GORENSTEIN $\pi[T]$-PROJECTIVITY WITH RESPECT TO A TILTING MODULE

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Abstract. Let $T$ be a tilting module. In this paper, Gorenstein $\pi[T]$-projective modules are introduced and some of their basic properties are studied. Moreover, some characterizations of rings over which all modules are Gorenstein $\pi[T]$-projective are given. For instance, on the $T$-cocoherent rings, it is proved that the Gorenstein $\pi[T]$-projectivity of all $R$-modules is equivalent to the $\pi[T]$-projectivity of $\sigma[T]$-injective as a module.

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1. Introduction

Throughout this paper, $R$ is an associative ring with non-zero identity, all modules are unitary left $R$-modules. First we recall some known notions and facts needed in the sequel. Let $R$ be a ring and $T$ an $R$-module. Then

(1) We denote by $\text{Prod} T$ (resp. $F.\text{Prod} T$), the class of modules isomorphic to direct summands of direct product of copies (resp. finitely many copies) of $T$.

(2) We denote by $\text{Add} T$ (resp. $F.\text{Add} T$), the class of modules isomorphic to direct summands of direct sum of copies (resp. finitely many copies) of $T$.

(3) Following [3], a module $T$ is called tilting (1-tilting) if it satisfies the following conditions:

(a) $\text{pd}(T) \leq 1$, where $\text{pd}(T)$ denotes the projective dimension of $T$.

(b) $\text{Ext}^i(T,T^{(\lambda)}) = 0$, for each $i > 0$ and for every cardinal $\lambda$.

(c) There exists the exact sequence $0 \to R \to T_0 \to T_1 \to \cdots \to T_n \to 0$, where $T_0, T_1 \in \text{Add} T$.

(4) By $\text{Copres}^n T$ (resp. $F.\text{Copres}^n T$) and $\text{Copres}^\infty T$ (resp. $F.\text{Copres}^\infty T$), we denote the set of all modules $M$ such that there exists exact sequences

$$0 \to M \to T_0 \to T_1 \to \cdots \to T_{n-1} \to T_n$$
and

\[0 \rightarrow M \rightarrow T_0 \rightarrow T_1 \rightarrow \cdots \rightarrow T_{n-1} \rightarrow T_n \rightarrow \cdots,\]

respectively, where \(T_i \in \text{Prod}T\) (resp. \(T_i \in \text{F.Prod}T\)), for every \(i \geq 0\).

(5) A module \(M\) is said to be cogenerated, by \(T\), denoted by \(M \in \text{Cogen}T\) (resp. generated, denoted \(M \in \text{Gen}T\)) by \(T\) if there exists an exact sequence \(0 \rightarrow M \rightarrow T^n\) (resp. \(T^{(n)} \rightarrow M \rightarrow 0\)), for some positive integer \(n\).

(6) Let \(C\) be a class of modules and \(M\) be a module. A right (resp. left) \(C\)-resolution of \(M\) is a long exact sequence \(0 \rightarrow M \rightarrow C_0 \rightarrow C_1 \rightarrow \cdots\) (resp. \(\cdots \rightarrow C_1 \rightarrow C_0 \rightarrow M \rightarrow 0\)), where \(C_i \in C\), for all \(i \geq 0\). It is said that a module \(M\) has right \(C\)-dimension \(n\) (briefly, \(C.dim(M) = n\)) if there exists a long exact sequence

\[0 \rightarrow M \rightarrow C_0 \rightarrow C_1 \rightarrow \cdots \rightarrow C_n \rightarrow 0\]

with \(C_i \in C\), for each \(i \geq 0\). In particular, the Prod\(T\)-resolution of \(M\) is called \(T\)-injective dimension of \(M\) and is denoted by \(T.i.dim(M)\). Note that for any tilting module \(M\), if \(M \in \text{Cogen}T\), then [6, Proposition 2.1] implies that \(\text{Cogen}T = \text{Copres}^\infty T\). This shows that any module cogenerated by \(T\) has an Prod\(T\)-resolution. The Prod\(T\)-resolutions and the relative homological dimension were studied by Nikmehr and Shaveisi in [6].

