ON S-EXTENDING MODULES

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ABSTRACT. Here we introduce and study the concept of relative superfluous injectivity, which is a generalization of relative injectivity. We show some of the properties that hold true for relative injectivity still hold for relative superfluous injectivity. We also introduce and characterize the new concept of superfluous extending modules. Finally, we make use of relative superfluous injectivity to study direct sums of superfluous extending modules.

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1. Introduction

Throughout all modules are unital right R-modules over an arbitrary associative ring R with unity. The generalization of relative injectivity for module was studied by many authors, and this has been by adding extra condition on the homomorphism in the defining diagram (cf. E. Mermut, C. Santa-Clara and P. F. Smith [6], H. Q. Dinh [3] and M. F. Yousif and Y. Zhou [8]). Here we introduce the concept of N being M-superfluous-injective, which is a generalization of N being M-injective for modules (Definition 2.3). We define in Definition 2.10, Definition 2.11 what is meant by Con-S-complement and Con-S-closed submodule of a module. In Theorem 2.16 we give an equivalent condition of the diagram in Definition 2.3 by a property on a decomposition form, in fact we prove that M_2 is M_1-superfluous-injective if and only if $M = C \oplus M_2$ holds for every Con-S-complement C of M_2 in M, whenever $M = M_1 \oplus M_2$; M_1 has the condition (S_*). This is analogue with the known fact about relative injectivity [2]. In Section 3, we introduce a generalization of extending modules. This has been done by relaxing the defining condition of such modules. In fact we concentrate on Con-S-closed submodules, and examine for being direct summands. We show in the case of S-extending modules in contrary of extending modules, that the class of summands does not coincide with the class of Con-S-closed submodules. In Remark 3.3, we mention when such classes coincide.

In [4], it was shown that a direct sum of extending modules M_1 and M_2 is extending if and only if every closed submodule with zero intersection with M_1 or with M_2 is a direct summand (Lemma 7.9). Here in Section 4, we prove that a direct sum of S-extending modules M_1 and M_2, with $0 \neq Rad(C) \trianglelefteq C$, for every proper Con-S-closed submodule C of M, with $C \cap M_i \neq 0$, $i = 1, 2$ is S-extending if and only if every Con-S-closed submodule with zero intersection with M_1 or with M_2 is a direct summand and also we prove that a direct sum of S-extending modules M_1 and M_2, with SIC-property, is S-extending if and only if every Con-S-closed submodule with zero intersection with M_1 or with M_2 is a direct summand. Finally in Theorem 4.9 and Theorem 4.10, we make use relatively superfluous injectivity to prove that a finite direct sum of S-extending modules is S-extending. We use $L \leq M$, $N \ll M$ and $K \trianglelefteq M$ to denote that L is a submodule, N is a superfluous submodule, and K is a large submodule of M, respectively, for more details see [1].

2. Relative superfluous injective modules

In this section we study and analyze the properties of the concept of superfluous injectivity.

Definition 2.1. Let N be a submodule of a module M. N is called a Con-S-submodule in M relative to L (denoted by Con-S_L-submodule in M) if there exists a non-zero superfluous submodule K of L, with $K \leq N$. If $L = M$, then we just say that N is a Con-S-submodule of M.

Definition 2.2. A monomorphism $f : N \to M$ is called a Con-S-monomorphism if $f(N)$ is a Con-S-submodule of M.

Definition 2.3. Let M be an R-module. A module N is said to be M-superfluous-injective if for every Con-S-monomorphism $f : L \to M$, any homomorphism $\varphi : L \to N$, there exists a homomorphism $\psi : M \to N$ such that $\psi f = \varphi$.

