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Investigation of the Motion of Scalar Particles in an External Electromagnetic Field via Asymptotic Iteration Method

Asimptotik İterasyon Metodu İle Bir Dış Elektromanyetik Alan İçerisindeki Skaler Parçacıkların Hareketinin İncelenmesi

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Abstract

In the present article, we study the energy eigenvalues and wavefuntions of the spinless particles in the existance of space-dependent electric and magnetic fields. The investigation is performed for two different orientations of the external fields by solving the Schrödinger and Klein-Gordon equations via Asymptotic Iteration Method (AIM). The obtained results are discussed numerically for first few quantum levels to understand the relativistic contributions by comparing the energy eigenvalues of Schrödinger and Klein-Gordon equations. By considering the quantum conditions for the resulting equations in the case of parallel and orthogonal orientations of the fields, we obtained the plots for the energy levels.

Keywords: Asymptotic Iteration Method, External Fields, Klein-Gordon Equation, Schrödinger Equation, Spinless Particles

Öz

Bu çalışmada, uzay-bağımlı elektrik ve manyetik alanların varlığında, spinsiz parçacıkların enerji özdeğerleri ve dalga fonksiyonları incelenmektedir. Çalışmada, dış alanların iki farklı yönelimi için Schrödinger ve Klein-Gordon denklemleri Asimptotik İterasyon Metodu (AIM) ile çözülmektedir. Elde edilen sonuçlar, Schrödinger ve Klein-Gordon denklemlerinin enerji özdeğerleri karşılaştırılarak göreceli katkıları görebilmek için, ilk birkaç kuantum seviyesi için, sayısal olarak tartışılmıştır. Alanların paralel ve ortogonal yönelimlerinde ortaya çıkan denklemlerin kuantum koşullarını göz önüne alarak, enerji seviyelerinin eğrisi elde edilmiştir.

***Anahtar kelimeler**: Asimptotik İterasyon Metodu, Dış Alanlar, Klein-Gordon Denklemi, Schrödinger Denklemi, Skaler Parçacıklar*

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1. Introduction

Electromagnetic fields have lots of important applications used in the medicine and technology for a long time. Depending on their usage in fundamental processes occurring in the engineering, particle and medical physics, important steps have been taken such as developments on the intense particle beams generated by laser sources or on the construction of accelerators. Besides, some observations in astrophysics such as the discovery of the pulsars got attention on the different configurations of parallel electric and magnetic fields (Chiu et al., 1969; Chiu and Canuto, 1969; Chiu and Occhionero, 1969). After these developments, exact solutions of the relativistic particle equations in external electromagnetic fields required considerable attention. Such studies have been accomplished for different configurations of the external fields (Redmond, 1965; Liboff, 1966; Occhionero and Demianski, 1969; Lam, 1971; Grewing and Heintzmann, 1972; Bergou and Ehlotzky, 1983; Ivanovski et al., 1993; Villalba and Pino, 2001; Chiang and Ho, 2001; Rutkowski and Poszwa., 2009; Sogut and Havare, 2014) in which the exact solutions of the non-relativistic and relativistic wave equations are obtained and principal informations regarding the corresponding quantum mechanical system are derived. These investigations have been very helpful in the interpretation of the physical processes such as Compton scattering by a laser source and the Brownian motion. There are less studies on the motion of the spinless particles in the coexistence of the electric and magnetic fields.

The aim of the present study is to obtain the exact solutions of the spinless particles for two orientations of exponentially varying electric and magnetic fields that are given by

$$A_\mu = \left(\frac{E_0}{\alpha+\beta z}\right)\delta_\mu^0 + \left(\frac{B_0}{\Gamma+\Lambda y}\right)\delta_\mu^1 \qquad (1)$$

$$A_\mu = \left(\frac{E_0}{\Gamma+\Lambda y}\right)\delta_\mu^0 + \left(\frac{B_0}{\Gamma+\Lambda y}\right)\delta_\mu^1 \qquad (2)$$

where $E_0, B_0, \alpha, \beta, \Gamma$ and Λ are constants and $y, z \in [0, \infty)$. Eqs. (1) and (2) represent the case of parallel fields ($\vec{E} \parallel \vec{B}$) and perpendicular fields ($\vec{E} \perp \vec{B}$), respectively.

There are different methods that can be used for the investigation of such a motion in quantum mechanics, such as Nikiforov-Uvarov (NU) method (Nikiforov and Uvarov, 1988), faztorization method (Dong, 2007) and Supersymmetric Quantum Mechanics (SUSYQM) (Valance et al., 1990). In this study, we use Asymptotic Iteration Method (AIM) which is widely used in recent years (Ciftci et al., 2003). AIM is a novel method that makes the calculations in quantum mechanics easier and faster. It can be applied to both exactly and approximately (numerically) solvable problems (Ciftci et al., 2005a, 2013; Ciftci and Kisoglu 2016; Benchiheub et al., 2015; Chabab et al., 2016; Demic et al., 2016). AIM can also be used in the scope of perturbation method (Ciftci et al., 2005b, 2013; Onate and Idiodi, 2015; Zhang et al., 2015; Alsadi, 2015; Kumaresan et al., 2015). Availability in the scope of perturbation method, besides the usage for exactly and approximately solvable problems, makes AIM a powerful method.

In present work, the motion of a spinless particle in the external fields given in Eq. (1) is investigated by means of Asymptotic Iteration Method (AIM). We obtain the exact solutions of both Schrödinger and Klein-Gordon equations for such a particle via the method. We also construct the energy spectrums for both equations.

According to organization of the paper, we introduce briefly the AIM in Section 2, and by applying this method to the Schrödinger and Klein-Gordon equations by considering Eqs. (1) and (2) in Section 3. Finally, we discuss the obtained results in Section 4.

2. A Brief Introduction to Asymptotic Iteration Method (AIM)

We summarize Asymptotic Iteration Method (AIM) in this section, while exhaustive information can be found in Ref. (Ciftci et al., 2003). AIM was provided for solving the second order linear differential equations in a form as follows

$$y''(x) = \lambda_0(x)y'(x) + s_0(x)y(x) \qquad (3)$$

where $\lambda_0(x)$, $s_0(x)$ and their derivatives are continuous functions within the boundaries of the given system. On the assumption that

$$\frac{s_n}{s_{n-1}} = \frac{\lambda_n}{\lambda_{n-1}} \equiv \zeta \tag{4}$$

is satisfied for $n \in Z^+$ (n is large enough), there is a general solution given by

$$y(x) = \exp(-\int^x \zeta(t)dt)\left[C_2 + C_1 \int^x \exp(\int^t (\lambda_0(\tau) + 2\zeta(\tau))d\tau)dt\right] \tag{5}$$

where C_1 and C_2 are constants and

$$\lambda_n = \lambda'_{n-1} + s_{n-1} + \lambda_0\lambda_{n-1}, \qquad s_n = s'_{n-1} + s_0\lambda_{n-1} \tag{6}$$

The (unknown) E energy eigenvalues of the eigenvalue problem are obtained from the following equation:

$$\delta_n(x,E) \equiv s_n(x,E)\lambda_{n-1}(x,E) - \lambda_n(x,E)s_{n-1}(x,E) = 0 \tag{7}$$

which is named as "termination condition", and is obtained through the Eq. (4).