(7) For any homomorphism \(f\), we denote by \(\ker f\) and \(\text{im} f\), the kernel and image of \(f\), respectively. Let \(A\) and \(M \in \text{Cogen}T\) be two modules. We define the functor

\[\mathcal{E}^n_T(A, M) := \frac{\ker \delta^a_n}{\text{im} \delta^a_{n-1}},\]

where

\[0 \rightarrow M \xrightarrow{\delta_0} T_0 \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_n} T_n \rightarrow \cdots\]

Prod\(T\)-resolution of \(M\) and \(\delta^a_i = \text{Hom}(id_B, \delta_n)\), for every \(i \geq 0\). See [6,9] for more details.

(8) Let \(M \in \text{Cogen}T\) and \(N\) be two modules. A similar proof to that of [7, Lemma 2.11] shows that \(\mathcal{E}^0_T(N, M) \cong \text{Hom}(N, M)\). Moreover, \(\mathcal{E}^1_T(-, M) = 0\) implies that \(M \in \text{Prod}T\), and if \(M \in \text{Gen}T\), then \(\mathcal{E}^1_T(M, -) = 0\) implies that \(M \in \text{Add}T\). It is clear that \(T.i.dim(M) = n\) if and only if \(n\) is the least non-negative integer such that \(\mathcal{E}^{n+1}_T(A, M) = 0\), for any module \(A\), see [6, Remark 2.2] for more details. So, \(T.i.dim(M) = n\) if and only if \(\mathcal{E}^{n+1}_T(A, M) = 0\) for every module \(A\) and every \(i \geq 1\). A module with zero \(T\)-injective dimension (resp. \(T\)-projective dimension) is called \(T\)-injective.
(resp. $T$-projective). A similar proof to that of [7, Proposition 2.3] shows that the definition of $E^+_T(C,M)$ is independent from the choice of Prod$T$-resolutions. For unexplained concepts and notations, we refer the reader to [2,6,8].

(9) For a module $T$, we denote by $\pi[T]$, the full subcategory of modules whose objects are of the form $\frac{B}{A} \leq \frac{T}{I}$ for some cardinal $I$ and some modules $A \leq B \leq T^I$. Also, the full subcategory $\sigma[T]$ of modules subgenerated by a given module $T$ (see [10]).

(10) $G$ is called Gorenstein $\sigma[T]$-injective if there exists an exact sequence of $\sigma[T]$-injective modules

$$A = \cdots \to A_1 \to A_0 \to A^0 \to A^1 \to \cdots$$

with $G = \ker(A^0 \to A^1)$ such that Hom($U$, $-$) leaves this sequence exact whenever $U \in \text{Pres}^1 T$ with T.p.dim($U$) $< \infty$ (see [9]).

(11) $M$ is said to be finitely cogenerated [2] if for every family $\{V_k\}_J$ of submodules of $M$ with $\bigcap_J V_k = 0$, there is a finite subset $I \subset J$ with $\bigcap_I V_k = 0$.

(12) $M$ is said to be finitely copresented if there is an exact sequence of $R$-modules $0 \to M \to E^0 \to E^1$, where each $E^i$ is a finitely cogenerated injective module, see [1,11,12].

Let $T$ be a tilting module. In this paper, we introduce the $\pi[T]$-projective modules, the $\pi[T]$-projective dimension and Gorenstein $\pi[T]$-projective modules.

Let $M \in \text{Gen} T$. Then, $M$ is called $\pi[T]$-projective if the functor $E^1_T(\cdot, -)$ vanishes on $\pi[T]$. Also, the $\pi[T]$-projective dimension of $M$ is defined to be

$$\pi[T].pd(M) = \inf\{n : E^{n+1}_T(M,N) = 0 \text{ for every } N \in \pi[T]\}.$$

We define a module $G$ to be Gorenstein $\pi[T]$-projective (GT-projective for short), if there exists an exact sequence of $\pi[T]$-projective modules

$$B = \cdots \to B_1 \to B_0 \to B^0 \to B^1 \to \cdots$$

with $G = \ker(B^0 \to B^1)$ such that Hom($- , U$) leaves this sequence exact whenever $U \in F.Copres^1 T$ with T.i.dim($U$) $< \infty$. In this paper, the GT-projective dimension of a module $G$ is denoted by $GT.pd(G)$.