Lemma 2.4. *The following are equivalent for modules N and M:*

(1) *N is M-superfluous-injective,*
(2) *For every Con-S-submodule L of M, any homomorphism $\varphi : L \to N$ can be extended to a homomorphism $\psi : M \to N$.*

Proof. $(2) \Rightarrow (1)$: Let $f : L \to M$ be a Con-S-monomorphism, and $\varphi : L \to N$ be a homomorphism. It follows that $f(L)$ is a Con-S-submodule of M. Hence, by (2), there is a homomorphism $\psi : M \to N$ such that ψ extends $\varphi f^{-1} \mid_{f(L)}$. Therefore $\psi f = \varphi$; and hence we have (1).

$(1) \Rightarrow (2)$: It is clear. $\qquad \square$

Remark 2.5. (a) Let M be an R-module, with no non-trivial superfluous submodules. Then every R-module N is M-superfluous-injective module. For examples:

(1) Every $\mathbb{Z}$-module N is a $\mathbb{Z}$-superfluous-injective module.

(2) If M is a semisimple R-module, then every R-module N is M-superfluous-injective module.

(3) If M is a semiprimitive R-module, then every R-module N is M-superfluous-injective module.

(b) Let M be an R-module, with $0 \neq Rad(M) \trianglelefteq M$. Then M-superfluous-injectivity and M-injectivity are equivalent.

The examples, which are given above, show that there are many M-superfluous-injective, which are not M-injective modules.

Proposition 2.6. *Let N and M be R-modules. Then the following are equivalent:*

(1) *N is M-superfluous-injective.*

(2) *N is M_1-superfluous-injective, for each submodule M_1 of M.*

(3) *N is M/L-superfluous-injective, for each superfluous submodule L of M.*

Proof. (1) $\Rightarrow$ (2): Let M_1 be a submodule of M, and $\varphi : L \to N$ a homomorphism, for some Con-S-submodule L of M_1. It is easy to show that L is Con-S-submodule of M, and hence, by (1), $\varphi : L \to N$ can be extended to a homomorphism $\psi : M \to N$. Therefore ψ restricted to M_1 extends φ.

(2) $\Rightarrow$ (3): Let L be a superfluous submodule of M. We show that N is M/L-superfluous-injective. To this end, let $\varphi : M_1/L \to N$ be a homomorphism, for some Con-S-submodule M_1/L of M/L. Hence M_1/L contains a nonzero superfluous submodule K/L in M/L. It is clear that K is superfluous in M; and hence M_1 is a Con-S-submodule of M. Let $\pi : M \to M/L$ be the natural epimorphism. As N is M-superfluous-injective, we have that $\varphi\pi \mid_{M_1}$ can be extended to a homomorphism $\psi : M \to N$; i.e. $\psi(m_1) = \varphi(m_1 + L)$, for all $m_1 \in M_1$. Since $\psi(l) = 0$, for all $l \in L$, we have that $\psi : M \to N$ defines a homomorphism $\alpha : M/L \to N$ by $\alpha(m + L) = \psi(m)$, for all $m + L \in M/L$. It is easy to show that $\alpha : M/L \to N$ extends $\varphi : M_1/L \to N$.

(3) $\Rightarrow$ (1): It is clear by taking $L = 0$. $\square$

Proposition 2.7. *Let N and M be R-modules. Then the following are equivalent:*

(1) *N is M-superfluous-injective.*

(2) *L is M-superfluous-injective, for each direct summand L of N.*

(3) *N' is M-superfluous-injective, for each isomorphic copy N' of N.*

(4) *N is M'-superfluous-injective, for each isomorphic copy M' of M.*

Proof. (1) $\Rightarrow$ (2): Let $N = L \oplus L^{'}$, and $\varphi : M_1 \to L$ be a homomorphism, for some *Con-S*-submodule M_1 of M. As N is M-superfluous-injective module, there is an extension homomorphism $\psi : M \to N$ of φ. It follows that $\pi_L \psi$ extends φ, where $\pi_L : N \to L$ is the projection of N onto L.

(2) $\Rightarrow$ (1): It is clear by taking $L = N$.

(1) $\Leftrightarrow$ (3): The proof is clear.

