

Some Exact Bianchi Types Cosmological Models in $f(R, T)$ Theory of Gravity

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Abstract

In this paper, we attempt to study spatially homogeneous Bianchi types-III, V, VI₀ & VI_h cosmological models in $f(R, T)$ theory of gravity. Here the models are obtained by assuming forms of the function $f(R, T)$ as $f(R, T) = R + 2f(T)$ and $f(R, T) = f_1(R) + f_2(T)$. The exact solutions of Einstein's field equations (EFEs) have been obtained for two different types of physically viable cosmologies using a special form of Hubble parameter (HP). The physical and geometrical properties of these models have been discussed and expressions for the Ricci scalar R in each case are obtained.

1. Introduction

General relativity (GR) or Einstein's theory of gravitation is the most successful theory in application to cosmology. Until recently, our mental picture of the universe was based more on our philosophical prejudices (or religious beliefs) than on observational data [1]. Cosmology is a study of the origin, structure evolution, and fate of the universe as a whole based on interpretations of astronomical observations at different wave-lengths through laws of physics. Relativistic cosmological models are described as the exact solutions of the EFEs that help in understanding the important features of our universe. Many generalizations of EFEs have been proposed in last few decades. Einstein's general theory of relativity (GR) is one of the most beautiful structures of theoretical physics. Among several theories of gravitation, GR has been designated as the most successful one. In fact, GR is regarded as a geometric theory of gravitation.

Einstein's theory of gravitation is characterized by mathematical elegance and outstanding formal beauty using tools of Riemannian geometry. It is also realized that it leads to gravitational action. In 1917, Einstein introduced the cosmological constant Λ as the universal repulsion to make the universe static in accordance with a generally accepted picture of that time.

Einstein's theory is modified in several ways for better understanding. The bimetric theory, scalar-tensor theory, etc to name a few. A modification was given in $f(R, T)$ theory [2, 3]. The $f(R, T)$ theory of gravitation is one of the most popular alternatives to Einstein's theory of gravitation. Harko et al. (2011) [4] proposed another extension of standard GR, called the $f(R, T)$ theory of gravity, by introducing an arbitrary function of the Ricci scalar R and the trace T of the energy-momentum tensor. The field equations are derived from the Hilbert-Einstein type variational principle [5, 6]. In $f(R, T)$ theory we assume that the gravitational part of the action still depends on a generic function of the Ricci scalar R , but also presents a generic dependence on T [7]. Such a dependence on T would come from the consideration of quantum effects [8]. In reality, $f(R, T)$ theory provides an alternative way to explain the current cosmic acceleration with no need of introducing either the existence of extra spatial dimension or an exotic component like dark energy [9, 10]. In this theory, the gravitational Lagrangian S_m is given by an arbitrary function of the Ricci scalar R and trace T . This theory can be applied to explore various issues of current interests and may lead to some good inferences [11].

Bianchi types models have a vital role in the description and understanding of the early stages of evolution of the universe. In view of the observation of microwave background radiation, it is found that the universe is not isotropic [3, 12, 13]. Thus, the study of Bianchi types-III, V, VI₀ & VI_h cosmological models is important in the sense that these models are homogeneous and anisotropic, from which the process of isotropization of the universe is studied through the passage of time.

In this paper, an attempt has been made to investigate the exact solutions for Bianchi types-III, V, VI₀ & VI_h cosmological models in the framework of two cases of $f(R, T)$ theory of gravity. The physical and geometrical behaviors of such models have also been discussed.

2. $f(R, T)$ theory of gravity

The $f(R, T)$ theory is a modification of GR. The field equations of $f(R, T)$ theory are derived from a Hilbert-Einstein type variational principle.

The action for modified $f(R, T)$ theory of gravity is given by

$$S = \frac{1}{16\pi} \int f(R, T) \sqrt{-g} d^4x + \int S_m \sqrt{-g} d^4x, \quad (2.1)$$

where $f(R, T)$ is an arbitrary smooth function of Ricci scalar R and the trace T of energy-momentum tensor. S_m is the matter Lagrangian density. The matter energy-momentum tensor T_{ij} from the Lagrangian S_m is defined as [14],

$$T_{ij} = \frac{-2}{\sqrt{-g}} \frac{\partial(\sqrt{-g} S_m)}{\partial g^{ij}}. \quad (2.2)$$

Let us assume that the dependence of matter Lagrangian density S_m is merely on the metric tensor g_{ij} instead of its derivatives. In this case, Equation (2.2) becomes

$$T_{ij} = g_{ij} S_m - 2 \frac{\partial S_m}{\partial g^{ij}}. \quad (2.3)$$

The variations of the metric determinant and Ricci scalar R are

$$\partial(\sqrt{-g}) = -\frac{1}{2} \sqrt{-g} g_{ij} \partial g^{ij}, \quad (2.4)$$

$$\partial(R) = \partial(g^{ij} R_{ij}) = R_{ij} \partial g^{ij} + g_{ij} \nabla^k \nabla_k \partial g^{ij} - \nabla_i \nabla_j \partial g^{ij}. \quad (2.5)$$

The field equations of $f(R, T)$ theory are obtained by varying the action S in Equation (2.1) and using the properties given in Equations (2.4) and (2.5)

$$\frac{\partial f(R, T)}{\partial R} R_{ij} - \frac{1}{2} f(R, T) g_{ij} + (g_{ij} \nabla^k \nabla_k - \nabla_i \nabla_j) \frac{\partial f(R, T)}{\partial R} = 8\pi T_{ij} - \frac{\partial f(R, T)}{\partial T} (T_{ij} + \Theta_{ij}), \quad i, j, k = 1, 2, 3, 4, \quad (2.6)$$

where ∇_i denotes the covariant derivative. We define the variation of T with respect to the metric tensor as

$$\frac{\partial(g^{kl} T_{kl})}{\partial g^{ij}} = T_{ij} + \Theta_{ij}, \quad (2.7)$$

where

$$\Theta_{ij} = g^{kl} \frac{\partial T_{kl}}{\partial g^{ij}}. \quad (2.8)$$

It is clear from Equations (2.3) and (2.7), the tensor Θ_{ij} give in Equation (2.8) lead to

$$\Theta_{ij} = -2T_{ij} + g_{ij} S_m - 2g^{kl} \frac{\partial^2 S_m}{\partial g^{ij} \partial g^{kl}}. \quad (2.9)$$

Note that when $f(R, T) = f(R)$, then Equations (2.6) reduces to the field equations of $f(R)$ gravity. Contraction of Equation (2.6) gives the following relation between the Ricci scalar R and the trace T of the stress-energy tensor

$$\frac{\partial f(R, T)}{\partial R} R + 3 \nabla^k \nabla_k \frac{\partial f(R, T)}{\partial R} - 2f(R, T) = 8\pi T - \frac{\partial f(R, T)}{\partial T} T - \frac{\partial f(R, T)}{\partial T} \Theta, \quad (2.10)$$

with $\Theta = g^{ij} \Theta_{ij}$, Equation (2.10) gives a relation between Ricci scalar R and the trace T of energy-momentum tensor T_{ij} . In the other way the matter Lagrangian S_m , can be taken as $S_m = -p$. Then with the use of Equation (2.9), we obtain Θ_{ij} as

$$\Theta_{ij} = -2T_{ij} - p g_{ij}. \quad (2.11)$$

Using Equation (2.11) in Equation (2.6) the field equations become

$$\frac{\partial f(R, T)}{\partial R} R_{ij} - \frac{1}{2} f(R, T) g_{ij} + (g_{ij} \nabla^k \nabla_k - \nabla_i \nabla_j) \frac{\partial f(R, T)}{\partial R} = \left(8\pi + \frac{\partial f(R, T)}{\partial T} \right) T_{ij} + \frac{\partial f(R, T)}{\partial T} p g_{ij}. \quad (2.12)$$