In Section 2, we study some basic properties of the Gorenstein $\pi[T]$-projective modules. Recall that a ring $R$ is said to be cocoherent if every finitely cogenerated module is finitely copresented. So, $R$ is a cocoherent ring if and only if Copres$^0 R = \text{Copres}^1 R$. For more information about the cocoherent rings, we refer the reader
Section 3 is devoted to some characterizations of $T$-cocoherent rings over which all modules are Gorenstein $\pi[T]$-projective. For instance, it is proved that every module is Gorenstein $\pi[T]$-projective if and only if every $T$-injective module is $\pi[T]$-projective if and only if every $\sigma[T]$-injective module is Gorenstein $\pi[T]$-projective. Finally, we give a sufficient condition under which every Gorenstein $\pi[T]$-projective module is $\pi[T]$-projective.

2. Gorenstein $\pi[T]$-projectivity

We start with the following definition.

Definition 2.1. Let $T$ be a tilting module. Then

1. $M$ is called $\pi[T]$-projective if $\mathcal{E}_1^T(M,N) = 0$, for every $N \in \pi[T]$.
2. Let $G \in \text{Gen}_T$. Then, $G$ is called Gorenstein $\pi[T]$-projective if there exists an exact sequence of $\pi[T]$-projective modules

$$B = \cdots \rightarrow B_1 \rightarrow B_0 \rightarrow B^0 \rightarrow B^1 \rightarrow \cdots$$

with $G = \ker(B^0 \rightarrow B^1)$ such that $\text{Hom}(-,U)$ leaves this sequence exact whenever $U \in \text{F.Copres}^1 T$ with $\text{T.i.dim}(U) < \infty$.

Remark 2.2. Let $T$ be a tilting module. Then

1. $\mathcal{E}_1^T(N,M) = 0$ for any $\pi[T]$-projective module $N$ and any $M \in \text{Copres}^0 T$.
2. If $A \in \text{Add} T$, then $A$ is $\pi[T]$-projective.

Lemma 2.3. Let $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be an exact sequence. Then

1. If $A$ is $T$-injective and $A,B,C \in \text{Cogen}T$, then $B = A \oplus C$.
2. If $A \in \text{F.Copres}^n T$ and $C \in \text{F.Copres}^T$, then $B \in \text{F.Copres}^n T$.
3. If $C \in \text{F.Copres}^n T$ and $B \in \text{F.Copres}^{n+1} T$, then $A \in \text{F.Copres}^{n+1} T$.
4. If $B \in \text{F.Copres}^n T$ and $A \in \text{F.Copres}^{n+1} T$, then $C \in \text{F.Copres}^n T$.

Proof. (1) If $A$ is $T$-injective and $A,B,C \in \text{Cogen}T$, then we deduce that the sequence

$$0 \rightarrow \text{Hom}(C,A) \xrightarrow{g^*} \text{Hom}(B,A) \xrightarrow{f^*} \text{Hom}(A,A) \rightarrow \mathcal{E}_1^T(C,A) = 0$$

is exact. So, there exists $h : B \rightarrow A$ such that $hf = 1_A$. 

We prove the assertion by induction on $n$. If $n = 0$, then the commutative diagram with exact rows

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
\scriptstyle o & \rightarrow & \scriptstyle A \\
\downarrow & \downarrow & \downarrow \\
\scriptstyle h_0' & \rightarrow & \scriptstyle B \\
\downarrow & \downarrow & \downarrow \\
\scriptstyle 0 & \rightarrow & \scriptstyle T_0' \\
\downarrow & \downarrow & \downarrow \\
\scriptstyle h_0 & \rightarrow & \scriptstyle T_0'' \\
\downarrow & \downarrow & \downarrow \\
\scriptstyle 0 & \rightarrow & \scriptstyle 0
\end{array}
\]

exists, where $T_0', T_0'' \in \text{F.Prod}T$, $i_0$ is the inclusion map, $\pi_0$ is a canonical epimorphism and $h_0 = i_0h_0'$ is endomorphism, by Five Lemma. Let $K_1' = \text{coker}(h_0')$, $K_1 = \text{coker}(h_0)$ and $K_1'' = \text{coker}(h_0'')$. It is clear that $(T_0' \oplus T_0'') \in \text{F.Prod}T$ and $K_1', K_1'' \in \text{F.Copres}^{n-1}T$; so, the induction implies that $K_1 \in \text{F.Copres}^{n-1}T$. Hence $B \in \text{F.Copres}^nT$.