(1) $\Rightarrow$ (4): Let L be a *Con-S*-submodule of $M^{'}$, then there exists a superfluous submodule K of $M^{'}$ in L and let $\varphi : L \to N$ be a homomorphism and $f : M^{'} \to M$ ba an isomorphism. Thus $f(K) \ll M$ and $f(K) \leq f(L)$. Then $f(L)$ is a *Con-S*-submodule of M. Since N is M-superfluous-injective, then there exists $g : M \to N$ such that $g(x) = \varphi f^{-1}(x)$, for all $x \in f(L)$. Therefore $gf(l) = \varphi f^{-1}(f(l)) = \varphi(l)$ for all $l \in L$. Thus gf extends φ.

By the similar way we can prove that (4) $\Rightarrow$ (1). $\qquad \square$

Proposition 2.8. *Let M be an R-module and $\{N_i : i \in I\}$ be a family of R-modules. Then $\Pi_{i \epsilon I} N_i$ is M-superfluous-injective if and only if N_i is M-superfluous-injective, for each $i \epsilon I$.*

Proof. It is clear. $\qquad \square$

Corollary 2.9. *Let M be an R-module and $\{N_i : i = 1, \ldots, n\}$ be a family of R-modules. Then $\oplus_{i=1}^{n} N_i$ is M-superfluous-injective if N_i is M-superfluous-injective, for each $i = 1, \ldots, n$.*

Definition 2.10. A submodule L of a module M is called a *Con-S*-complement of a submodule K in M, if it is a *Con-S*-submodule of M, and maximal with respect to $L \cap K = 0$.

Definition 2.11. A submodule L of a module M is called a *Con-S*-closed submodule of M, if it is a *Con-S*-submodule of M, and has no proper large extension submodules of M.

Remark 2.12. Let M be a module, we have the following:

(1) Every maximal large extension of a *Con-S*-submodule of a module M is *Con-S*-closed in M.

(2) Let N be a *Con-S*-submodule, and K be a submodule of a module M; with $N \cap K = 0$. Then N is a *Con-S*-complement of K if and only if N is *Con-S*-closed, and $N \oplus K$ is large in M.

Definition 2.13. A module M is said to have the condition (S_*), if for every superfluous submodule K of M and $K \leq N \trianglelefteq M$, we have that $K \ll N$.

Example 2.14. Any semiprimitive module and any semisimple module has (S_*).

Lemma 2.15. *Let $M = M_1 \oplus M_2$ be a module where M_1 has (S_*), N be a Con-S-submodule of M_1, and $f \colon N \to M_2$ be a homomorphism. Let $N \leq X \leq M_1$, and $\varphi \colon X \to M_2$ be a maximal extension of f. Then the submodule $X^* = \{x + \varphi(x) : x \in X\}$ of M is a Con-S-complement of M_2 in M.*

Proof. It is clear that $X^* \cap M_2 = 0$. Let $\pi_i : M \to M_i$, $i = 1, 2$, be the natural projections. Since N is a Con-S-submodule of M_1, then there exists $0 \neq K \leq N$, with $K \ll M_1$. Hence $K^* = \{k + \varphi(k) : k \in K\} \leq \pi_1(K^*) + \pi_2(K^*) = K + \varphi(K)$. Since $K \ll M_1$, $K \leq N \leq X \trianglelefteq M_1$ and M_1 has (S_*), then $K \ll X$ and since homomorphic images of superfluous submodules are superfluous, then $\varphi(K) \ll M_2$ and so, $\varphi(K) \ll M$. As finite sums of superfluous submodules are superfluous, thus $K + \varphi(K) \ll M$, hence $K^* \ll M$ and $K^* \leq X^*$; i.e. X^* is a Con-S-submodule of M. Now let $X^* \leq D$, with $D \cap M_2 = 0$. Clearly $\pi_1|_D$ is monomorphism and $X = \pi_1(X^*) \leq \pi_1(D)$, define $\psi : \pi_1(D) \to M_2$ by $\psi(\pi_1(d)) = \pi_2(d)$ for all $d \in D$. Then ψ extends $\varphi : X \to N$. Thus $X = \pi_1(D)$ by maximality of φ, and hence $X^* = D$. Which shows that X^* is a Con-S-complement of M_2. $\qquad\square$