Following, Harko et al. (2011) [4] to obtain some particular classes of $f(R, T)$ modified gravity models by specifying functional forms of $f(R, T)$ as

$$f(R, T) = \begin{cases} R + 2f(T), \\ f_1(R) + f_2(T), \\ f_1(R) + f_2(R) f_3(T). \end{cases} \quad (2.13)$$

In this paper, the attempt is to explore the first and the second cases of Equation (2.13) to study the exact solutions for Bianchi-III, V, VI₀ & VI_h in $f(R, T)$ theory of gravity.

3. The metric and the field equations

The spatially homogeneous (SH) and anisotropic Bianchi types space-times are given by,

$$ds^2 = dt^2 - A_1^2 dx^2 - e^{-2x} A_2^2 dy^2 - e^{-2mx} A_3^2 dz^2, \tag{3.1}$$

where A_1, A_2 and A_3 are called cosmic scale factors which are functions of time t , so the equation (3.1) represents different Bianchi types as,

1. Bianchi type-III if $m = 0$,
2. Bianchi type-V if $m = 1$,
3. Bianchi type-VI₀ if $m = -1$,
4. Bianchi type-VI_h for all other $m = h = -1$.

The computation of Ricci tensor R_{ij} and its spur was done using Mathematica [15] and [16]; the non-vanishing independent components are,

$$R_{11} = 1 + m^2 + A_1 \left[\dot{A}_1 \left(-\frac{\dot{A}_2}{A_2} - \frac{\dot{A}_3}{A_3} \right) - \ddot{A}_1 \right], \tag{3.2}$$

$$R_{14} = \frac{(m+1)\dot{A}_1}{A_1} - \frac{\dot{A}_2}{A_2} - \frac{m\dot{A}_3}{A_3}, \tag{3.3}$$

$$R_{22} = \frac{e^{-2x} A_2 \left[-(m+1)A_2 A_3 - A_1 (\dot{A}_2 (A_3 \dot{A}_1 + A_1 \dot{A}_3) + A_1 A_3 \dot{A}_2) \right]}{A_1^2 A_3}, \tag{3.4}$$

$$R_{33} = \frac{e^{-2mx} A_3 \left[-A_1^2 \dot{A}_2 \dot{A}_3 + A_2 (m(m+1)A_3 - A_1 (\dot{A}_1 \dot{A}_3 + A_1 \dot{A}_3)) \right]}{A_1^2 A_2}, \tag{3.5}$$

$$R_{44} = \frac{\ddot{A}_1}{A_1} + \frac{\ddot{A}_2}{A_2} + \frac{\ddot{A}_3}{A_3}, \tag{3.6}$$

and

$$R = 2 \left[\frac{\ddot{A}_1}{A_1} + \frac{\ddot{A}_2}{A_2} + \frac{\ddot{A}_3}{A_3} + \frac{\dot{A}_1 \dot{A}_2}{A_1 A_2} + \frac{\dot{A}_1 \dot{A}_3}{A_1 A_2 A_3} + \frac{\dot{A}_2 \dot{A}_3}{A_2 A_3} - \frac{m^2 + m + 1}{A_1^2} \right], \tag{3.7}$$

where an overhead dot denotes derivative with respect to time t . The energy-momentum tensor for a perfect fluid is given by

$$T_{ij} = (\rho + p)u_i u_j - p g_{ij}, \tag{3.8}$$

where ρ is the proper energy density, p is the isotropic pressure and $u_i = (0, 0, 0, 1)$ is 4-velocity of the fluid particles which satisfies the condition $u^i u_i = 1$. The EFEs are given by

$$R_{ij} - \frac{1}{2} R g_{ij} = -8\pi T_{ij} + \Lambda g_{ij}, \tag{3.9}$$

where Λ is the cosmological constant. The average scale factor $a(t)$ and spatial volume V are defined by

$$V = a^3 = \prod_{i=1}^3 A_i. \tag{3.10}$$

Mean HP is given by

$$H = \frac{1}{3} \frac{\dot{V}}{V} = \frac{\dot{a}}{a} = \frac{1}{3} \sum_{i=1}^3 H_i = \frac{1}{3} \left(\frac{\dot{A}_1}{A_1} + \frac{\dot{A}_2}{A_2} + \frac{\dot{A}_3}{A_3} \right), \tag{3.11}$$

in which HPs in the directions of x, y and z -axes are obtained as

$$H_i = \frac{\dot{A}_i}{A_i}, \quad i = 1, 2, 3 \quad (\text{no sum}). \tag{3.12}$$

The scalar expansion θ is given by

$$\theta = \left(\frac{\dot{A}_1}{A_1} + \frac{\dot{A}_2}{A_2} + \frac{\dot{A}_3}{A_3} \right) = 3H.$$

Moreover, the shear σ^2 is given by

$$\sigma^2 = \frac{1}{2} \sigma_{ij} \sigma^{ij} = \frac{1}{2} \left[\sum_{i=1}^3 H_i^2 - 3H^2 \right],$$

the shear parameter is given by

$$\Sigma^2 = \Sigma_+^2 + \Sigma_-^2 = \frac{\sigma^2}{3H^2} = \frac{1}{6} \Sigma_{ij} \Sigma^{ij}, \quad \text{with } \Sigma_{ij} = \frac{\sigma_{ij}}{H}.$$

The density parameter Ω is given by

$$\Omega = 1 - \Sigma^2 - K \geq 0,$$

where the curvature parameter K is given by

$$K = \frac{{}^3R}{6H^2} = \frac{1}{12} \left(\sum_i N_i^2 - 2 \sum_{i < j} N_i N_j \right), \quad i, j = 1, 2, 3, \tag{3.13}$$

i.e.,

Group class	Bianchi type	N_1	N_2	N_3
Class A, ($a = 0$)	VI_0	0	+	-
	V	0	0	0
Class B, ($a \neq 0$)	VI_h	0	-	+
	III	0	+	-