(3) Let $B \in \text{F.Pres}^{n+1}T$ and $C \in \text{F.Pres}^nT$, then the following commutative diagram with exact rows:

\[
\begin{array}{ccc}
0 & 0 \\
\downarrow & \downarrow \\
0 & \rightarrow & \scriptstyle A \\
\downarrow & \downarrow \\
0 & \rightarrow & \scriptstyle B \\
\downarrow & \downarrow & \text{||} \\
0 & \rightarrow & \scriptstyle C \\
\downarrow & \downarrow \\
0 & 0
\end{array}
\]

where $T_0 \in \text{F.Prod}T$ and $L \in \text{F.Copres}^nT$. By (2), $D \in \text{F.Copres}^nT$. So, we deduce that $A \in \text{F.Copres}^{n+1}T$.

(4) Let $A \in \text{F.Pres}^{n+1}T$ and $B \in \text{F.Pres}^nT$, then the following commutative diagram with exact rows:

\[
\begin{array}{ccc}
0 & 0 \\
\downarrow & \downarrow \\
0 & \rightarrow & \scriptstyle A \\
\downarrow & \downarrow \\
0 & \rightarrow & \scriptstyle B \\
\downarrow & \downarrow & \text{||} \\
0 & \rightarrow & \scriptstyle C \\
\downarrow & \downarrow \\
0 & 0
\end{array}
\]
where $T_0, T'_0 \in \text{F.Prod}_T$ and $L \in \text{F.Copres}^{n-1}T$. Since $T'_0$ is $T$-injective, we have that $T_0 = T'_0 \oplus D$ by (1), and $D \in \text{Cogen}_T$. Thus for any $N \in \text{Cogen}_T$, we have

$$E^1_T(T_0, N) = E^1_T(T'_0 \oplus D, N) = E^1_T(T'_0, N) \oplus E^1_T(D, N) = 0.$$ 

Hence $D \in \text{F.Prod}_T$. On the other hand, $L \in \text{F.Copres}^{n-1}T$. Therefore, we conclude that $C \in \text{F.Copres}^nT$. □

In the following theorem, we show that in the case of $T$-cocoherent rings, the existence of $\pi[T]$-projective complex of a module is sufficient to be Gorenstein $\pi[T]$-projective.

**Theorem 2.4.** Let $R$ be a $T$-cocoherent ring and $G \in \text{Gen}_T$ be a module. Then $G$ is Gorenstein $\pi[T]$-projective if and only if there is an exact sequence

$$0 \to T_0 \to T'_0 \to L' \to 0$$

where $T_0, T'_0 \in \text{F.Prod}_T$ and $L \in \text{F.Copres}^{n-1}T$. Since $T'_0$ is $T$-injective, we have that $T_0 = T'_0 \oplus D$ by (1), and $D \in \text{Cogen}_T$. Thus for any $N \in \text{Cogen}_T$, we have

$$E^1_T(T_0, N) = E^1_T(T'_0 \oplus D, N) = E^1_T(T'_0, N) \oplus E^1_T(D, N) = 0.$$ 

Hence $D \in \text{F.Prod}_T$. On the other hand, $L \in \text{F.Copres}^{n-1}T$. Therefore, we conclude that $C \in \text{F.Copres}^nT$. □

Theorem 2.4. Let $R$ be a $T$-cocoherent ring and $G \in \text{Gen}_T$ be a module. Then $G$ is Gorenstein $\pi[T]$-projective if and only if there is an exact sequence

$$B = \cdots \to B_1 \to B_0 \to B^0 \to B^1 \to \cdots$$

of $\pi[T]$-projective modules such that $G = \ker(B^0 \to B^1)$.

