# A Note on Entropy Subsethood Relationship

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*Abstract: We comment on subsethood measures defined by Kosko and Young and give some new aspects of these measures. Finally we would like to discard the entropy subsethood relationship established by the authors. We present some properties of subsethood measure from set theoretic approach and also from axiomatic approach with the expectation that these would help in removing the shortcomings that currently exist in these definitions.*

## 1. Introduction

Since fuzzy set was introduced by Zadeh [1], in 1965, inclusion measure and fuzzy entropy have become two important topics in fuzzy set theory and have successfully been applied in many fiekds such as image processing, fuzzy neural networks, fuzzy reasoning and fuzzy control.

Inclusion measure of fuzzy sets indicates the degree to which a fuzzy set A is contained in another fuzzy set B. Zadeh [1] first gave the definition of fuzzy set inclusion in a crisp relation , in another word, a fuzzy set A is either included or not included in fuzzy set B. This is defined as follows

A fuzzy set A in the Universe U is a subset of another fuzzy set B if for every element x in U, its membership degree in A is less than or equal to the membership degree in B. This can formally be stated as

$$A \subseteq B \Longrightarrow \mu_A(x) \leq \mu_B(x) \qquad (1)$$

The graphical representation of this definition is, he defines the fuzzy set A as a subset of another fuzzy set B if graph of A never goes above the graph of B.

Later on it was realized that defining fuzzy subsethood in this way is though highly appreciable and useful but it still against the spirit of fuzzy set theory in the sense that it presents a crisp decision about being a subset or not. For many researchers working in the area of fuzzy subsethood considered this definition to be too rigid and so remained very interested in assigning a degree of inclusion of one fuzzy set into another. As a result more variant and expressions of fuzzy subsethood for two fuzzy sets were studied (see for example [9-11]).

It is known that Kosko [2] and Young [3] had considered the connection between fuzzy subsethood and entropy of fuzzy sets. It is for this reason in this article, we shall discuss about the two most commonly used subsethood measures of which one is set theoretic approach given by Kosko and the other is the axiomatization principle of Young.

Kosko criticizes the definition of fuzzy set containment given by

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Zadeh, pointing out that if this inequality hold for all but for just a few x, we can still consider A to be subset of B to some degree. He proposed a definition of subsethood of two fuzzy sets from the point of view of set theoretic approach and this is as follows:

$$S(A,B) = \frac{M(A \cap B)}{M(A)} \qquad (2)$$

where S(A,B) stands for the degree of subsethood of A in B and $M(A)$ and $M(A \cap B)$ are the cardinalities of the fuzzy sets A and $A \cap B$ respectively which are defined in the following way

$$M(A) = \sum_i \mu_A(x_i), \forall x_i \epsilon X \qquad (3)$$

and

$$M(A \cap B) = \sum_i \mu_{A \cap B}(x_i), \forall x_i \epsilon X \qquad (4)$$

After that Sinha and Doughtery [4], introduced axiomatic definition of inclusion measure of fuzzy sets.

On the basis of Kosko's subsethood measure, fuzzy entropy and Willmott's work, Young defined the concept of subsethood measure and weak subsethood measure, which can be summarised as follows:

A real function $c: F(X) \times F(X) \rightarrow [0,1]$ is called a subsethood measure, if c has the following properties

(C1) C(A,B)=1 if and only if $A \subseteq B$, i.e $\mu_A(x) \leq \mu_B(x)$

(C2) If $\left[\frac{1}{2}\right] \subseteq A$ then $c(A, A^c) = 0$, if and only if A=X.

(C3)If $A \subseteq B \subseteq C$ then $c(C,A) \leq c(B,A)$ and if $A \subseteq B$,

then $c(C,A) \leq c(C,B)$

He further mentioned that if c satisfies (C2) and (C3), then c is called a weak subsethood measure,

Moreover, the Kosko's subsethood measure complies with Young's axiomatic characterization of a fuzzy subsethood measure but does not fit into the framework of Sinha and Doughtery [4].

Entropy of fuzzy set discusses the fuzziness degree of fuzzy set and was first mentioned in 1965 by Zadeh. Several scholars have studied it from different point of view. For example, De Luca and Termini [5] introduced some axioms to describe the fuzziness degree of fuzzy set. Another way given by Yager [6] was to view

the fuzziness degree of fuzzy set in terms of lack of distinction between fuzzy set and its complement. Aimed at these two concepts, Kosko investigated fuzzy entropy in relation to subsethood measure. Kosko [2] has defined the entropy of a fuzzy set A as:

$$E(A) = \frac{M(A \cap A^c)}{M(A \cup A^c)} \qquad (5)$$

Where $M(A \cap A^c)$ and $M(A \cup A^c)$ denote the cardinalities of the sets $A \cap A^c$ and $A \cup A^c$.