Table 1: Canonical Structure Constants for Different Bianchi Types

The three structure constants N_1, N_2 and N_3 are the eigenvalue of the symmetric matrix, $N^{ij} = \text{diag}(N_1, N_2, N_3)$. In another way, we can get the form

$$\Omega + K + \Sigma^2 = 1.$$

An important observable quantity in cosmology is deceleration parameter (DP) defined as

$$q = -1 - \frac{\dot{H}}{H^2} = -\frac{\ddot{a}a}{\dot{a}^2}. \tag{3.14}$$

The evolution of H is describe by

$$\dot{H} = -(1+q)H^2.$$

It is worth mentioning here that “the name DP and the negative sign are historical. Initially, q was supposed to be positive but recent observations from the supernova experiments suggest that it is negative”. To solve an integral part in the aforementioned equation, we may refer to the power-law assumption. Many kinds of researchers have used the power-law relation. For instance, Johri and Desikan [17] in the context of Robertson-Walker Brans-Dicke models, have already used the power-law relation between scale factor and scalar field. We use a well-known relation [18] between the mean HP and average scale factor a , given as

$$H = la^{-n}, \quad \forall n, \tag{3.15}$$

where $l > 0$. This is an important relation because it gives the constant value of the DP. Using Equations (3.11) and (3.15), we get

$$\dot{a} = la^{1-n}, \tag{3.16}$$

and consequently, from Equation (3.14) the DP turns out to be

$$q = n - 1,$$

which is obviously a constant. Integrating Equation (3.16), it follows that

$$a = \begin{cases} k_1 e^{lt}, & \text{for } n = 0, \\ (nlt + k_2)^{\frac{1}{n}}, & \text{for } n \neq 0, \end{cases} \tag{3.17}$$

where k_1 and k_2 are constants of integration, thus we obtain two values of the average scale factor a , that correspond to two different models of the universe. In this paper, we consider the average scale factor a when $n = 0$ in the first case of $f(R, T)$ theory and $n \neq 0$ in the second case of $f(R, T)$ theory.

4. Exact solutions for some Bianchi types

Here we first develop some important cosmological parameters and EFEs for Bianchi types III, V, VI_0 & VI_h space-times and then find the exact solutions of EFEs for constant and non-constant curvature case.

4.1. Solution for $f(R, T) = R + 2\lambda T$

We consider the case in which the function $f(R, T)$ is given by $f(R, T) = R + 2\lambda T$, where λ is a constant. Thus the field Equation (2.12) take the form

$$G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij} = (8\pi + 2\lambda)T_{ij} + \lambda(2p + T)g_{ij}. \tag{4.1}$$

This form looks like EFEs in GR, the term $\lambda(2p + T)$ may play the role of cosmological parameter Λ of the GR field equations, that is

$$\Lambda = \Lambda(T) = \lambda(2p + T),$$

which supports the suggestion by Poplawski [19] where the dependence of the cosmological parameter Λ on T . The researchers like Magnano [20] have suggested that the $\Lambda(T)$ gravity is more general than the gravity in Palatini $f(R)$ theory and could be reduced to it if the pressure of the matter is neglected. Considering the perfect fluid case, the trace $T = \rho - 3p$, hence Equation (4.8) becomes

$$\Lambda = \lambda(\rho - p).$$

Thus we rewrite Equation (4.1) as

$$R_{ij} - \frac{1}{2}Rg_{ij} = (8\pi + 2\lambda)T_{ij} + \Lambda g_{ij}. \tag{4.2}$$

Now using Equations (4.2), and (3.2) to (3.9) we obtain a set of differential equations for Bianchi types-III, V, VI₀ & VI_h space-times

$$\begin{aligned} \frac{\ddot{A}_2}{A_2} + \frac{\ddot{A}_3}{A_3} + \frac{\dot{A}_2\dot{A}_3}{A_2A_3} - \frac{m}{A_1^2} &= (8\pi + 2\lambda)p - \Lambda, \\ \frac{\dot{A}_1}{A_1} + \frac{\ddot{A}_2}{A_2} + \frac{\dot{A}_1\dot{A}_2}{A_1A_2} - \frac{1}{A_1^2} &= (8\pi + 2\lambda)p - \Lambda, \\ \frac{\dot{A}_1}{A_1} + \frac{\ddot{A}_3}{A_3} + \frac{\dot{A}_1\dot{A}_3}{A_1A_3} - \frac{m^2}{A_1^2} &= (8\pi + 2\lambda)p - \Lambda, \\ \frac{\dot{A}_1\dot{A}_2}{A_1A_2} + \frac{\dot{A}_2\dot{A}_3}{A_2A_3} + \frac{\dot{A}_1\dot{A}_3}{A_1A_3} - \frac{m^2 + m + 1}{A_1^2} &= -\Lambda - (8\pi + 2\lambda)\rho, \\ (m + 1)\frac{\dot{A}_1}{A_1} - \frac{\dot{A}_2}{A_2} - \frac{m\dot{A}_3}{A_3} &= 0. \end{aligned} \tag{4.3}$$

Integrating Equation (4.3) and absorbing the integrating constant into A_2 or A_3 , we get

$$A_1^{m+1} = A_2A_3^m. \tag{4.4}$$

Using Equation (3.10) in Equation (4.4) we get

$$A_1^{m+2} = a^3A_3^{m-1}. \tag{4.5}$$

4.1.1. Cosmological solutions

We now obtain physically factual cosmological models to describe the decelerating and accelerating phases of the universe. Setting $A_3 = V^d$, where d is any constant, then from Equation (4.5), we get

$$\begin{aligned} A_1^{m+2} &= a^3V^{d(m-1)}, \\ &= a^{3(1+md-d)}. \end{aligned} \tag{4.6}$$

Here we will consider the value of average scale factor a for $n = 0$ only (see (3.17)). Using Equations (3.10), (3.17), (4.3) and (4.6) the metric coefficients $A_i (i = 1, 2, 3)$ turn out to be

$$A_i(t) = \left(k_1 e^{lt}\right)^{\xi_i}, \quad i = 1, 2, 3 \text{ (no sum)}, \tag{4.7}$$

where

$$\xi_1 = \frac{3(1+md-d)}{m+2}, \quad \xi_2 = \frac{3(1+m-d-2md)}{m+2}, \quad \xi_3 = 3d.$$

Using these in (3.1), we get the following form of the metric (3.1) as

$$ds^2 = dt^2 - \left(k_1 e^{lt}\right)^{\xi_1} dx^2 - e^{-2x} \left(k_1 e^{lt}\right)^{\xi_2} dy^2 - e^{-2mx} \left(k_1 e^{lt}\right)^{\xi_3} dz^2.$$

4.1.2. Physical and geometrical properties of the solution

In this subsection, we will compute the relevant physical and geometrical properties of the space-time. The necessary computations were done using Mathematica, necessary programming was done by us. The spatial volume and the average scale factor $a(t)$ are

$$V = (k_1 e^{lt})^3 = a^3.$$

Mean HP and DP are

$$H = l, \quad q = \frac{-\ddot{a}}{aH^2} = -1,$$

from Equation (3.12), the HPs in the directions of x, y and z -axes are

$$H_i = \frac{\dot{A}_i}{A_i} = l \xi_i, \quad i = 1, 2, 3 \text{ (no sum)}.$$

The scalar expansion is

$$\theta = 3l = 3H.$$

The shear scalar is

$$\begin{aligned} \sigma^2 &= \frac{l^2}{2} [\xi_1^2 + \xi_2^2 + \xi_3^2 - 3], \\ &= \frac{l^2}{(m+2)^2} [3(1-3d)^2(m^2+m+1)]. \end{aligned}$$

The shear parameter is given by

$$\begin{aligned} \Sigma^2 &= \frac{1}{6} [\xi_1^2 + \xi_2^2 + \xi_3^2 - 3], \\ &= \frac{1}{3(m+2)^2} [3(1-3d)^2(m^2+m+1)]. \end{aligned}$$