**Proof.** ($\Rightarrow$): This is a direct consequence of definition.

($\Leftarrow$): By definition, it suffices to show that $\text{Hom}(B, U)$ is exact for every module $U \in \text{F.Copres}^1T$ with $\text{T.i.dim}(U) = m < \infty$. To prove this, we use the induction on $m$. The case $m = 0$ is clear. Assume that $m \geq 1$. Since $U \in \text{F.Copres}^1T$, there exists an exact sequence $0 \to U \to T_0 \to I \to 0$ with $T_0 \in \text{F.Prod}_T \subseteq \text{F.Copres}^0T$. Now, from the $T$-cocoherence of $R$ and Lemma 2.3, we deduce that $I, T_0 \in \text{F.Copres}^1T$. Also, $\text{T.i.dim}(I) \leq m - 1$ and $\text{T.i.dim}(T_0) = 0$. Thus by Remark 2.2, the following short exact sequence of complexes exists:
By induction, Hom(\(B\), \(T\)) and Hom(\(B\), I) are exact, hence Hom(\(B\), U) is exact by [8, Theorem 6.10]. Therefore, \(G\) is Gorenstein \(\pi[T]\)-projective. □

It is worthy to mention that the notion of \(T\)-injectivity (\(T\)-projectivity) is different from the notion of an \(M\)-injective (\(M\)-projective) module in [2].

**Corollary 2.5.** Let \(R\) be a \(T\)-cocoherent ring and \(G \in \text{Gen}T\) be a module. Then the following assertions are equivalent:

1. \(G\) is Gorenstein \(\pi[T]\)-projective;
2. There is an exact sequence \(0 \to G \to B^0 \to B^1 \to \cdots\) of modules, where every \(B^i\) is \(\pi[T]\)-projective;
3. There is a short exact sequence \(0 \to G \to M \to I \to 0\) of modules, where \(M\) is \(\pi[T]\)-projective and \(I\) is Gorenstein \(\pi[T]\)-projective.

**Proof.** (1) \(\Rightarrow\) (2) and (1) \(\Rightarrow\) (3) follow from definition.

(2) \(\Rightarrow\) (1) For module \(G \in \text{Gen}T\), [6, Proposition 2.1] implies that \(\text{Gen}T = \text{Pres}^\infty T\). So, there is an exact sequence

\[\cdots \to T_1 \to T_0 \to G \to 0\]

where any \(T_i\) is \(\pi[T]\)-projective by Remark 2.2. Thus, the exact sequence

\[\cdots \to T_1 \to T_0 \to B^0 \to B^1 \to \cdots\]

of \(\pi[T]\)-projective modules exists, where \(G = \ker(B^0 \to B^1)\). Therefore, \(G\) is Gorenstein \(\pi[T]\)-projective, by Theorem 2.4.
(3) ⇒ (2) Assume that the exact sequence
\[ 0 \rightarrow G \rightarrow M \rightarrow I \rightarrow 0 \] (1)
exists, where \( M \) is \( \pi[T] \)-projective and \( I \) is Gorenstein \( \pi[T] \)-projective. Since \( I \) is Gorenstein \( \pi[T] \)-projective, there is an exact sequence
\[ 0 \rightarrow I \rightarrow C^0 \rightarrow C^1 \rightarrow \cdots \] (2)
where every \( C^i \) is \( \pi[T] \)-projective. Assembling the sequences (1) and (2), we get the exact sequence
\[ 0 \rightarrow G \rightarrow M \rightarrow C_0 \rightarrow C_1 \rightarrow \cdots , \]
where \( M \) and every \( C^i \) are \( \pi[T] \)-projective, as desired.

**Proposition 2.6.** For any module \( G \in \text{Gen} T \), the following statements hold.

1. If \( G \) is Gorenstein \( \pi[T] \)-projective, then \( \mathcal{E}^r_T(G,U) = 0 \) for all \( i > 0 \) and every module \( U \in \text{F.Copres}^1 T \) with \( T.i.\dim(U) < \infty \).
2. If \( 0 \rightarrow N \rightarrow G_{n-1} \rightarrow \cdots \rightarrow G_0 \rightarrow G \rightarrow 0 \) is an exact sequence of modules where every \( G_i \) is a Gorenstein \( \pi[T] \)-projective and \( G_i \in \text{Gen} T \), then \( \mathcal{E}^r_T(N,U) = \mathcal{E}^{n+i}_T(G,U) \) for any \( i > 0 \) and any module \( U \in \text{F.Copres}^1 T \) with \( T.i.\dim(U) < \infty \).