Theorem 2.16. *Let $M = M_1 \oplus M_2$ where M_1 has (S_*). Then the following are equivalent:*

(1) *M_2 is M_1-superfluous-injective.*
(2) *If N is a Con-S-submodule of M, with $N \cap M_2 = 0$, then $M = C \oplus M_2$, for some submodule C containing N.*
(3) *$M = C \oplus M_2$, for each Con-S-complement C of M_2 in M.*

Proof. (1) $\Rightarrow$ (2): Let N be a Con-S-submodule of M with $N \cap M_2 = 0$. As N is a Con-S-submodule of M; i.e. N contains a non-zero submodule K superfluous in M, it follows that $\pi_1(K)$ is superfluous in M_1, where $\pi_1 : M \to M_1$ is the projection onto M_1. Hence by (1), there exists a homomorphism $\varphi : M_1 \to M_2$ such that $\varphi\pi_1 \mid_N = \pi_2 \mid_N$, where $\pi_2 : M \to M_2$ is the projection onto M_2. Hence, for all $n \in N$, we have $n = \pi_1(n) + \pi_2(n) = \pi_1(n) + \varphi\pi_1(n) \in M_1^* = \{m_1 + \varphi(m_1) : m_1 \in M_1\}$, where $M = M_1^* \oplus M_2$.
(2) $\Rightarrow$ (3): Let C be a Con-S-complement of M_2 in M. Then $C \cap M_2 = 0$, so by (2) there exists $L \geq C$ such that $M = L \oplus M_2$. Hence $C = L$ (due to $L \cap M_2 = 0$).
(3) $\Rightarrow$ (1): Let N be a Con-S-submodule of M_1, and $f : N \to M_2$ be a homomorphism. Let $\varphi : N \leq X \to M_2$ be a maximal extension of f. By Lemma 2.15, we have that $X^* = \{x + \varphi(x) : x \in X\}$ is a Con-S-complement of M_2. Hence,

by (3), $M = X^* \oplus M_2$. It is easy to show that $X = M_1$, and therefore M_2 is M_1-superfluous-injective. $\qquad\square$

3. Superfluous extending modules

In this section we study simultaneously the dual concepts of superfluousness and largeness on modules, namely the condition that each Con-S-submodule of a module M is large in a direct summand of M. Such modules will be called S-extending modules.

Definition 3.1. A module M is called a superfluous extending (for short S-extending) module, if every Con-S-submodule of M is large in a direct summand of M.

Theorem 3.2. *The following are equivalent for a module M:*

(1) *M is S-extending.*
(2) *Every S-closed submodule of M is a direct summand.*
(3) *For every Con-S-submodule A of M, there exists a decomposition*
    $M = M_1 \oplus M_2$ such that M_1 contains A, and $A \oplus M_2$ is large in M.
(4) *If A is a closed submodule of M and B a direct summand of M; with $A \cap B$*
    Con-S-closed submodule in B, then $A \cap B$ is a direct summand of M.

Proof. (1)$\Rightarrow$ (2): Follows from definition of Con-S-closed submodule.

(2)$\Rightarrow$ (3): Let A be a Con-S-submodule of M, it follows from Remark 2.12 that a maximal large extension C of A in M is Con-S-closed. Hence, by (2), C is a direct summand of M; and therefore $M = C \oplus M_2$, with $A \oplus M_2$ is large in M, for some submodule M_2 of M.