In this subsection, we take $\lambda = 0.1, l = 5, m = 0, \pm 1, n = 0.5, d = 0.1$ and $k_1 = 1$, for all graphs. The energy density ρ in the model is obtained as

$$\begin{aligned} \rho &= \frac{1}{8(8\pi^2 + 6\lambda\pi + \lambda^2)} \left[\frac{9l^2\lambda \left(-3dm(1+m) + (1+m)^2 + 3d^2(1+m+m^2) \right)}{(m+2)^2} \right. \\ &\quad \left. - \frac{9l^2(8\pi + 3\lambda) \left(1+m-d(-2+3d)(1+m+m^2) \right)}{(m+2)^2} - \frac{\lambda m - (8\pi + 3\lambda)(m^2+m+1)}{(k_1 e^{lt})^{\frac{6(1+md-d)}{(m+2)}}} \right]. \end{aligned}$$

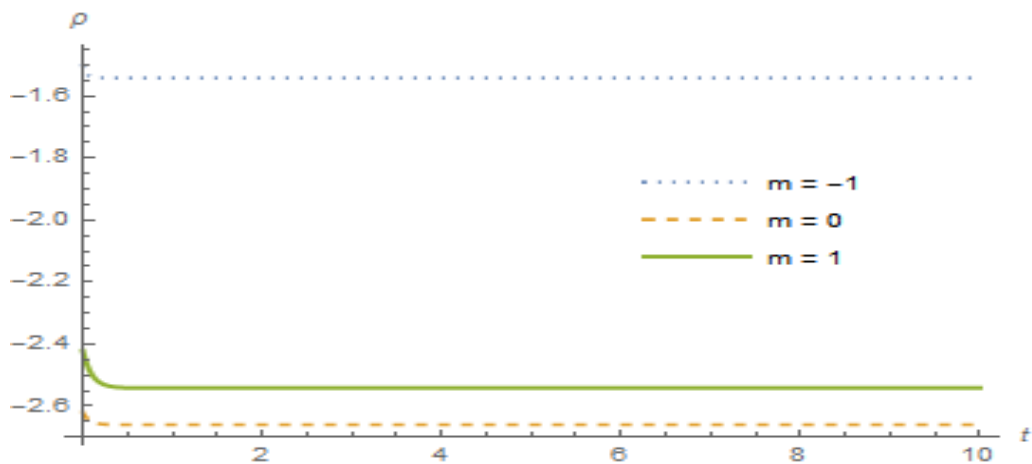


Figure 4.1: The Evolution of Energy Density ρ Versus Cosmic Time t

Figure 4.1, shows ρ as a decreasing function for $0 \leq t < 1$ and constant for $t \geq 1$. The expressions for isotropic pressure p in the model is given by

$$\begin{aligned} p &= \frac{1}{8(8\pi^2 + 6\lambda\pi + \lambda^2)} \left[\frac{9l^2(8\pi + 3\lambda) \left(-3dm(1+m) + (1+m)^2 + 3d^2(1+m+m^2) \right)}{(m+2)^2} \right. \\ &\quad \left. - \frac{9l^2\lambda \left(1+m-d(-2+3d)(1+m+m^2) \right)}{(m+2)^2} - \frac{m(8\pi + 3\lambda) - (m^2+1)\lambda}{(k_1 e^{lt})^{\frac{6(1+md-d)}{(m+2)}}} \right]. \end{aligned}$$

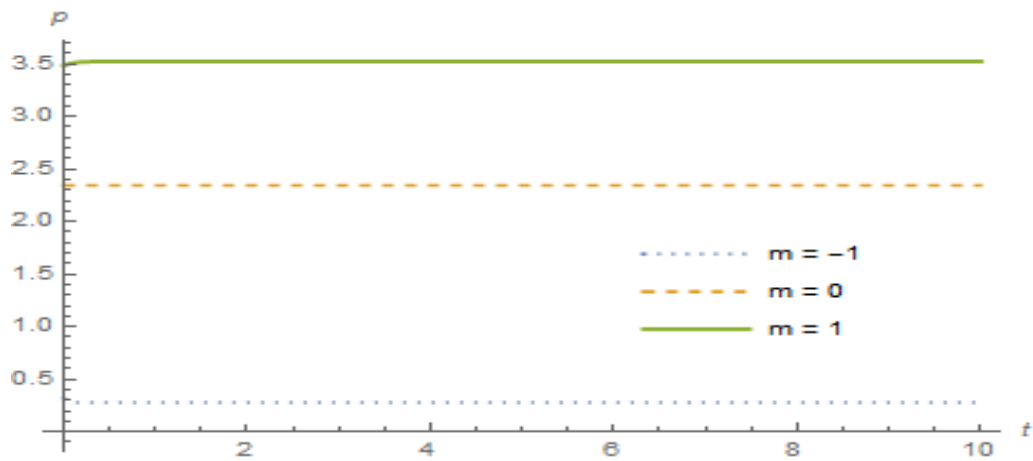


Figure 4.2: The Evolution of Pressure p Versus Cosmic Time t

From Figure 4.2, we observe that the pressure is an increasing function for $0 \leq t < 1$ and constant for $t \geq 1$. The cosmological parameter Λ is

$$\Lambda = \frac{\lambda}{8(8\pi^2 + 6\lambda\pi + \lambda^2)} \left[\frac{-9l^2(8\pi + 2\lambda) \left(6d^2 + 6md^2 + 6m^2d^2 + m + m^2 - 5m^2d - 5md - 2d \right)}{(m+2)^2} - \frac{(8\pi + 3\lambda)(m^2 + 2m + 1) - \lambda(m^2 + m + 1)}{(k_1 e^{lt})^{\frac{6(1+md-d)}{(m+2)}}} \right].$$

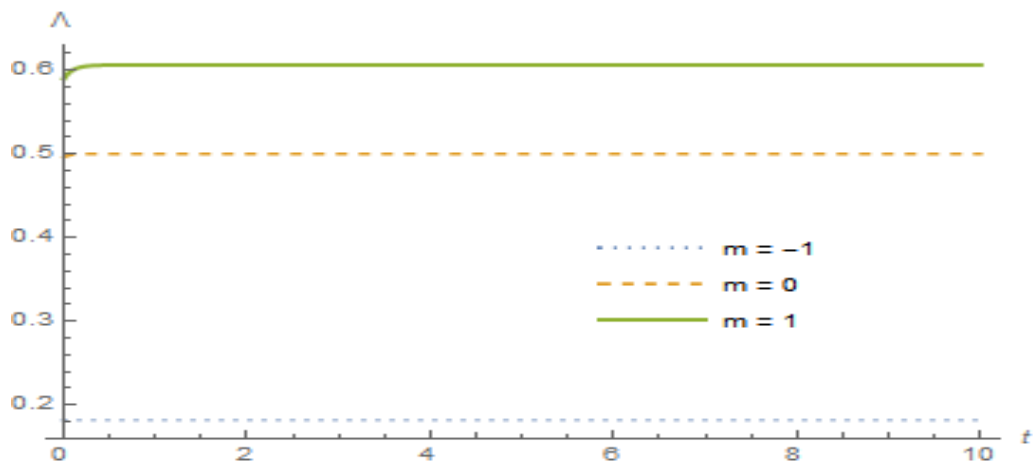


Figure 4.3: The Evolution of Cosmological Constant Λ Versus Cosmic Time t

From Figure 4.3, we observe that the cosmological term Λ is an increasing function for $0 \leq t < 1$ and constant for $t \geq 1$. The density parameter Ω is given by

$$\Omega = 1 - \frac{1}{3(m+2)^2} \left[3(1 - 3d)^2(m^2 + m + 1) \right] - K \geq 0,$$

where K is the curvature parameter, as defined in Equation (3.13). The Ricci scalar R for Bianchi types-III, V, VI₀ & VI_h cosmological models are given by Equations (3.7) and (4.7), it follows that

$$R = \frac{2l^2}{(m+2)^2} \left[(27 - 18d - 18md + 27m + 27d^2 + 27md^2) + m^2(9 - 18d + 27d^2) \right] - 2(m^2 + m + 1) \left(k_1 e^{lt} \right)^{\frac{-6(1+md-d)}{(m+2)}}.$$