**Proof.** (1) Let \( G \) be a Gorenstein \( \pi[T] \)-projective module, and \( T.i.\dim(U) = m < \infty \). Then by hypothesis, the following \( \pi[T] \)-projective resolution of \( G \) exists:
\[ 0 \rightarrow G \rightarrow B^0 \rightarrow \cdots \rightarrow B^{m-1} \rightarrow N \rightarrow 0. \]

By Remark 2.2, \( \mathcal{E}^r_T(B_j,U) = 0 \) for every \( i > 0 \) and every \( 0 \leq j \leq m - 1 \). Since \( T.i.\dim(U) = m \), we deduce that \( \mathcal{E}^r_T(G,U) \cong \mathcal{E}^{n+i}_T(N,U) = 0 \).

(2) Setting \( G_n = N \) and \( K_j = \ker(G_j \rightarrow G_{j-1}) \), for every \( 0 \leq j \leq n \), the short exact sequence \( 0 \rightarrow K_j \rightarrow G_j \rightarrow K_{j-1} \rightarrow 0 \) exists. Thus by (1), the induced exact sequences
\[ 0 = \mathcal{E}^r_T(G_j,U) \rightarrow \mathcal{E}^r_T(K_j,U) \rightarrow \mathcal{E}^{r+1}_T(K_{j-1},U) \rightarrow \mathcal{E}^{r+1}_T(G_j,U) = 0 \]
exists and so \( \mathcal{E}^r_T(K_j,U) \cong \mathcal{E}^{r+1}_T(K_{j-1},U) \), for every \( r \geq 0 \). Since \( K_{n-1} = N \), we have
\[ \mathcal{E}^{n+i}_T(G,U) \cong \mathcal{E}^{n+i-1}_T(K_0,U) \cong \cdots \cong \mathcal{E}^r_T(N,U), \]
as desired.

Next, we study the Gorenstein \( \pi[T] \)-projectivity of modules on \( T \)-cocoherent rings, in short exact sequences.
Proposition 2.7. Let \( R \) be \( T \)-cocoherent and consider the exact sequence \( 0 \to N \to B \to G \to 0 \), where \( B \) is \( \pi[T] \)-projective. Then \( \text{GT-pd}(G) \leq \text{GT-pd}(N) + 1 \). In particular, if \( G \) is Gorenstein \( \pi[T] \)-projective, so is \( N \).

**Proof.** We shall show that \( \text{GT-pd}(G) \leq \text{GT-pd}(N) + 1 \). In fact, we may assume that \( \text{GT-pd}(N) = n < \infty \). Then, by definition, \( N \) admits a Gorenstein \( \pi[T] \)-projective resolution:

\[
0 \to B_n \to B_{n-1} \to \cdots \to B_0 \to N \to 0.
\]

Assembling this sequence and the short exact sequence \( 0 \to N \to B \to G \to 0 \), the following commutative diagram is obtained:

\[
\begin{array}{cccccccc}
0 & \to & B_n & \to & \cdots & \to & B_1 & \to & B_0 & \to & B & \to & G & \to & 0 \\
\downarrow & & \uparrow & & & & \downarrow & & \uparrow & & \downarrow & & \uparrow & & \downarrow & & \uparrow \\
N & \equiv & N & & & & & & & & & & & & & & \equiv & K & \equiv & K \\
0 & & 0 & & & & & & & & & & & & & & 0 & & 0
\end{array}
\]

which shows that \( \text{GT-pd}(G) \leq n + 1 \). The particular case follows from Corollary 2.5.

Proposition 2.8. Let \( R \) be a \( T \)-cocoherent ring and \( 0 \to N \to G \to B \to 0 \) be an exact sequence, where \( N, B \in \text{Gen} T \). If \( N \) is Gorenstein \( \pi[T] \)-projective and \( B \) is \( \pi[T] \)-projective, then \( G \) is Gorenstein \( \pi[T] \)-projective.