This termination condition has a crucial role to get the eigenvalues. If the unknown E energy eigenvalues (roots of Eq. (7)) can be obtained independently from the variable x, the problem is exactly solvable. If not, the problem is approximately (or numerically) solvable and an acceptable initial $x \equiv x_0$ value is needed to initiate the AIM iterations. This initial value may be obtained from $\lambda_0 = 0$ (Ciftci et al., 2003, 2013; Aygun et al., 2007). Besides that, eigenfunctions of the problem are obtained by using the function generator given as follows (Bayrak and Boztosun, 2006; Bayrak et al. 2007):

$$y_n(x) = C_2 \exp\left(-\int^x \frac{s_n(t)}{\lambda_n(t)}dt\right) \tag{8}$$

3. Application of AIM to the Problem

In this section, we apply AIM to Schrödinger and Klein-Gordon equations individually, for a scalar particle subjected to external electric and magnetic fields. For each equation, both $\vec{E} \parallel \vec{B}$ and $\vec{E} \perp \vec{B}$ cases are tackled, and exact energy eigenvalues and eigenfunctions for each case are obtained.

3.1. Solution of Schrödinger Equation

The Schrödinger equation for a particle moving in external electric and magnetic fields is

$$\left[\frac{(\vec{p}-e\vec{A})^2}{2m}\right]\Psi = \left(i\frac{\partial}{\partial t} - eA_0\right)\Psi \tag{9}$$

in natural units ($\hbar = c = 1$) where *e* and *m* are charge and mass of the particle, respectively. $A_\mu = (A_0, \vec{A})$ is electromagnetic four-vector potential (Greiner, 1997, 2001).

3.1.1. The Case of $\vec{E} \parallel \vec{B}$

In the case of $\vec{E} \parallel \vec{B}$, since the electromagnetic vector potential is independent of the x variable, we can write the wavefunction as follow

$$\Psi_{\|}^{Sch}(x,y,z,t) = e^{i(xk_x-\varepsilon t)}F(y)G(z)$$

where $x, y, z \in [0,\infty)$. By inserting this form of the wavefunction into Eq. (9), we obtain

$$\left[\hat{U}(y)+\hat{V}(z)\right]F(y)G(z) = 0 \tag{10}$$

where $\hat{U}(y)$ and $\hat{V}(z)$ operators are defined as follows

$$\hat{U}(y) = \hat{p}_y^2 + \left(k_x - \frac{eB_0}{\Gamma+\Lambda y}\right)^2 \tag{11}$$

$$\hat{V}(z) = \hat{p}_z^2 - 2m\varepsilon + \frac{2meE_0}{\alpha+\beta z} \tag{12}$$

Eq. (10) can be separated into two ordinary differential equations as follows

$$\left[\hat{U}(y)+a^2\right]F(y) = 0 \tag{13}$$

$$\left[\hat{V}(z)-a^2\right]G(z) = 0 \tag{14}$$

where a^2 is separation constant. If we change the variable as $\rho = \Gamma + \Lambda y$, then Eq. (13) reduces to following form

$$\left[\frac{d^2}{d\rho^2} - \frac{e^2B_0^2}{\Lambda^2}\frac{1}{\rho^2} + \frac{2ek_xB_0}{\Lambda^2}\frac{1}{\rho} - \pi^2\right]F(\rho) = 0 \tag{15}$$

where $\pi^2 = \frac{k_x^2+a^2}{\Lambda^2}$.

Regarding to domain of the problem, F(ρ) can be assumed as

$$F(\rho) = \rho^{\sigma+1}e^{-\pi\rho}g(\rho) \tag{16}$$

where

$$\sigma = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{e^2B_0^2}{\Lambda^2}} \tag{17}$$

Substituting Eq. (16) into Eq. (15), we get

$$g''(\rho) = \left[2\pi - \frac{2(\sigma+1)}{\rho}\right]g'(\rho) + \left[\frac{2\pi(\sigma+1)}{\rho} - \frac{2ek_xB_0}{\Lambda^2}\frac{1}{\rho}\right]g(\rho) \tag{18}$$

This equation is in appropriate form to start the AIM iterations with the functions

$$\lambda_0(\rho) = \left[2\pi - \frac{2(\sigma+1)}{\rho}\right] \tag{19}$$

$$s_0(\rho) = \left[\frac{2\pi(\sigma+1)}{\rho} - \frac{2ek_xB_0}{\Lambda^2}\frac{1}{\rho}\right] \tag{20}$$

First few iterations of AIM give

$$\pi_0 = \frac{ek_xB_0}{\Lambda^2(\sigma+1)}, \quad \pi_1 = \frac{ek_xB_0}{\Lambda^2(\sigma+2)}, \quad \pi_2 = \frac{ek_xB_0}{\Lambda^2(\sigma+3)} \quad \text{and} \quad \pi_3 = \frac{ek_xB_0}{\Lambda^2(\sigma+4)}$$

So, we can generalize π as

$$\pi_n = \frac{ek_x B_0}{\Lambda^2(n+\sigma+1)} \qquad (n = 0,1,2,3,\dots) \tag{21}$$

Thus, using $\pi^2 = \frac{k_x^2+a^2}{\Lambda^2}$, the separation constant a^2 can be obtained as

$$a_n^2 = k_x^2\left[\frac{e^2B_0^2}{\Lambda^2(n+\sigma+1)^2} - 1\right] \tag{22}$$

As for Eq. (14), it is written in the explicit form as below

$$\left[\frac{d^2}{d\gamma^2} - \frac{2meE_0}{\beta^2}\frac{1}{\gamma} - \eta^2\right]G(\gamma) = 0 \tag{23}$$

where we defined a new variable, $\gamma = \alpha + \beta z$ and η as

$$\eta^2 = -\frac{a^2+2m\varepsilon}{\beta^2} \tag{24}$$

One can choose *G(γ)* , in Eq. (23), as follows

$$G(\gamma) = \gamma e^{-\eta\gamma} f(\gamma)$$

Then it yields

$$f''(\gamma) = \left[2\eta - \frac{2}{\gamma}\right]f'(\gamma) + \left[\frac{2\eta}{\gamma} + \frac{2meE_0}{\beta^2}\frac{1}{\gamma}\right]f(\gamma) \tag{25}$$