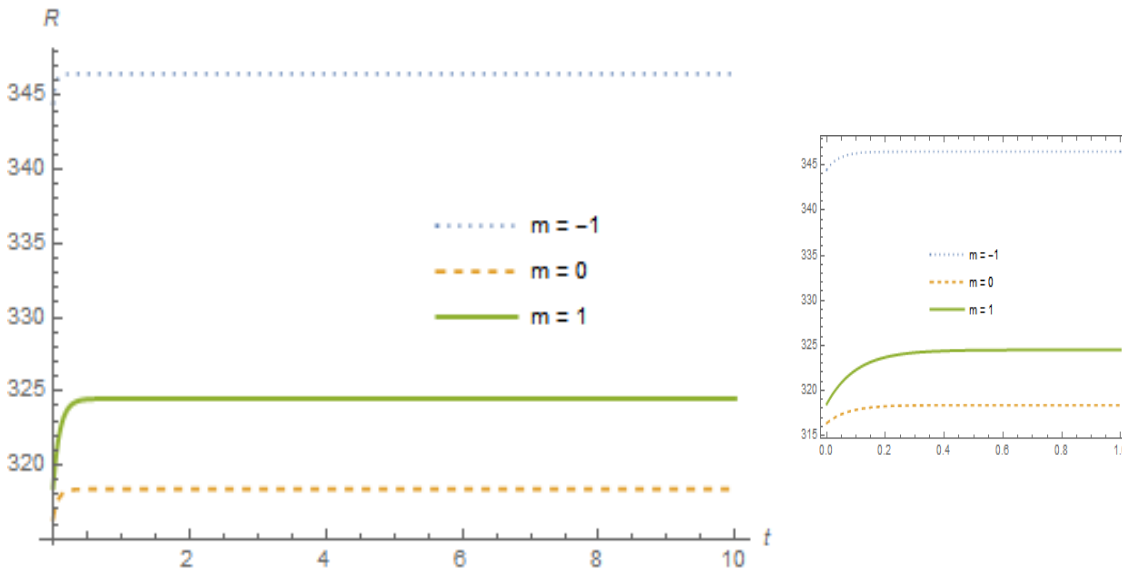


Figure 4.4: The Evolution of Ricci Scalar R Versus Cosmic Time t

From Figure 4.4, we observe that the Ricci scalar R is an increasing function for $0 \leq t < 1$ and constant for $t \geq 1$. The function $f(R, T)$ of Ricci scalar R and the trace T , can be found as

$$f(R, T) = R + 2\lambda(\rho - 3p).$$

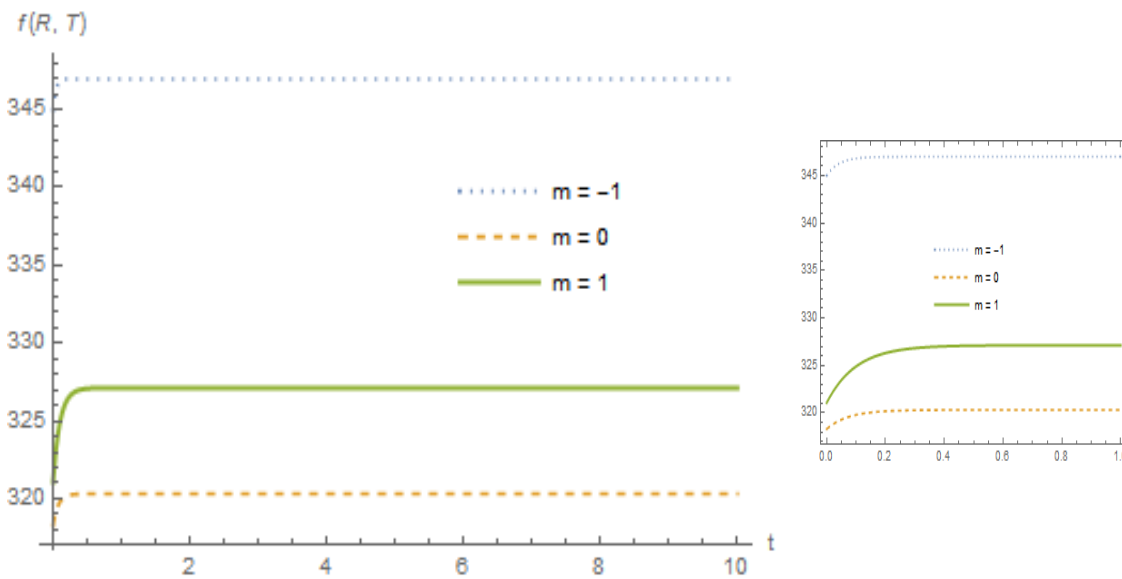


Figure 4.5: The Evolution of $f(R, T)$ Versus Cosmic Time t

Figure 4.5, shows that the $f(R, T)$ is an increasing function for $0 \leq t < 1$ and constant for $t \geq 1$.

4.2. Solution for $f(R, T) = \lambda(R + T)$

We consider the case in which the function $f(R, T)$ is given by $f(R, T) = \lambda(R + T)$, where λ is an arbitrary parameter. Thus the field equation (2.12) take the form

$$\lambda Ri_j - \frac{1}{2}\lambda(R + T)g_{ij} + (g_{ij}\nabla^k\nabla_k - \nabla_i\nabla_j)\lambda = (8\pi + \lambda)T_{ij} + \left(p + \frac{1}{2}T\right)\lambda g_{ij},$$

setting $(g_{ij}\nabla^k\nabla_k - \nabla_i\nabla_j)\lambda = 0$, we get

$$\lambda Ri_j - \frac{1}{2}\lambda Rg_{ij} = (8\pi + \lambda)T_{ij} + \left(p + \frac{1}{2}T\right)\lambda g_{ij}. \tag{4.8}$$

The Einstein tensor G_{ij} is defined by

$$G_{ij} = Ri_j - \frac{1}{2}Rg_{ij}.$$

Equation (4.8) becomes

$$\lambda G_{ij} = (8\pi + \lambda)T_{ij} + \left(p + \frac{1}{2}T\right)\lambda g_{ij}. \tag{4.9}$$

This form looks like EFEs in GR, we choose a negative small value for the arbitrary λ so that we have the same sign of the RHS of Equation (4.9) we keep this choice of λ throughout the discussion. The term $(p + \frac{1}{2}T)$ can now be regarded as a cosmological parameter Λ . Hence

$$\Lambda = p + \frac{1}{2}T,$$

so Equation (4.9) becomes

$$G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij} = \frac{(8\pi + \lambda)}{\lambda}T_{ij} + \Lambda g_{ij}, \tag{4.10}$$

which supports the suggestion by Poplawski [19] where the dependence of the cosmological parameter Λ on T . The researchers like Magnano [20] have suggested that the Λ gravity is more general than the gravity in Palatini $f(R)$ theory and could be reduced to it if the pressure of matter is neglected. Considering the perfect fluid case, the trace $T = \rho - 3p$, hence

$$\Lambda = \frac{1}{2}(\rho - p).$$

The field equations, in this case, are similar to those written earlier (4.3) to (4.3) with only change due to rights side of (4.10), thus the coefficient $(8\pi + 2\lambda)$ is replaced by $(\frac{8\pi+2\lambda}{\lambda})$. The field equation for G_{14} is same as Equation (4.3). Also following same procedure we get relations (4.4) and (4.5).