(3)$\Rightarrow$ (4): Let A be a closed submodule of M, and B be a direct summand of M; with $A \cap B$ Con-S-closed submodule of B, then $A \cap B$ is a Con-S-closed submodule of M. To avoid triviality, we may consider $A, B \neq M$. Now, by (3), there is a decomposition $M = M_1 \oplus M_2$ such that M_1 contains $A \cap B$, and $A \cap B \oplus M_2$ is large in M. As $A \cap B$ is a closed submodule in M, we have that $A \cap B = M_1$.

(4)$\Rightarrow$ (1): Let A be a Con-S-submodule of M. Hence, from Remark 2.12, a maximal large extension of A in M is Con-S-closed in M. Then apply (4) with $B = M$, we have (1). $\qquad\square$

Remark 3.3. It is clear that if a module M has large radical, then Con-S-submodules and submodules of M are the same, i.e. M being extending and M being S-extending are equivalent.

In the following we give simple examples of modules M which are S-extending modules, while they are not extending:

(1) The $\mathbb{Z}$-module $M = C(p) \oplus \mathbb{Z}$, for a prime number p is S-extending (due to M with no Con-S-submodules), while M is not extending (due to $C(p)$ is not injective as it is shown in [5, Corollary 2]).

(2) A more general than the example in (1), is that the $\mathbb{Z}$-module $M = C(p^n) \oplus \mathbb{Z}$, for a prime number p, $n \in \mathbb{Z}^+$ is S-extending (due to any Con-S-closed submodule A of M must contain $C(p^n)$, and hence either $A = C(p^n)$, or $A = M$), while M is not extending, for the same reason as in the example in (1).

In the following we show that the S-extending condition on a module is inherited by a direct summand.

Lemma 3.4. *Let M be a module, and N be a direct summand of M. If M is S-extending, then so is N.*

Proof. Let A be a Con-S-closed submodule of N. It is clear that A is a Con-S-closed submodule of M. As M is an S-extending module, it follows that A is a direct summand of M and hence of N. $\qquad\square$

In the following we show that isomorphic copy of an S-extending module is S-extending.

Lemma 3.5. *Let M be an S-extending module, and let $f : M \to N$ be an isomorphism. Then N is an S-extending module.*

Proof. Let C be a Con-S-closed submodule of N. It is easy to show that $f^{-1}(C)$ is a Con-S-closed submodule of M, and hence $M = f^{-1}(C) \oplus M_1$. Thus $N = C \oplus f(M_1)$. $\qquad\square$

Proposition 3.6. *Let M be an S-extending module, with $0 \neq Rad(M)$ not large in M. Then M has a decomposition $M = M_1 \oplus M_2$, where $Rad(M_2) = 0$ and M_1 is extending and M_2-superfluous-injective.*

Proof. Let M_1 be a maximal large extension of $Rad(M)$ in M. As M is an S-extending module, we have that $M = M_1 \oplus M_2$, where $Rad(M_2) = 0$, and hence, by Remark 2.5, M_1 is M_2-superfluous-injective. It is clear that $Rad(M_1) = Rad(M) \trianglelefteq M_1$. By Lemma 3.4, M_1 is S-extending. By Remark 3.3, M_1 is extending. $\qquad\square$

Lemma 3.7. ([7, Theorem 4]) *A ring R is right noetherian if and only if every extending right R-module is a direct sum of uniform submodules.*

Theorem 3.8. *Let M be a module over a right noetherian ring R, and let $0 \neq Rad(M)$ not large in M. If M is S-extending, then $M = M_1 \oplus M_2$, where $Rad(M_2) = 0$; and M_1 is extending and a direct sum of uniform submodules.*

Proof. The proof follows from Proposition 3.6 and Lemma 3.7. $\qquad\square$

4. Relative superfluous injective and S-extending modules

In this section we study direct sums of S-extending modules which are, in pairs, relatively superfluous injective, and give the necessary and sufficient conditions for such direct sums to be S-extending.