In order to follow the procedures of AIM, we use the following functions

$$\lambda_0(\gamma) = \left[2\eta - \frac{2}{\gamma}\right] \tag{26}$$

$$s_0(\gamma) = \left[\frac{2\eta}{\gamma} + \frac{2meE_0}{\beta^2}\frac{1}{\gamma}\right] \tag{27}$$

According to first few AIM iterations, we have

$$\eta_0 = -\frac{meE_0}{\beta^2},\ \ \eta_1 = -\frac{meE_0}{2\beta^2},\ \ \eta_2 = -\frac{meE_0}{3\beta^2} \ \text{ and } \ \eta_3 = -\frac{meE_0}{4\beta^2}$$

Then we obtain the exact form of generalized η

$$\eta_l = -\frac{meE_0}{(l+1)\beta^2} \qquad (\mathrm{l} = 0,1,2,3,\dots) \tag{28}$$

Reminding Eqs. (22) and (24), we obtain the exact energy eigenvalues of Schrödinger equation for the case $\vec{E} \parallel \vec{B}$ as

$$\varepsilon_{||n,l}^{Sch} = -\frac{1}{2m}\left\{\frac{m^2e^2E_0^2}{(l+1)^2\beta^2} + k_x^2\left[\frac{e^2B_0^2}{\Lambda^2(n+\sigma+1)^2} - 1\right]\right\} \tag{29}$$

The wave functions belonging to Eqs. (11) and (12) are determined in two steps. The functions given in Eqs. (19) and (20) are used to obtain the function generator $g_n(\rho) = \exp\left(-\int^{\rho}\frac{s_n(t)}{\lambda_n(t)}dt\right)$. We get the following results according to first few AIM iterations

$$g_0 = 1,$$
$$g_1 = -(2\sigma+2)(\sigma+2)\left[1-\frac{2\pi_1\rho}{2\sigma+2}\right],$$
$$g_2 = (2\sigma+2)(2\sigma+3)(\sigma+3)^2\left[1-\frac{4\pi_2\rho}{2\sigma+2}+\frac{4\pi_2^2\rho^2}{(2\sigma+2)(2\sigma+3)}\right],$$
$$g_3 = -(2\sigma+2)(2\sigma+3)(2\sigma+4)(\sigma+4)^3\left[1-\frac{6\pi_3\rho}{2\sigma+2}+\frac{12\pi_3^2\rho^2}{(2\sigma+2)(2\sigma+3)}-\frac{8\pi_3^3\rho^3}{(2\sigma+2)(2\sigma+3)(2\sigma+4)}\right],$$
$$g_4 = (2\sigma+2)(2\sigma+3)(2\sigma+4)(2\sigma+5)(\sigma+5)^4\left[1-\frac{8\pi_4\rho}{2\sigma+2}+\frac{24\pi_4^2\rho^2}{(2\sigma+2)(2\sigma+3)}-\frac{32\pi_4^3\rho^3}{(2\sigma+2)(2\sigma+3)(2\sigma+4)}+\frac{16\pi_4^4\rho^4}{(2\sigma+2)(2\sigma+3)(2\sigma+4)(2\sigma+5)}\right]$$

So, the function generator can be generalized as

$$g_n(\rho) = (-1)^n(\sigma+n+1)^n[\prod_{k=2}^{n+1}(2\sigma+k)]_1F_1(-n;2\sigma+2;2\pi_n\rho) \tag{30}$$

where ${}_1F_1(a;b;z)$ is the confluent hypergeometric function of the first kind. By using Eq. (30), the eigenfunction of Eq. (15) is obtained as

$$F_n(\rho) = (-1)^n\rho^{\sigma+1}e^{-\pi_n\rho}(\sigma+n+1)^n[\prod_{k=2}^{n+1}(2\sigma+k)]_1F_1(-n;2\sigma+2;2\pi_n\rho) \tag{31}$$

The eigenfunction of Eq. (23) is determined in the same manner we followed in Eq. (15). Using the functions in Eqs. (26) and (27), AIM iterations give the following expressions for the $f_l(\gamma)$ function generator:

$$f_0 = 1,$$
$$f_1 = 2\cdot 2\left[1-\frac{2\eta_1\gamma}{2}\right],$$
$$f_2 = 2\cdot 3\cdot 3^2\left[1-\frac{4\eta_2\gamma}{2}+\frac{8\eta_2^2\gamma^2}{2\cdot 3\cdot 2!}\right],$$
$$f_3 = 2\cdot 3\cdot 4\cdot 4^3\left[1-\frac{6\eta_3\gamma}{2}+\frac{24\eta_3^2\gamma^2}{2\cdot 3\cdot 2!}-\frac{48\eta_3^3\gamma^3}{2\cdot 3\cdot 4\cdot 3!}\right]$$

Then we generalize the $f_l(\gamma)$ as follows

$$f_l(\gamma) = (l+1)^l[\prod_{t=2}^{l+1}k]_1F_1(-l;2;2\eta_l\gamma) \tag{32}$$

So, the eigenfunction of Eq. (23) is

$$G_l(\gamma) = \gamma e^{-\eta_l\gamma}(l+1)^l[\prod_{t=2}^{l+1}k]_1F_1(-l;2;2\eta_l\gamma) \tag{33}$$

and the wavefunction of Schrödinger equation for the case $\vec{E} \parallel \vec{B}$ is obtained as below

$$\psi_{||n,l}^{Sch} = N_1e^{i(xk_x-\varepsilon t)}\gamma\rho^{\sigma+1}e^{-(\pi_n\rho+\eta_l\gamma)}(l+1)^l[-(\sigma+n+1)]^n\left(\prod_k^{n+1}2\sigma+k\right)\left(\prod_{t=2}^{l+1}t\right)$$
$${}_1F_1(-n;2\sigma+2;2\pi_n\rho)\ {}_1F_1(-l;2;2\eta_l\gamma) \tag{34}$$

where N_1 is normalization constant and *n*,*l*=0, 1, 2, 3....