To relate subsethood measure with fuzzy entropy, Kosko proposes the following expression, given a subsethood measure c; the entropy e generated by c is defined as
It can be put in form

$$e(A) = c(A \cup A^c, A \cap A^c) \qquad (6)$$

Further, Young [3] sets out her axiomatization imposing that the measure

$$E(A) = S(A \cup A^c, A \cap A^c) \qquad (7)$$

will be a fuzzy entropy.

It is important to mention here that all the results discussed above were obtained with the help of the existing definition of complementation of fuzzy sets which is defined in the following manner:

The membership function of a the complement of a fuzzy set A is defined as

$$\mu_{A^c}(x) = 1 - \mu_A(x), x \epsilon X \qquad (8)$$

In this article, our main intention is to revisit these two definitions from the standpoint of the new definition of complementation and thereafter we would like to put forward some suggestions in this regard. But before proceeding further with our work, we would like to discuss in brief about the new definition of complementation of fuzzy sets as introduced by Baruah [7, 8].

## 2. New Definition of Complementation of Fuzzy Sets

The existence of two functions of which one is called membership function and the other is reference function are required to define a fuzzy set, Baruah [7, 8]. The membership value is the difference between the membership function and reference function with the condition that the membership function should be greater that the reference function.
Accordingly,
A fuzzy set

$$A = \{x, \mu(x), x \epsilon X\} \qquad (9)$$

would be defined in this way as

$$A = \{x, \mu(x), 0, x \epsilon X\} \qquad (10)$$

so that the complement would become

$$A^c = \{x, 1, \mu(x), x \epsilon X\} \qquad (11)$$

It is important to mention here the fact that these definitions would give us the following results
$A \cup A^c = the\ Universal\ set$
And
$A \cap A^c = the\ null\ set$
which violates Zadeh's initial conception that fuzzy sets don't obey excluded middle laws.

Again it is important to note here that according to our definition $A \cap A^c$ is a null set and $A \cup A^c$ is the universal set and since a null set is always a subset of the universal set then we have

$$A \cap A^c \subseteq A \cup A^c \qquad (12)$$

The above result discard Kosko's claim that the universal set can be a subset of any of its subset to some degree.

This concept of complementation of fuzzy sets plays a key role in developing some requirements of subsethood measure of fuzzy sets and thereby effects the relationship between entropy and subsethood established so far.

## 3. Comment on The Entropy Subsethood Relationship

From the above analysis we think that the better form of Young's definition may be the following:
A real function $c: F(X) \times F(X) \to [0,1]$ is called a subsethood measure, if c has the following properties

(C1) c(A,B)=1, if and only if $\mu_A(x) \leq \mu_B(x)$   i.e

the membership values of the fuzzy set A is less than or equal to the membership values of the fuzzy set B.

(C2) $c(A, A^c) = 0, \forall x \epsilon X$

(C3) $c(A \cup A^c, A \cap A^c) = 0, \forall x \epsilon X$

(C4) If $A \subseteq B \subseteq C$ then $c(C, A) \leq c(B, A)$ and if $A \subseteq B$,

then $c(C, A) \leq c(C, B)$

It is important to mention here that the second condition of Young's definition is replaced by a new one and another additional condition is added to make it logical. The reason behind such a proposal can be described by the fact that the membership value of a fuzzy set can never remain included in the membership value of its complement set. The property (C2) comes in this way. Regarding (C3) we have discussed in the previous section.

If these conditions are taken for granted for subsethood then the entropy subsethood relationship proposed by the author would give us no idea of fuzziness of the fuzzy set concerned.

As regards Kosko's entropy subsethood relationship, we would like to draw attention to the following few lines.

By this formula it was derived that $A \cup A^c$ is a sbset of $A \cap A^c$ to some degree and considered this as the unique feature of fuzzy set theory. Thus we can see that the Universe of discourse can also become a subset of any of its own subset to some degree. This is not desirable. Furthermore, the measure of fuzziness which was introduced by the author is also not free from defects as has been discussed in Dhar [12-15].

Another reason for which we would like to discard the subsethood theorem is due to the fact that. It was derived with the help of geometrical representation of fuzzy sets which is again controversial from the standpoints of the new definition of complementation on the basis of reference function. Let us have a look at it in brief in the following

Consider a universe of discourse containing two elements, $U = \{x_1, x_2\}$ The Universal set is represented by the point (1, 1) with the membership function $\mu_A(x_1) = 1$ and $\mu_A(x_2) = 1$. The point (1, 0) represents the set $\{x_1\}$ and the point (0, 1) represents the set. $\{x_2\}$ Similarly, a fuzzy set defined in that universe of discourse $A = \left\{(x_1, \frac{1}{5}), (x_2, \frac{3}{5})\right\}$ is represented by the membership function

$$\mu_A(x_1) = \frac{1}{5} \text{ and } \mu_A(x_2) = \frac{3}{5}$$

Then the representation takes the following form
Then the point A can be represented as a point in two dimensional unit hypercube which is a square. This square represents all possible fuzzy sets of both elements; vertices of the square

represent crisp set.