4.2.1. Cosmological solutions

We now obtain physically factual cosmological models to describe the decelerating and accelerating phases of the universe. Setting $A_3 = V^d$, where d is any constant, then from Equation (4.5), we get

$$\begin{aligned} A_1^{m+2} &= a^3 V^{d(m-1)} \\ &= a^{3(1+md-d)}. \end{aligned} \tag{4.11}$$

Here we will consider the value of average scale factor a for $n \neq 0$ only. Using Equations (3.10), (3.17), (4.3) and (4.11) the metric coefficients A_i ($i = 1, 2, 3$) turn out to be

$$A_i(t) = (nlt + k_2)^{\frac{\xi_i}{n}}, i = 1, 2, 3 \text{ (no sum)}, \tag{4.12}$$

where

$$\xi_1 = \frac{3(1+md-d)}{m+2}, \quad \xi_2 = \frac{3(1+m-d-2md)}{m+2}, \quad \xi_3 = 3d.$$

Using these in (3.1), we get the following form of the metric (3.1)

$$ds^2 = dt^2 - (nlt + k_2)^{\frac{\xi_1}{n}} dx^2 - e^{-2x} (nlt + k_2)^{\frac{\xi_2}{n}} dy^2 - e^{-2mx} (nlt + k_2)^{\frac{\xi_3}{n}} dz^2.$$

4.2.2. Physical and geometrical properties of the solution

In this subsection, we will compute the relevant physical and geometrical properties of the space-time. The necessary computations were done using Mathematica, necessary programming was done by us. The spatial volume and the average scale factor $a(t)$ are

$$V = (nlt + k_2)^{\frac{3}{n}} = a^3.$$

In this subsection, we take $\lambda = 0.1$, $l = 5$, $m = 0, \pm 1$, $n = 0.5$, $d = 0.1$ and $k_2 = 1$ for all graphs.

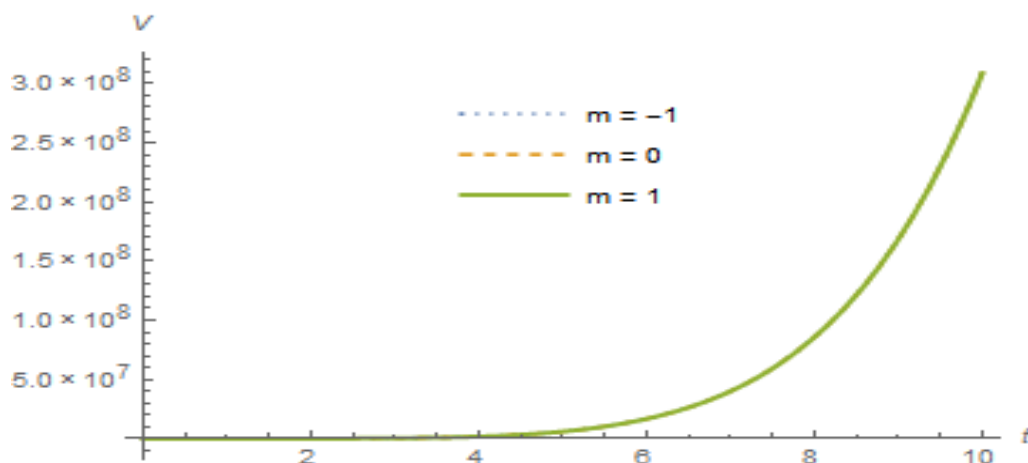


Figure 4.6: The Evolution of Volume V Versus Cosmic Time t

Figure 4.6, shows that volume V is an increasing function of time t . Mean HP and DP are

$$H = \frac{l}{nlt + k_2}, \quad q = \frac{-\ddot{a}}{aH^2} = n - 1,$$

in which HPs in the directions of x, y and z -axes are

$$H_i = \frac{\dot{A}_i}{A_i} = \frac{\xi_i l}{nlt + k_2}, \quad i = 1, 2, 3 \text{ (no sum)}.$$

The scalar expansion becomes

$$\theta = \frac{3l}{nlt + k_1}.$$

The shear scalar is

$$\begin{aligned} \sigma^2 &= \frac{l^2}{2(nlt + k_2)^2} [\xi_1^2 + \xi_2^2 + \xi_3^2 - 3] \\ &= \frac{l^2}{(m+2)^2(nlt + k_2)^2} [3(1-3d)^2(m^2 + m + 1)]. \end{aligned}$$

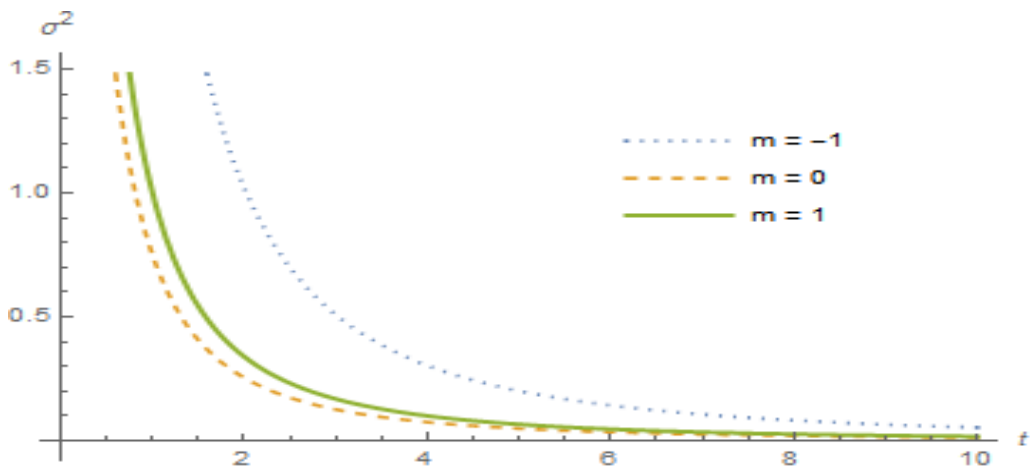


Figure 4.7: The Evolution of Shear Scalar σ^2 Versus Cosmic Time t

Figure 4.7, shows σ^2 as a decreasing function of time t . The shear parameter is given by

$$\begin{aligned} \Sigma^2 &= \frac{1}{6} [\xi_1^2 + \xi_2^2 + \xi_3^2 - 3], \\ &= \frac{1}{3(m+2)^2} [3(1-3d)^2(m^2 + m + 1)]. \end{aligned}$$