3.1.2. The case of $\vec{E} \perp \vec{B}$

For perpendicular orientation of the external fields, the particle is assumed as moving freely on *xz*-plane, and the wave function of the particle can be written as

$$\Psi_{\perp}^{Sch}(x,y,z,t) = e^{i(xk_x+zk_z-\varepsilon t)}\phi(y)$$

The Schrödinger equation for this wavefunction is

$$\left[\frac{d^2}{d\theta^2} - \frac{e^2B_0^2}{\Lambda^2}\frac{1}{\theta^2} + \frac{2e(k_xB_0-mE_0)}{\Lambda^2}\frac{1}{\theta} - \Sigma^2\right]\phi(\theta) = 0 \tag{35}$$

where $\theta = \Gamma + \Lambda y$ and $\Sigma^2 = \frac{k_x^2+k_z^2-2m\varepsilon}{\Lambda^2}$. By defining $\phi(\theta)$ as $\phi(\theta) = \theta^{\tau+1}e^{-\Sigma\theta}f(\theta)$

Eq. (35) yields

$$f''(\theta) = \left[2\Sigma - \frac{2(\tau+1)}{\theta}\right]f'(\theta) + \left[\frac{2\Sigma(\tau+1)}{\theta} - \frac{2e(k_xB_0-mE_0)}{\Lambda^2}\frac{1}{\theta}\right]f(\theta) \tag{36}$$

where $\tau = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{e^2B_0^2}{\Lambda^2}}$. This equation is in suitable form for the AIM iterations with

$$\lambda_0(\theta) = \left[2\Sigma - \frac{2(\tau+1)}{\theta}\right] \tag{37}$$

$$s_0(\theta) = \left[\frac{2\Sigma(\tau+1)}{\theta} - \frac{2e(k_xB_0-mE_0)}{\Lambda^2}\frac{1}{\theta}\right] \tag{38}$$

Following expressions are acquired with reference to first few AIM iterations

$$\Sigma_0 = \frac{e(k_xB_0-mE_0)}{\Lambda^2(\tau+1)}, \quad \Sigma_1 = \frac{e(k_xB_0-mE_0)}{\Lambda^2(\tau+2)},$$
$$\Sigma_2 = \frac{e(k_xB_0-mE_0)}{\Lambda^2(\tau+3)}, \quad \Sigma_3 = \frac{e(k_xB_0-mE_0)}{\Lambda^2(\tau+4)}$$

So, the general form of Σ is

$$\Sigma_n = \frac{e(k_xB_0-mE_0)}{\Lambda^2(n+\tau+1)} \qquad (\mathrm{n} = 0,1,2,3,\ldots) \tag{39}$$

Thus, we get the energy eigenvalues for the case $\vec{E} \perp \vec{B}$ as follows

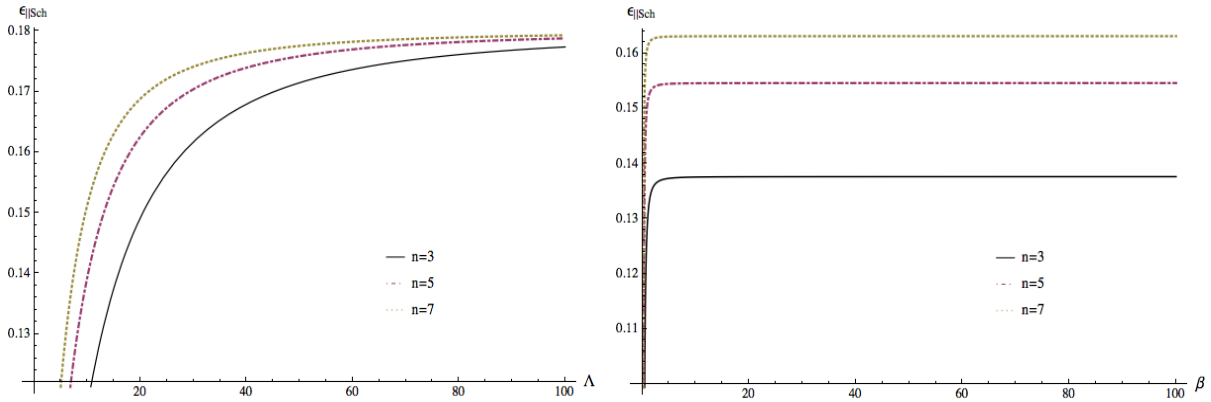
$$\varepsilon_{\perp n}^{Sch} = -\frac{1}{2m}\left\{\left[\frac{e(k_xB_0-mE_0)}{\Lambda(n+\tau+1)}\right]^2 - (k_x^2+k_z^2)\right\} \tag{40}$$

The wavefunction for the case of $\vec{E} \perp \vec{B}$ can easily be determined by using the fact that Eq. (39) and Eq. (21) are in the same form. So, we can obtain the below given eigenfunction of the case $\vec{E} \perp \vec{B}$ in the same manner we used for the case of $\vec{E} // \vec{B}$

$$\Psi_{\perp n}^{Sch} = N_2 e^{i(xk_x+zk_z-\varepsilon t)}\theta^{\tau+1}e^{-\Sigma_n\theta}(-n-\tau-1)^n\left[\prod_{k=2}^{n+1}(2\tau+k)\right]{}_1F_1(-n;2\tau+2;2\Sigma_n\theta) \tag{41}$$

where N_2 is normalization constant and *n*=0,1,2,3,...

Figure 1 depicts the variation of energy eigenvalues versus the parameters Λ and β for the case $\vec{E}||\vec{B}$, whereas, Figure 2 illustrates Λ dependency of energy eigenvalues for the case $\vec{E} \perp \vec{B}$.



(a) (b)

Figure 1. Variation of the energy eigenvalues versus the parameters Λ (a) and β (b) for the case $\vec{E}||\vec{B}$. We set $m = q = 1, k_x \equiv p_x = 0.6, E_0 = 0.5, B_0 = 50$ taking $\beta = 10$ (a) and $\Lambda = 15$ (b).