Lemma 4.1. *Let $M = A + B$, A, B and S be submodules of a module M, and $A \cap B \ll M$. If $S \ll M$, then $A \cap (B + S) \ll M$.*

Proof. Let $A \cap (B + S) + C = M$ for some submodule C in M. Then by the modular law $A = A \cap M = A \cap (B + S) + (C \cap A)$, hence

$$M = A + B = A \cap (B + S) + (C \cap A) + B.$$

Since $A \cap (B + S) + B \subseteq B + S$, then $M \subseteq B + S + (C \cap A) \subseteq M$, hence $M = B + S + (C \cap A)$. Since $S \ll M$, then $M = B + (C \cap A)$. By the modular law, we have that $A = (B \cap A) + (C \cap A)$, and $A \cap (B + S) = (B \cap A) + ((C \cap A) \cap (B + S))$. It follows that

$$M = A \cap (B + S) + C = (B \cap A) + ((C \cap A) \cap (B + S)) + C = (B \cap A) + C.$$

Since $A \cap B \ll M$, then $C = M$. Therefore $A \cap (B + S) \ll M$. $\qquad\square$

Lemma 4.2. *Let $M = A \oplus B$ be a module where A has (S_*). Then the following are equivalent:*

(1) *B is A-superfluous-injective, and A is S-extending.*
(2) *If C is a Con-S-closed submodule of M, with $C \cap B = 0$, then $M = C \oplus A' \oplus B$, for some $A' \leq A$.*

Proof. $(1) \Rightarrow (2)$: Let C be a *Con-S*-closed submodule of M, with $C \cap B = 0$. Let $0 \neq K \leq C$, with $K \ll M$. By Lemma 4.1 we have that $(K \oplus B) \cap A \ll A$. Since $0 \neq (K \oplus B) \cap A \leq (C \oplus B) \cap A$, it follows that $(C \oplus B) \cap A$ is a *Con-S*-submodule of A. Since A is S-extending, then $A = A_1 \oplus A'$ such that $(C \oplus B) \cap A \trianglelefteq A_1$. Hence $[(C \oplus B) \cap A] \oplus B \trianglelefteq A_1 \oplus B$. As $C \oplus B = (C \oplus B) \cap (A \oplus B) = [(C \oplus B) \cap A] \oplus B$, we have that $(C \oplus B) \trianglelefteq A_1 \oplus B$, and that C is closed in $A_1 \oplus B$. Thus, by Remark 2.12, it follows that C is a *Con-S*-complement of B in $A_1 \oplus B$. Since B

is A-superfluous-injective, it follows, by Proposition 2.6, that B is A_1-superfluous-injective. Then, by Theorem 2.16, we have that $A_1 \oplus B = C \oplus B$. Therefore $M = A \oplus B = A_1 \oplus A\backslash \oplus B = A\backslash \oplus C \oplus B$.

$(2) \Rightarrow (1)$: First we show that A is S-extending. To this end let C be a Con-S-closed submodule of A. As A is a direct summand of M, it follows that C is a Con-S-closed submodule of M, with $C \cap B = 0$. By (2), C is a direct summand of M, and hence is a direct summand of A. Now we show that B is A-superfluous-injective. Let D be a Con-S-complement of B in M, hence, by Remark 2.12, D is Con-S-closed submodule of M; with $D \cap B = 0$. By (2), we have that $M = A\backslash \oplus D \oplus B$, for some submodule $A\backslash$ of A. As D is a Con-S-complement of B in M, and $(D \oplus A\backslash) \cap B = 0$, we have that $D = D \oplus A\backslash$; i.e. $M = D \oplus B$. Therefore, by Theorem 2.16, B is A-superfluous-injective. $\qquad\square$

Theorem 4.3. *Let $M = M_1 \oplus M_2$ be a module where M_1 has (S_*). Then the following are equivalent:*

(1) *M_i is M_j-superfluous-injective ($i \neq j = 1, 2$), and M_i is S-extending ($i = 1, 2$).*