The energy density ρ in the model is obtained as

$$\rho = -\frac{9\lambda l^2(1+m-d(-2+3d)(1+m+m^2))}{2(4\pi+\lambda)(m+2)^2(nlt+k_2)^2} - \frac{\lambda^2}{6(4\pi+\lambda)^2} \left[\frac{3l^2(3-n)}{(nlt+k_2)^2} - \frac{2(m^2+m+1)}{(nlt+k_2)^{\frac{6(1+md-d)}{n(m+2)}}} \right] + \frac{\lambda(m^2+m+1)}{2(4\pi+\lambda)(nlt+k_2)^{\frac{6(1+md-d)}{n(m+2)}}}.$$

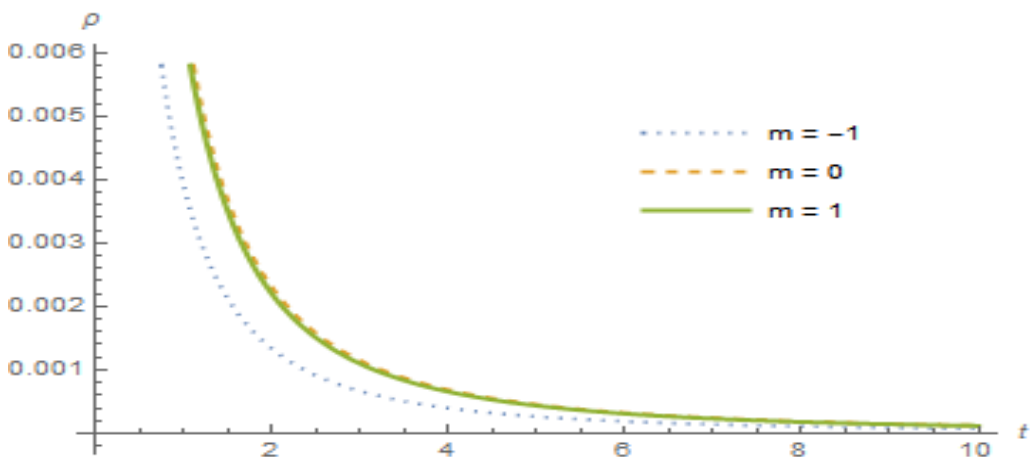


Figure 4.8: The Evolution of Energy Density ρ Versus Cosmic Time t

Figure 4.8, shows that energy density ρ is a decreasing function of time t . The expressions for isotropic pressure p in the model is given by

$$p = -\frac{9\lambda l^2(1+m-d(-2+3d)(1+m+m^2))}{2(4\pi+\lambda)(m+2)^2(nlt+k_2)^2} + \frac{\lambda(16\pi+3\lambda)}{6(4\pi+\lambda)^2} \left[\frac{3l^2(3-n)}{(nlt+k_2)^2} - \frac{2(m^2+m+1)}{(nlt+k_2)^{\frac{6(1+md-d)}{n(m+2)}}} \right] + \frac{\lambda(m^2+m+1)}{2(4\pi+\lambda)(nlt+k_2)^{\frac{6(1+md-d)}{n(m+2)}}}.$$

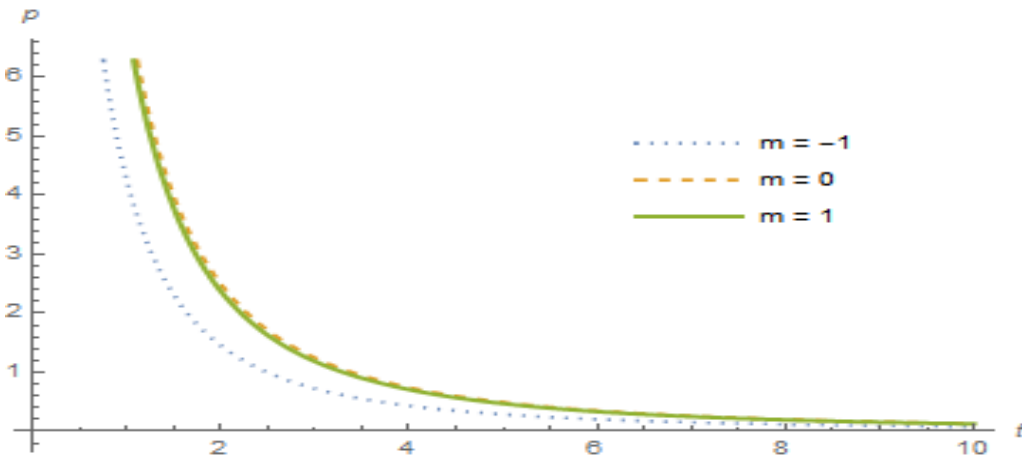


Figure 4.9: The Evolution of Pressure p Versus Cosmic Time t

Figure 4.9, shows that the pressure is a decreasing function of time. The cosmological parameter Λ is

$$\Lambda = \frac{1}{2}(\rho - p) = \frac{\lambda}{6(4\pi+\lambda)} \left[\frac{3l^2(3-n)}{(nlt+k_2)^2} - \frac{2(m^2+m+1)}{(nlt+k_2)^{\frac{6(1+md-d)}{n(m+2)}}} \right].$$

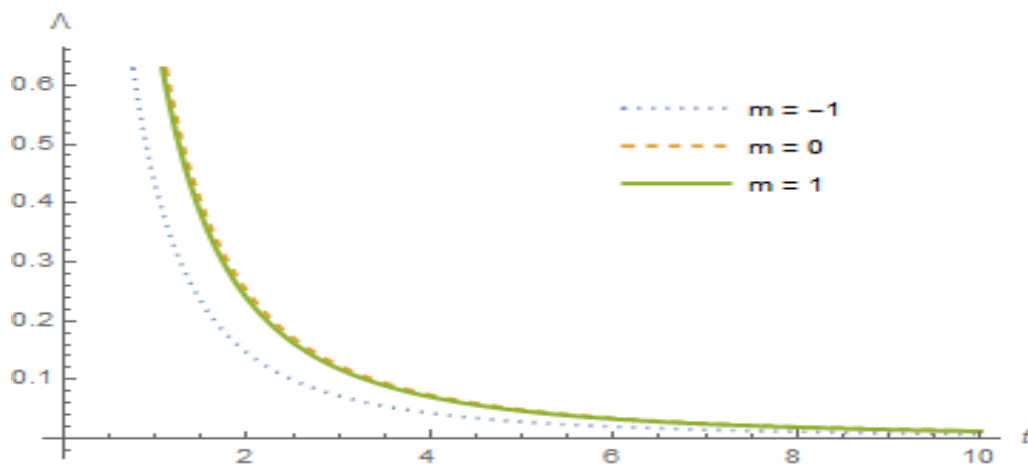


Figure 4.10: The Evolution of Cosmological Constant Λ Versus Cosmic Time t

Figure 4.10, shows that the cosmological term Λ is a decreasing function of time t . The density parameter Ω , is given by

$$\Omega = 1 - \frac{1}{3(m+2)^2} \left[3(1-3d)^2(m^2+m+1) \right] - K \geq 0,$$

where K is the curvature parameter, as defined in Equation (3.13). The Ricci scalar R for Bianchi types-III, V, VI₀ & VI_h cosmological models are given by Equations (3.7) and (4.12) it follows that

$$\begin{aligned} R &= -\frac{2l^2}{(nlt+k_2)^2} \left[\xi_1^2 + \xi_2^2 + \xi_3^2 - n(\xi_1 + \xi_2 + \xi_3) + (\xi_1 \xi_2 + \xi_1 \xi_3 + \xi_2 \xi_3) \right] - 2(m^2+m+1)(nlt+k_2)^{-\frac{2\xi_1}{n}}, \\ &= \frac{2l^2}{(m+2)^2(nlt+k_2)^2} \left[(m+1)(27-18d+27d^2-12n) + m^2(9-18d+27d^2-3n) \right] - 2(m^2+m+1)(nlt+k_2)^{-\frac{6(1+md-d)}{n(m+2)}}. \end{aligned}$$