The idea that the fuzzy set located at the vertex has the zero entropy and the fuzzy set located at the midpoint has the maximum entropy, led Kosko to define fuzzy entropy in the following form:

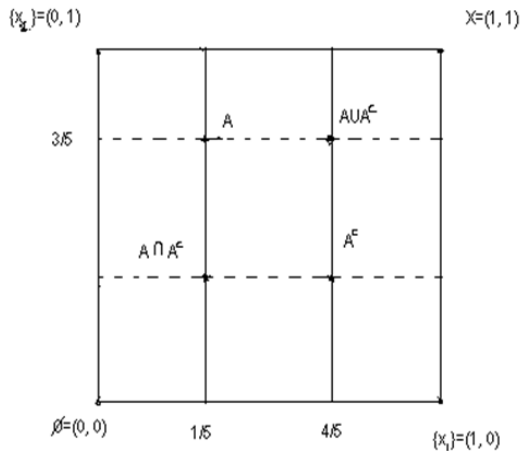$$E(A) = \frac{M(A \cap A^c)}{M(A \cup A^c)} \qquad (13)$$

**Figure 1.** 2- Geometrical representation of fuzzy sets

where $A^c$ stands for the complement of the fuzy set A and M(A) stands for the cardinality of the fuzzy set A.

The geometrical interpretation of fuzzy entropy theorem which is the outcome of geometrical representation of fuzzy sets was presented in the following form:
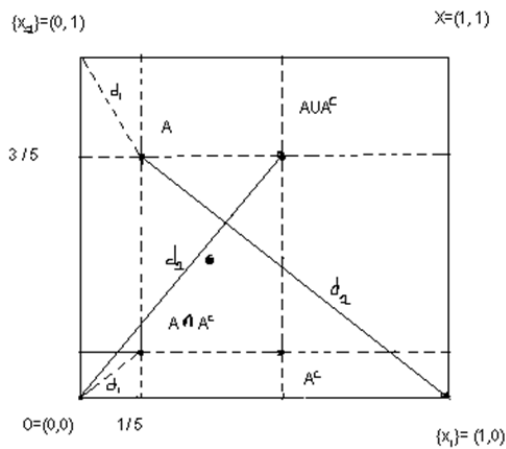
**Figure 2.** Geometrical representation of fuzzy entropy theorem

In the figure, $d_1$ represents M $(A \cap A^c)$ and $d_2$ represents M $(A \cup A^c)$

From the standpoint of the new definition of complementation $A \cap A^c$ coincides with the corner point $\emptyset$ and $A \cup A^c$ coincides with the other corner point X having cardinalities zero and $|X|$ respectively In order to get rid of such types of controversies, here we would like to suggest some properties of subsethood measure of fuzzy sets which is based on membership value.

The concept of new definition of complementation of fuzzy sets has led us to redefine the properties of subsethood of fuzzy sets when expressed in set theoretic form in the following way

(E1)$0 \leq S(A,B) \leq 1$

(E2)S(A,B)=1 if and only if and only if $\mu_A(x) \leq \mu_B(x)$ i.e the membership values of the fuzzy set A is less than or equal to the membership values of the fuzzy set B.

(E3) $S(A, A^c) = 0, \forall x \epsilon X$

(E4) $S(A \cup A^c, A \cap A^c) = 0, \forall x \epsilon X$

(E5) If$A \subseteq B \subseteq C$ then $S(C,A) \leq S(B,A)$ and if $A \subseteq B$,

then $S(C,A) \leq S(C,B)$ At first blush, inclusion measure and entropy of fuzzy sets don't seem to be related. However with respect to specific pair of entropy and inclusion measure of fuzzy sets Kosko [2] and Young showed the result

$$E(A) = S(A \cup A^c, A \cap A^c) \qquad (14)$$

But from the above discussion it is clear that the degree of subsethood of $A \cup A^c$ in $A \cap A^c$is zero and hence we shall get no result from the said entropy subsethood relationship established so far.

## 4. Conclusions

In this article, we have commented on the two most popular entropy subsethood relationships and have given some suggestions. As regard subsethood some new properties are proposed which in turn discard the entropy-subsethood relationship.

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