(2) *If C is a Con-S-closed submodule of M, with $C \cap M_i = 0$ ($i = 1$ or 2), then $M = C \oplus M_j^\backslash \oplus M_i$, for some $M_j^\backslash \leq M_j$, $j \neq i$.*

Proof. The proof follows from Lemma 4.2. $\qquad\square$

Corollary 4.4. *Let $M = M_1 \oplus M_2$ be a module, with M_i ($i = 1, 2$) uniform. If M_i is M_j-superfluous-injective ($i \neq j = 1, 2$), then M is S-extending.*

Proof. The proof follows from Theorem 4.3, and from the fact that every Con-S-closed submodule C of M, different from M, is uniform; and hence $C \cap M_i = 0$ for some $i = 1, 2$. $\qquad\square$

Remark 4.5. As an application of Corollary 4.4, we have the following examples:

(1) Direct sums of two S-extending modules need not, in general, be S-extending modules; for the $\mathbb{Z}$-module $C(p) \oplus C(p^3)$, for a prime number p, is not S-extending (due to $C(p)$ not a $C(p^3)$-superfluous-injective).

(2) The $\mathbb{Z}$-module $C(p^n) \oplus \mathbb{Z}$, where $n \in \mathbb{Z}^+$ and p is a prime number, is an S-extending $\mathbb{Z}$-module (due to $C(p^n)$ and $\mathbb{Z}$ relatively superfluous-injective), which is not extending as we have mentioned and given the reason before.

Definition 4.6. Let $M = M_1 \oplus M_2$ be a module. M is said to satisfy the superfluous closed intersection property (for short M satisfy the SIC property), if for any

$0 \neq K \ll M$, with $K \cap M_i = 0$, for some $i = 1, 2$, then $C \cap M_i = 0$ for each proper closed submodule C of M containing K.

It is clear that if M and K, the same as in Definition 4.6, with $K \cap M_i = 0$, then $C \cap M_i = 0$ for each maximal large extension C of K in M.

Proposition 4.7. *Let $M = M_1 \oplus M_2$ be a module, which satisfies the SIC property. Then the following are equivalent:*

(1) *M is an S-extending module.*

(2) *Every Con-S-closed submodule C of M, with $C \cap M_i = 0$, for some $i = 1$ or 2, is a direct summand of M.*

Proof. $(1) \Rightarrow (2)$: It is clear.

$(2) \Rightarrow (1)$: It is clear that each *Con-S*-closed submodule C in M_i is a *Con-S*-closed submodule of M, with $C \cap M_j = 0$, $i \neq j$. Hence M_i $(i = 1, 2)$ is an *S*-extending module. Now let C be a *Con-S*-closed submodule of M, with $C \cap M_i \neq 0$, $i = 1, 2$. As C is a *Con-S*-submodule of M, there exists a submodule K in C with $0 \neq K \ll M$. Since M satisfies the *SIC* Property, and since $C \cap M_1 \neq 0$, we have that $0 \neq K \cap M_1 \leq C \cap M_1$ is a *Con-S*-submodule of M. Let $C^\backprime$ be a maximal large extension of $C \cap M_1$ in C. It is clear that $C^\backprime$ is a *Con-S*-closed submodule of M, with $C^\backprime \cap M_2 = 0$. Hence, by (2), $M = C^\backprime \oplus H$, for some submodule H of M. It follows that $C = C' \oplus C''$, where $C'' = C \cap H$. Let $\pi : M \to H$ be the projection onto H. Again, since M satisfies the *SIC* Property, we have that $0 \neq K \cap M_2 \ll M$. As $(K \cap M_2) \cap C^\backprime = 0$, we have that $0 \neq \pi(K \cap M_2) \leq C^\backprime{}^\backprime$, with $\pi(K \cap M_2) \ll M$. Hence $C^\backprime{}^\backprime$ is a *Con-S*-closed submodule of M, with $C^\backprime{}^\backprime \cap M_1 = 0$. Thus, by (2), $C^\backprime{}^\backprime$ is a direct summand of M, and therefore C is a direct summand of M. $\square$