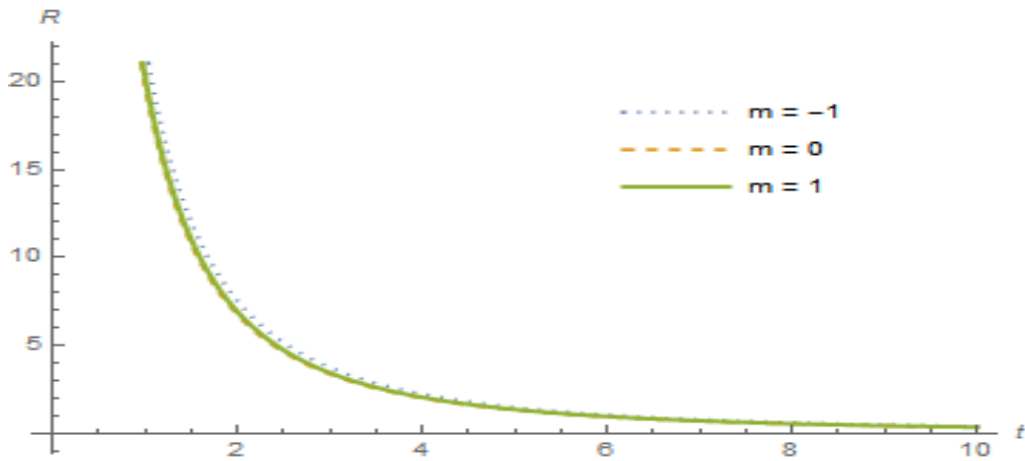


Figure 4.11: The Evolution of Ricci Scalar R Versus Cosmic Time t

Figure 4.11, shows that the curvature is positive through the whole evolution of the universe. The function $f(R, T)$ of Ricci scalar R and the trace T can be found as

$$f(R, T) = \lambda_1 R + \lambda_2(\rho - 3p).$$

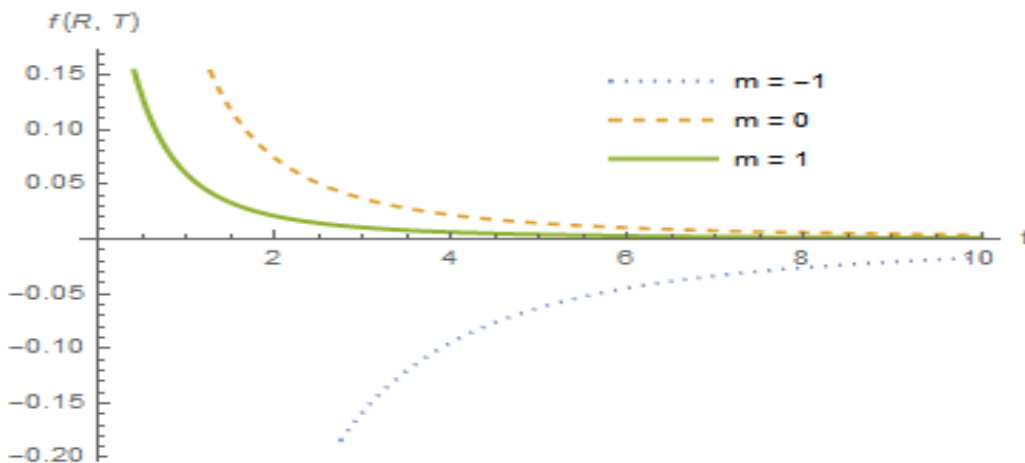


Figure 4.12: The Evolution of $f(R, T)$ Versus Cosmic Time t

Figure 4.12, shows that the $f(R, T)$ is an increasing function when $m = 0, 1$ and decreasing function when $m = -1$ of time t .

5. Conclusion

In this paper, we have extended the study of exact solutions of EFEs for Bianchi types-III, V, VI₀ & VI_h space-times in $f(R, T)$ theory and obtained the exact solutions corresponding to singularity point $n \neq 0$, and regular point $n = 0$. The exact solutions to the corresponding field equations are obtained in quadrature form. The behaviors of the cosmological parameter Λ have been discussed in each case. We have also examined the well-known physical and geometrical properties of our models in two different viable cosmologies. It is shown that our models represent expanding, shearing, non-rotating and accelerating universe in each case. In the first case of $f(R, T)$ theory, when $n = 0$ with $a = k_1 e^{lt}$, the model has no singularity point. The volume V is finite and blows to infinite at $t \rightarrow \infty$. The generalized HP H is constant and accordingly, expansion scalar θ is constant. The HPs $H_i, i = 1, 2, 3$ are finite for all finite values of t . The shear scalar σ^2 and shear parameter Σ^2 are constant as $t \rightarrow \infty$. The energy density ρ (Figure 4.1) is constant as $t \rightarrow \infty$ and the Figure 4.1, shows that ρ is negative, its physical interpretation may be debatable however this is mathematically consistent. The isotropic pressure p (Figure 4.2), density parameter Ω and Ricci scalar R (Figure 4.4) are constant as $t \rightarrow \infty$. The function $f(R, T)$ of the Ricci scalar R and trace T is finite (Figure 4.5) at non-singularity.

In the second case of $f(R, T)$ theory, when $n \neq 0$ with $a = (nlt + k_2)^{\frac{1}{n}}$, it is observed that the spatial volume $V \rightarrow \infty$ as $t \rightarrow \infty$ (Figure 4.6), and the volume scalar factor vanishes at the singularity point. The generalized HP is zero at the singularity. The expansion scalar $\theta \rightarrow 0$ as $t \rightarrow \infty$, as well as it is observed that θ starts with infinite value at $t = 0$ and then rapidly becomes constant after some finite time. The direction HPs $H_i, i = 1, 2, 3$ are zero at the singularity point. The shear scalar σ^2 (Figure 4.7) and shear parameter Σ^2 are zero as $t \rightarrow \infty$. The isotropic pressure p (Figure 4.9), energy density ρ (Figure 4.8) are constant at $t \rightarrow \infty$. The Ricci scalar R is infinite $t \rightarrow \infty$ (Figure 4.11). The density parameter Ω is constant as $t \rightarrow \infty$. The function $f(R, T)$ of the Ricci scalar R and trace T is infinite (Figure 4.12) at the singularity.

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