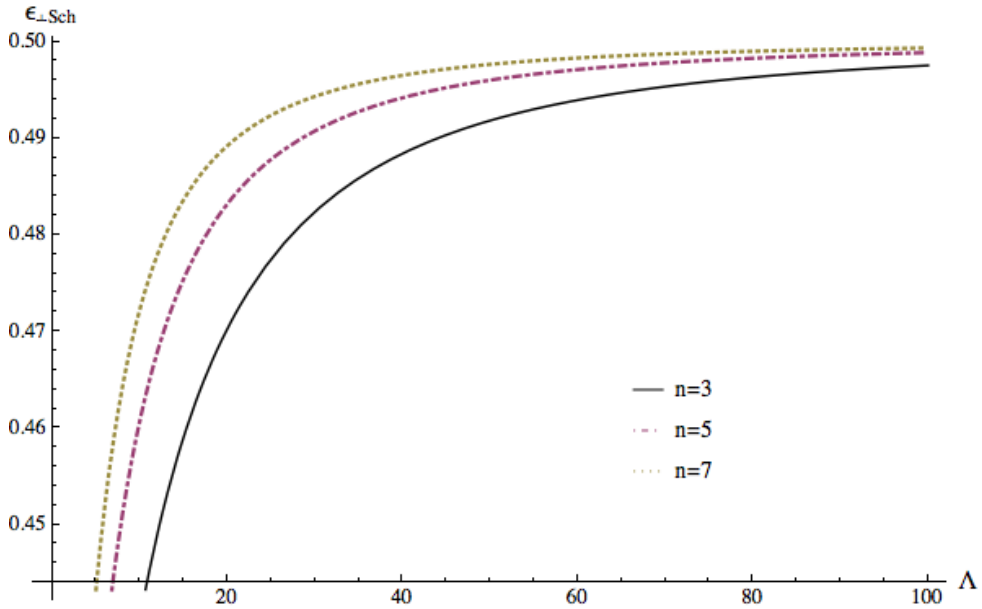


Figure 2. Λ dependency of the energy eigenvalues for the case $\vec{E} \perp \vec{B}$ taking $m = q = 1, k_x \equiv p_x = 0.6,\ k_z \equiv p_z = 0.8,\ E_0 = 0.5, B_0 = 50$ and $\beta = 10$.

3.2. Solutions of Klein-Gordon Equation

Klein-Gordon equation for any relativistic scalar particle interacting with external electric and magnetic fields is given by (Greiner, 1997) ($\hbar = c = 1$)

$$\left[\left(\vec{p} - e\vec{A}\right)^2 + m^2\right]\Psi = (p_0 - eA_0)^2\Psi$$

where e and m are charge and mass of the spin-zero particle, respectively, and $A_\mu = \left(A_0, \vec{A}\right)$ is four-vector electromagnetic potential.

3.2.1. The case of $\vec{E} \parallel \vec{B}$

Regarding to the selected electromagnetic potential, the wavefunction is written as

$$\Psi_{\|}^{KG}(x, y, z, t) = e^{i(xk_x - \varepsilon t)}H(y)R(z)$$

By using Eq.(2), the Klein-Gordon equation yields

$$\left[\hat{D}(y) - b^2\right]H(y) = 0 \tag{42}$$

$$\left[\hat{O}(z) + b^2\right]R(z) = 0 \tag{43}$$

where

$$\hat{D}(y) = \hat{p}^2(y) + \left(k_x - \frac{eB_0}{\Gamma+\Lambda y}\right)^2 + m^2 \tag{44}$$

$$\hat{O}(z) = \hat{p}^2(z) - \left(\varepsilon - \frac{eE_0}{\alpha+\beta z}\right)^2 \tag{45}$$

and b^2 is separation constant. By changing the variable such as $\delta = \Gamma + \Lambda y$, Eq. (42) can be given as

$$\left[\frac{d^2}{d\delta^2} - \frac{e^2B_0^2}{\Lambda^2}\frac{1}{\delta^2} + \frac{2ek_xB_0}{\Lambda^2}\frac{1}{\delta} - \xi^2\right]H(\delta) = 0 \tag{46}$$

where $\xi^2 = \frac{k_x^2+m^2-b^2}{\Lambda^2}$. By writing $H_{(\delta)}$ as

$$H(\delta) = \delta^{\mu+1}e^{-\xi\delta}f(\delta)$$

Eq. (46) is reduced to

$$f''(\delta) = \left[2\xi - \frac{2(\mu+1)}{\delta}\right]f'(\delta) + \left[\frac{2\xi(\mu+1)}{\delta} - \frac{2ek_xB_0}{\Lambda^2}\frac{1}{\delta}\right]f(\delta) \tag{47}$$

where $\mu = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{e^2B_0^2}{\Lambda^2}}$. So, AIM iterations are initiated using

$$\lambda_0(\delta) = \left[2\xi - \frac{2(\mu+1)}{\delta}\right] \tag{48}$$

$$s_0(\delta) = \left[\frac{2\xi(\mu+1)}{\delta} - \frac{2ek_xB_0}{\Lambda^2}\frac{1}{\delta}\right] \tag{49}$$

First few AIM iterations give the following results

$$\xi_0 = \frac{ek_xB_0}{\Lambda^2(\mu+1)}, \quad \xi_1 = \frac{ek_xB_0}{\Lambda^2(\mu+2)}, \quad \xi_2 = \frac{ek_xB_0}{\Lambda^2(\mu+3)} \quad \text{and} \quad \xi_3 = \frac{ek_xB_0}{\Lambda^2(\mu+4)}$$

With the usage of these expressions, the analytical form of ξ is achieved as

$$\xi_n = \frac{ek_xB_0}{\Lambda^2(n+\mu+1)} \qquad (\mathrm{n} = 0,1,2,3,\ldots) \tag{50}$$

If we use the relation $\xi^2 = \frac{k_x^2+m^2-b^2}{\Lambda^2}$, b^2 separation constant is derived as

$$b_n^2 = k_x^2\left[1 - \frac{e^2B_0^2}{\Lambda^2(n+\mu+1)^2}\right] + m^2 \tag{51}$$

For Eq.(43), changing the variable as $\nu = \alpha + \beta z$ gives follows

$$\left[\frac{d^2}{d\nu^2} + \frac{e^2E_0^2}{\beta^2}\frac{1}{\nu^2} - \frac{2e\varepsilon E_0}{\beta^2}\frac{1}{\nu} - \Delta^2\right]R(\nu) = 0 \tag{52}$$

where $\Delta^2 = \frac{b^2-\varepsilon^2}{\beta^2}$.