**Proof.** Since \( N \) is Gorenstein \( \pi[T] \)-projective, by Corollary 2.5, there exists an exact sequence of \( 0 \to N \to B' \to K \to 0 \), where \( B' \) is \( \pi[T] \)-projective and \( K \) is Gorenstein \( \pi[T] \)-projective. Now, we consider the following diagram:

\[
\begin{array}{cccccccc}
0 & & 0 & & & & & & & & & & & & & & 0 & & 0 \\
\downarrow & & \downarrow & & & & & & & & & & & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & N & \to & G & \to & B & \to & 0 \\
\downarrow & & \downarrow & & \parallel & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & B' & \to & D & \to & B & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K & \equiv & K & & & & & & & & & & & & & & 0 & & 0 \\
\end{array}
\]

The exactness of the middle horizontal sequence with \( B \) and \( B' \), \( \pi[T] \)-projective, implies that \( D \) is \( \pi[T] \)-projective. Hence from the middle vertical sequence and Corollary 2.5, we deduce that \( G \) is Gorenstein \( \pi[T] \)-projective. \( \square \)
3. Gorenstein $\pi[T]$-projective modules on $T$-cocoherent rings

This section is devoted to $T$-cocoherent rings over which every module is Gorenstein $\pi[T]$-projective.

**Lemma 3.1.** Let $T$ be a tilting module and $G \in \text{Gen}_T$. Then, $G \in \text{Cogen}_T$.

**Proof.** Let $G \in \text{Gen}_T$. Then, the short exact sequence $0 \to K \to T(I) \to G \to 0$ exists. We have $K \subseteq T(I) \subseteq T^J$. So, $K \in \text{Cogen}_T$. By [6, Proposition 2.1], $\text{Cogen}_T = \text{Copres}_T^\infty$, since $T$ is tilting. Thus by Lemma 2.3, $G \in \text{Copres}_T^m$, and hence $G \in \text{Cogen}_T$.

**Proposition 3.2.** Let $R$ be a ring. The following assertions are equivalent:

1. Every module belong $\text{Gen}_T$, is Gorenstein $\pi[T]$-projective;
2. The ring satisfies the following two conditions:
   (i) Every $T$-injective module is $\pi[T]$-projective.
   (ii) $E^1_T(N,U) = 0$ for any $N \in \text{Gen}_T$ and any $U \in \text{F.Copres}_T^\infty$ with $\text{T.i.dim}(U) < \infty$.

**Proof.** (1) $\Rightarrow$ (2) The condition (i) follows from this fact that every $T$-injective module $M$ is Gorenstein $\pi[T]$-projective. So, the following $\pi[T]$-projective resolution of $M$ exists:

$$0 \to M \to B^0 \to B^1 \to \cdots.$$ 

Since $M$ is $T$-injective, $M$ is $\pi[T]$-projective as a direct summand of $B^0$. Also, Proposition 2.6(1) and (1) imply that $E^1_T(N,U) = 0$ for any module $N \in \text{Gen}_T$ and any module $U \in \text{F.Copres}_T^\infty$ with finite $T$-injective dimension. So the condition (ii) follows.

(2) $\Rightarrow$ (1) Let $G \in \text{Gen}_T$. Then by Lemma 3.1, $G \in \text{Cogen}_T$. So, a Add$T$-resolution $\cdots \to T_1 \to T_0 \to G \to 0$ and a Prod$T$-resolution $0 \to G \to T^0 \to T^1 \to \cdots$ of $G$ exists. By Remark 2.2, any $T_i$ is $\pi[T]$-projective and any $T^i$ is $T$-injective. Hence by (2), every $T^i$ is $\pi[T]$-projective. Assembling these resolutions, we get the following exact sequence of $\pi[T]$-projective modules:

$$B = \cdots \to T_1 \to T_0 \to T^0 \to T^1 \to \cdots,$$

where $G = \ker(T^0 \to T^1)$. So by (2)(ii), $\text{Hom}(B,U)$ is exact for any module $U \in \text{F.Copres}_T^\infty$ with finite $T$-injective dimension. Hence $G$ is Gorenstein $\pi[T]$-projective.

