S_1 \times 0$ is a $\psi^1 \times \psi^2$-secondary submodule of $M$, where $S_1$ is a $\psi^1$-secondary submodule of $M_1$ and $\psi^2(0) = 0$.

Proof. Let $(r_1, r_2) \in R$ and $K_1 \times K_2$ be a submodule of $M$ such that $(r_1, r_2)(S_1 \times 0) \subseteq K_1 \times K_2$ and
\[
(r_1, r_2)((\psi^1 \times \psi^2)(S_1 \times 0)) = r_1\psi^1(S_1) \times r_2\psi^2(0) = r_1\psi^1(S_1) \times 0 \nsubseteq K_1 \times K_2.
\]
Then $r_1S_1 \subseteq K_1$ and $r_1\psi^1(S_1) \nsubseteq K_1$. Hence, $(r_1)^nS_1 = 0$ for some $n \in \mathbb{N}$ or $S_1 \subseteq K_1$ since $S_1$ is a $\psi^1$-secondary submodule of $M_1$. Therefore, $(r_1, r_2)^n(S_1 \times 0) = 0 \times 0$ or $S_1 \times 0 \subseteq K_1 \times K_2$, as requested.

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