One can assume $R(\upsilon)$ as

$$R(\nu) = \nu^{\Omega+1} e^{-\Delta\nu} g(\nu)$$

with $\Omega = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{e^2E_0^2}{\beta^2}}$. Then Eq. (52) reduces to

$$g''(\nu) = \left[2\Delta - \frac{2(\Omega+1)}{\nu}\right] g'(\nu) + \left[\frac{2\Delta(\Omega+1)}{\nu} + \frac{2e\varepsilon E_0}{\beta^2}\frac{1}{\nu}\right] g(\nu) \tag{53}$$

Which is in AIM form, where

$$\lambda_0(\nu) = \left[2\Delta - \frac{2(\Omega+1)}{\nu}\right] \tag{54}$$

$$s_0(\nu) = \left[\frac{2\Delta(\Omega+1)}{\nu} + \frac{2e\varepsilon E_0}{\beta^2}\frac{1}{\nu}\right] \tag{55}$$

in accordance to Eq. (3) and Eq. (53). Then first few AIM iterations give

$$\Delta_0 = -\frac{e\varepsilon E_0}{\beta(\Omega+1)},\ \ \Delta_1 = -\frac{e\varepsilon E_0}{\beta(\Omega+2)},\ \ \Delta_2 = -\frac{e\varepsilon E_0}{\beta(\Omega+3)} \text{ and } \Delta_3 = -\frac{e\varepsilon E_0}{\beta(\Omega+4)}$$

and the general form of Δ is

$$\Delta_l = -\frac{e\varepsilon E_0}{\beta^2(l+\Omega+1)} \qquad (l = 0,1,2,3,\dots) \tag{56}$$

So, using $\Delta^2 = \frac{b^2-\varepsilon^2}{\beta^2}$, following exact energy eigenvalues are got

$$\varepsilon^{KG}_{||n,l} = \pm\sqrt{\frac{b_n^2[\beta(l+\Omega+1)]^2}{e^2E_0^2+[\beta(l+\Omega+1)]^2}} \tag{57}$$

for the case of $\vec{E} \parallel \vec{B}$ where b_n^2 is given as Eq. (51).

One can attain the wave function for this case through the same procedures of Schrödinger equation. If $\lambda_0(\delta)$ and $s_0(\delta)$, given in Eqs. (48) and (49), are used for AIM iterations one can get the $f_n(\delta) = \exp\left(-\int^{\delta} \frac{s_n(t)}{\lambda_n(t)}dt\right)$ function generator for Eq. (47). Then following results are obtained

$f_0 = 1,$

$f_1 = -(2\mu+2)(\mu+2)\left[1 - \frac{2\xi_1\delta}{2\mu+2}\right],$

$f_2 = (2\mu+2)(2\mu+3)(\mu+3)^2\left[1 - \frac{4\xi_2\delta}{2\mu+2} + \frac{4\xi_2^2\delta^2}{(2\mu+2)(2\mu+3)}\right],$

$f_3 = -(2\mu+2)(2\mu+3)(2\mu+4)(\mu+4)^3\left[1 - \frac{6\xi_3\delta}{2\mu+2} + \frac{12\xi_3^2\delta^2}{(2\mu+2)(2\mu+3)} - \frac{8\xi_3^3\delta^3}{(2\mu+2)(2\mu+3)(2\mu+4)}\right]$

Thus, the general form of $f_n(\delta)$ is obtained as

$$f_n(\delta) = (-1)^n(n+\mu+1)^n[\textstyle\prod_{k=2}^{n+1}(2\mu+k)]_1F_1(-n; 2\mu+2; 2\xi_n\delta) \tag{58}$$

and eigenfunction of Eq. (46) is

$$H_n(\delta) = (-1)^n e^{-\xi_n \delta} \delta^{\mu+1} (n+\mu+1)^n [\prod_{k=2}^{n+1} (2\mu+k)]_1 F_1(-n; 2\mu+2; 2\xi_n \delta) \quad (59)$$

The wavefunction related to the motion in *z*-direction is obtained via $g_n(\upsilon) = \exp\left(-\int^{\upsilon} \frac{s_l(t)}{\lambda_l(t)} dt \right)$ function generator for Eq. (53). Thus, $\lambda_0(\nu)$ and $s_0(\nu)$ functions given in Eqs. (54) and (55) should be used in AIM iterations. We have following expressions

$$g_0 = 1,$$
$$g_1 = (2\Omega+2)(\Omega+2)\left[1 - \frac{2\Delta_1 \nu}{2\Omega+2}\right],$$
$$g_2 = (2\Omega+2)(2\Omega+3)(\Omega+3)^2\left[1 - \frac{4\Delta_2 \nu}{2\Omega+2} + \frac{4\Delta_2^2 \nu^2}{(2\Omega+2)(2\Omega+3)}\right],$$
$$g_3 = (2\Omega+2)(2\Omega+3)(2\Omega+4)(\Omega+4)^3\left[1 - \frac{6\Delta_3 \nu}{2\Omega+2} + \frac{12\Delta_3^2 \nu^2}{(2\Omega+2)(2\Omega+3)} - \frac{8\Delta_3^3 \nu^3}{(2\Omega+2)(2\Omega+3)(2\Omega+4)}\right]$$

pursuant to AIM iterations. Then, the general form of $g_l(\nu)$ is got as follow

$$g_l(\nu) = (l+\Omega+1)^l \left[\prod_{d=2}^{l+1} (2\Omega+d)\right]_1 F_1(-l; 2\Omega+2; 2\Delta_l \nu) \quad (60)$$

So, the eigenfunction of Eq. (52) is found as

$$R_l(\nu) = \nu^{\Omega+1} e^{-\Delta_l \nu} (l+\Omega+1)^l \left[\prod_{d=2}^{l+1} (2\Omega+d)\right]_1 F_1(-l; 2\Omega+2; 2\Delta_l \nu) \quad (61)$$

Therefore, the wavefunction for the case $\vec{E} \parallel \vec{B}$ is achieved as below

$$\psi_{||n,l}^{KG} = N_3 e^{i(xk_x - \varepsilon t)} \delta^{\mu+1} \nu^{\Omega+1} e^{-(\xi_n \delta + \Delta_l \nu)} (l+\Omega+1)^l [-(\mu+n+1)]^n \left(\prod_{k=2}^{n+1} 2\mu+k \right)$$
$$\left(\prod_{d=2}^{l+1} 2\Omega+d\right)\ {}_1F_1(-n; 2\mu+2; 2\xi_n \delta)\ {}_1F_1(-l; 2\Omega+2; 2\Delta_l \nu) \quad (62)$$

where N_3 is normalization constant and *n*,*l*=0,1,2,3,....