The next theorem shows that if $R$ is a $T$-cocoherent ring and every $\sigma[T]$-injective module is Gorenstein $\pi[T]$-projective, then every module is Gorenstein $\pi[T]$-projective.
Theorem 3.3. Let $R$ be a $T$-cocoherent ring. Then the following are equivalent:

1. Every module is Gorenstein $\pi[T]$-projective;
2. Every Gorenstein $\sigma[T]$-injective module is Gorenstein $\pi[T]$-projective;
3. Every $\sigma[T]$-injective module is Gorenstein $\pi[T]$-projective;
4. Every $T$-injective module is $\pi[T]$-projective.

Proof. (1) $\Rightarrow$ (2) This is clear.

(2) $\Rightarrow$ (3) Let $G$ be a $\sigma[T]$-injective module. Every $\sigma[T]$-injective module is Gorenstein $\sigma[T]$-injective (see,[9]). Since $G$ is Gorenstein $\sigma[T]$-injective, we deduce that $G$ is Gorenstein $\pi[T]$-projective by hypothesis.

(3) $\Rightarrow$ (4) Let $G$ be a $T$-injective module. Then $G$ is $\sigma[T]$-injective, and so $G$ is Gorenstein $\pi[T]$-projective by hypothesis. By Corollary 2.5, there exists an exact sequence $0 \rightarrow G \rightarrow B \rightarrow N \rightarrow 0$, where $B$ is $\pi[T]$-projective. Thus the sequence splits. Hence $G$ is $\pi[T]$-projective as a direct summand of $B$.

(4) $\Rightarrow$ (1) Let $G \in \text{Gen} T$. Then by Lemma 3.1, there is an exact sequence $0 \rightarrow G \rightarrow T^0 \rightarrow T^1 \rightarrow \cdots$, where any $T^i$ is $T$-injective. Then by (5), every $T^i$ is $\pi[T]$-projective. Hence Corollary 2.5 completes the proof.

We denote the right $\pi[T]$-projective dimension of any module $M$ by $\pi[T].pd(M)$, and $\pi[T].pd(M) = \inf\{n : E^n_{\pi}(M,N) = 0 \text{ for every } N \in \pi[T]\}$.

Example 3.4. Let $R$ be a 1-Gorenstein ring and $0 \rightarrow R \rightarrow E^0 \rightarrow E^1 \rightarrow 0$ be the minimal injective resolution of $R$. Then, $\pi[T].pd(E^0) = \pi[T].pd(E^1) = 0$. Since by [4], $T = E_0 \oplus E_1$ is a tilting module. So, any $E^i$ is $\pi[T]$-projective and hence, any $E^i$ is Gorenstein $\pi[T]$-projective for $i = 0, 1$.

Definition 3.5. We define the global $\pi[T]$-projective dimension of any ring $R$ to be:

$$\text{gl.}\pi[T].pd(R) = \sup\{\pi[T].pd(M) | M \text{ is a module}\}.$$

Clearly, every $\pi[T]$-projective module is Gorenstein $\pi[T]$-projective. But the converse is not true in general. We finish this paper with the following theorem which determines a sufficient condition under which the converse holds.

Theorem 3.6. If $\text{gl.}\pi[T].pd(R) < \infty$, then every Gorenstein $\pi[T]$-projective module is $\pi[T]$-projective.
Proof. Suppose that $\text{gl.} \pi[T].\text{pd}(R) = m < \infty$, and $G$ is a Gorenstein $\pi[T]$-projective module. If $m = 0$, then $\mathcal{E}^T_1(M, N) = 0$ for any $N \in \pi[T]$, and hence $G$ is $\pi[T]$-projective. For $m \geq 1$, since $G$ is Gorenstein $\pi[T]$-projective, there exists an exact sequence $0 \to G \to B^0 \to B^1 \to \cdots$ with each $B^i$ is $\pi[T]$-projective. Let $L = \text{coker}(B^{m-2} \to B^{m-1})$. Then

$$0 \to G \to B^0 \to B^1 \to \cdots \to B^{m-2} \to B^{m-1} \to L \to 0$$

is exact, and hence $G$ is $\pi[T]$-projective since $\pi[T].\text{pd}(L) \leq m$. $\square$

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References


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