Lemma 4.8. *Let $M = M_1 \oplus M_2$ be a module, with $0 \neq Rad(C) \trianglelefteq C$, for every proper Con-S-closed submodule C of M, with $C \cap M_i \neq 0$, $i = 1, 2$. Then the following are equivalent:*

(1) *M is an S-extending module.*

(2) *Every Con-S-closed submodule C of M, with $C \cap M_i = 0$, for some $i = 1$ or 2, is a direct summand of M.*

Proof. $(1) \Rightarrow (2)$: It is clear.

$(2) \Rightarrow (1)$: It is clear that each *Con-S*-closed submodule C in M_i is a *Con-S*-closed submodule of M, with $C \cap M_j = 0$, $i \neq j$. Hence M_i $(i = 1, 2)$ is an *S*-extending module. Now let C be a proper *Con-S*-closed submodule of M, with $C \cap M_i \neq 0$,

$(i = 1, 2)$, and with $0 \neq Rad(C) \trianglelefteq C$. It follows that $C \cap M_1$ is a Con-S-submodule of M; and hence $M = C^\backslash \oplus L$, where $C^\backslash$ is a maximal large extension of $C \cap M_1$ in C, and L is a submodule of M. It follows that $C = C^\backslash \oplus (L \cap C)$. If $L \cap C = 0$, then we have done. On the other hand, if $C \cap L \neq 0$, then $C \cap L$ is a Con-S-closed submodule of M, with $(C \cap L) \cap M_1 = 0$ (due to $Rad(C) \trianglelefteq C$). Again, by (2), $C \cap L$ is a direct summand of M, and therefore C is a direct summand of M. $\quad\square$

Theorem 4.9. *Let $M = M_1 \oplus M_2$ be a module, with $0 \neq Rad(C) \trianglelefteq C$, for every proper Con-S-closed submodule C of M, with $C \cap M_i \neq 0$, $(i = 1, 2)$. Let M_i be M_j-superfluous-injective, $(i \neq j = 1, 2)$. Then the following are equivalent:*

  (1) *M is an S-extending module.*
  (2) *M_1 and M_2 both are S-extending modules.*

Proof. $(1) \Rightarrow (2)$: Follows from Lemma 3.4.

$(2) \Rightarrow (1)$: By Lemma 4.8, it is enough to show that every Con-S-closed submodule C of M, with $C \cap M_i = 0$ for some $i = 1$ or 2, is a direct summand of M. To this end, let C be a Con-S-closed submodule of M, with $C \cap M_1 = 0$. Since M_1 is M_2-superfluous-injective and M_2 is S-extending, we have, by Lemma 4.2, that $M = C \oplus M_2^\backslash \oplus M_1$, for some $M_2^\backslash \leq M_2$. $\quad\square$

Theorem 4.10. *Let $M = M_1 \oplus M_2$ be a module, which satisfies the SIC property. Let M_i be M_j-superfluous-injective, $(i \neq j = 1, 2)$. Then the following are equivalent:*

  (1) *M is an S-extending module.*
  (2) *M_1 and M_2 both are S-extending modules.*

Proof. The proof follows from Lemma 3.4, Proposition 4.7 and Lemma 4.2. $\quad\square$

Corollary 4.11. *Let $M = \oplus_{i=1}^{n} M_i$ be a module, $0 \neq Rad(C) \trianglelefteq C$, for every proper Con-S-closed submodule C of M, with $C \cap M_i \neq 0$, $(i = 1, 2, \ldots, n)$. Let M_i be M_j-superfluous-injective, $(i \neq j = 1, 2, \ldots, n)$. Then the following are equivalent:*

  (1) *M is an S-extending module.*
  (2) *M_i is S-extending modules, for each $i = 1, 2, \ldots, n$.*

Proof. The proof follows by Theorem 4.9, and induction on n. $\quad\square$

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