3.2.2. The case of $\vec{E} \perp \vec{B}$

By introducing the wavefunction as

$$\Psi_{\perp}^{KG}(x,y,z,t) = e^{i(xk_x + zk_z - \varepsilon t)} D(y)$$

and using the electromagnetic potential given by Eq. (2), The Klein-Gordon equation for the case $\vec{E} \perp \vec{B}$ is reduced to the form

$$\left[\frac{d^2}{d\omega^2} - \frac{e^2(B_0^2 - E_0^2)}{\Lambda^2}\frac{1}{\omega^2} + \frac{2e(k_x B_0 - \varepsilon E_0)}{\Lambda^2}\frac{1}{\omega} - \chi^2\right] D(\omega) = 0 \quad (63)$$

where we introduced a new variable $\omega = \Gamma + \Lambda y$, and made a definition $\chi^2 = \frac{k_x^2 + k_z^2 + m^2 - \varepsilon^2}{\Lambda^2}$. Taking into consideration the domain of the problem, the wavefunction $D(\omega)$ can be taken as

$$D(\omega) = \omega^{\varpi+1} e^{-\chi\omega} f(\omega)$$

with $\varpi = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{e^2(B_0^2 - E_0^2)}{\Lambda^2}}$. Then, the following equation can be obtained

$$f''(\omega) = \left[2\chi - \frac{2(\varpi+1)}{\omega}\right] f'(\omega) + \left[\frac{2\chi(\varpi+1)}{\omega} - \frac{2e(k_x B_0 - \varepsilon E_0)}{\Lambda^2}\frac{1}{\omega}\right] f(\omega) \tag{64}$$

in which the AIM iterations can be initialized with the following functions

$$\lambda_0(\omega) = \left[2\chi - \frac{2(\varpi+1)}{\omega}\right] \tag{65}$$

$$s_0(\omega) = \left[\frac{2\chi(\varpi+1)}{\omega} - \frac{2e(k_x B_0 - \varepsilon E_0)}{\Lambda^2}\frac{1}{\omega}\right] \tag{66}$$

First few AIM iterations give the below results

$$\chi_0 = \frac{e(k_x B_0 - \varepsilon E_0)}{\Lambda^2(\varpi+1)}, \quad \chi_1 = \frac{e(k_x B_0 - \varepsilon E_0)}{\Lambda^2(\varpi+2)}$$
$$\chi_2 = \frac{e(k_x B_0 - \varepsilon E_0)}{\Lambda^2(\varpi+3)}, \quad \chi_3 = \frac{e(k_x B_0 - \varepsilon E_0)}{\Lambda^2(\varpi+4)}$$

from which, the general expression for χ is derived as

$$\chi_n = \frac{e(k_x B_0 - \varepsilon E_0)}{\Lambda^2(n+\varpi+1)} \qquad (\mathrm{n} = 0,1,2,3,\ldots) \tag{67}$$

Taking into consideration the definition $\chi^2 = \frac{k_x^2 + k_z^2 + m^2 - \varepsilon^2}{\Lambda^2}$, one can obtain the following second order linear equation for ε

$$c_1\varepsilon^2 + c_2\varepsilon + c_3 = 0 \tag{68}$$

So, the exact energy eigenvalues for the case $\vec{E} \perp \vec{B}$ is obtained as follows

$$\varepsilon_{\perp n}^{KG} = \frac{-c_2 \pm \sqrt{c_2^2 - 4c_1c_3}}{2c_1} \tag{69}$$

where

$c_1 = e^2E_0^2 + \Lambda^2(n+\varpi+1)^2$, $c_2 = -2e^2k_xB_0E_0$

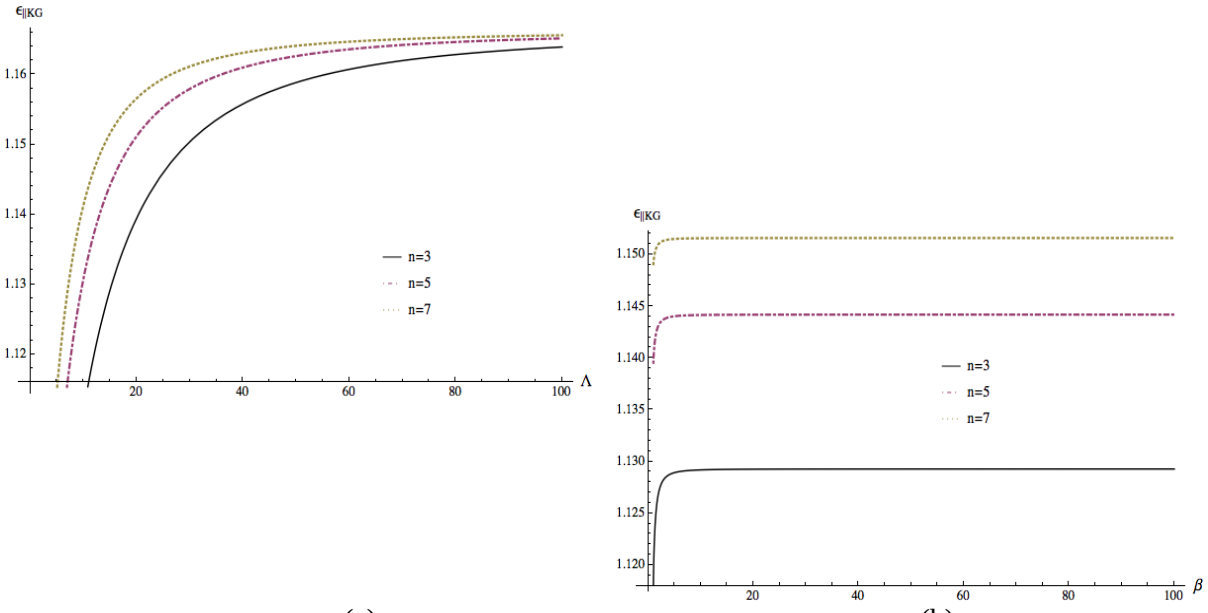
$c_3 = e^2k_x^2B_0^2 - (k_x^2 + k_z^2 + m^2)\Lambda^2(n+\varpi+1)^2$

The wavefunction for the case of $\vec{E} \perp \vec{B}$ can be achieved by comparing the similar forms of Eq. (50) and Eq. (67). So, one can suppose the following eigenfunction for the case $\vec{E} \perp \vec{B}$

$$\Psi_{\perp n}^{KG} = N_4 e^{i(xk_x + zk_z - \varepsilon t)} \omega^{\varpi+1} e^{-\chi_n \omega} (-n-\varpi-1)^n \left[\prod_{k=2}^{n+1}(2\varpi+k)\right]_1 F_1$$
$$(-n; 2\varpi+2; 2\chi_n\omega) \tag{70}$$

where N_4 is normalization constant.

Variations of energy eigenvalues versus the parameters Λ and β for the case $\vec{E}||\vec{B}$ are shown in Figure 3. Figure 4 shows Λ dependency of energy eigenvalues for the case $\vec{E} \perp \vec{B}$.



(a) (b)

Figure 3: Variations of energy eigenvalues of Klein-Gordon equation versus the parameters Λ (a) and β (b) for the case $\vec{E} \parallel \vec{B}$. We set $m = q = 1, k_x = p_x = 0.6, E_0 = 0.5, B_0 = 50$ taking $\beta = 10$ (a) and Λ=15 (b).

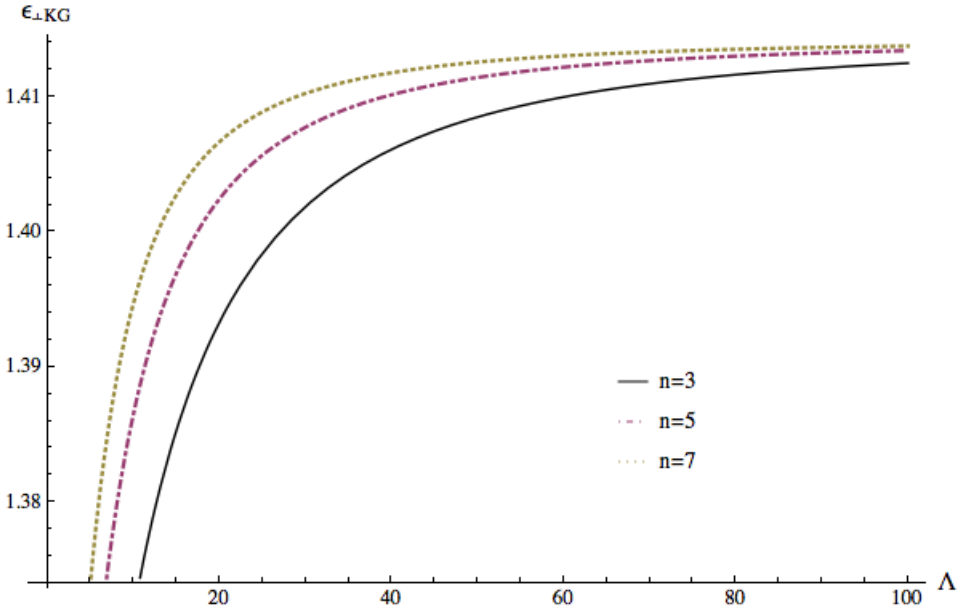


Figure 4: Λ dependency of the energy eigenvalues of Klein-Gordon equation for the case $\vec{E} \perp \vec{B}$ taking $m = q = 1, k_x = p_x = 0.6, k_z = p_z = 0.8, E_0 = 0.5, B_0 = 50$ and $\beta = 10$.

Figure 3 and Figure 4 show us that energy eigenvalues have weak commitment to the parameter β. This is the case which we have also identified for the non-relativistic equation. We compare the energy eigenvalues of Schrödinger and Klein-Gordon equations for the both $\vec{E} \parallel \vec{B}$ and $\vec{E} \perp \vec{B}$ cases in Table 1 and Table 2, respectively. The eigenvalues are written in terms of Λ parameter in these tables.

Table 1. Energy eigenvalues of Schrödinger and Klein-Gordon equations for certain values of Λ. We set $m = q = 1, p_x = 0.6, E_0 = 0.5, B_0 = 50$ and $\beta = 10$.

n	Λ	$\varepsilon_{\|\|Sch}$	$\varepsilon_{\|\|KG}$
0	2.5	0.007528	1.007470
	5	0.015872	1.015700
	15	0.045254	1.044160
1	2.5	0.024018	1.023720
	5	0.043877	1.042930
	15	0.0953811	1.091170
2	2.5	0.037718	1.037020
	5	0.064891	1.062890
	15	0.121830	1.115160
3	2.5	0.049615	1.048430
	5	0.081339	1.078260
	15	0.137554	1.129190

Table 2. Energy eigenvalues of Schrödinger and Klein-Gordon equations for certain values of Λ. We set $m = q = 1, p_x = 0.6, p_z = 0.8, E_0 = 0.5, B_0 = 50$ and $\beta = 10$.

n	Λ	$\varepsilon_{\|\|Sch}$	$\varepsilon_{\|\|KG}$
0	2.5	0.334438	1.293110
	5	0.342506	1.299300
	15	0.370917	1.320840
1	2.5	0.349476	1.304620
	5	0.368679	1.319150
	15	0.418480	1.356060
2	2.5	0.362555	1.314540
	5	0.388830	1.334220
	15	0.443887	1.374480
3	2.5	0.374001	1.323150
	5	0.404676	1.345940
	15	0.459033	1.385330

4. Conclusion

We study the motion of the spinless non-relativistic and relativistic particles in the presence of parallel and orthogonal electric and magnetic fields, which can be derived from Eqs. (1) and (2), via Asymptotic Iteration Method (AIM). The resulting wave-functions of the studied quantum systems show us that the relativistic effects arise for the motion on both *y*- and *z*-directions.

On the other hand, the energy eigenvalues against the beta and lambda parameters are drawn for non-relativistic and relativistic cases in Figures 1-4. A first look at the plots shows us that while the motion on the *y*-direction contributes to increase of the energy, the motion on the *z*-direction has no effect on the increase of energy spectrum, namely energy spectrum becomes constant after a certain value of β. Therefore, while reduction of the magnetic field due to the increase of Λ has no effect in the increase of the energy spectrum, the reduction of electric field due to the increase of the β causes the energy eigenvalues to be constant. Such a variation on the energy eigenvalues upon the Λ is due to the term $-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{e^2 B_0^2}{\Lambda^2}}$ in the denominator. This outcome applies for the both Schrödinger and Klein-Gordon equations. In addition to this, energy eigenvalues of Klein-Gordon equation are about one mass greater than the ones obtained for Schrödinger equation as expected. This can also be clearly seen from Table 1 and Table 2. From Figures 1-4 and Table 1 and Table 2, it can be deduced in the case of $\vec{E} \parallel \vec{B}$ that the external magnetic field bounds the system, since the

energy eigenvalues specifically vary with respect to the Λ. In this case, it can also be said that the time-dependent electric field is responsible for the particle production, as expected (Sogut and Havare, 2015). As for the $\vec{E} \perp \vec{B}$ case, one could not interpret which field component dominate the energy eigenvalues since the Λ is used for both electric and magnetic fields. Besides, one can refer the differences on the energy eigenvalues of the particle for the cases where the electric and magnetic fields are perpendicular and parallel to each other, seperately (see in Table 1 and Table 2), can be refered to the Poynting Theorem in the classical electromagnetic theory (Griffiths, 1991).

Finally, it is also seen from Table 1 and Table 2 that as the Λ parameter increases, the difference between Klein-Gordon and Schrödinger energy eigenvalues falls below the mass difference (i.e., m = 1) between these two equations. This can be explained by that Schrödinger equation approaches to the relativistic case (i.e., Klein-Gordon case) as the parameter Λ increases.

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