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Korovkin Type Approximation Theorem for Functions of Two Variables Through $\alpha\beta-$Statistical Convergence

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Abstract

In this paper, we introduce the concepts of $\alpha\beta-$statistical convergence and strong $\alpha\beta-$summability of double sequences and investigate the relation between these two new concepts. Moreover, statistical convergence and $\alpha\beta-$statistical convergence of double sequences are compared under some certain assumptions. Finally, as an application, we prove Korovkin type approximation theorem for a function of two variables by using the notion of $\alpha\beta-$statistical convergence.

1. Introduction

The idea of statistical convergence for sequences of real and complex numbers was introduced by Fast [1] and Steinhaus [2] independently in the same year 1951 as follows. Let $K \subseteq \mathbb{N}$, the set of natural numbers and $K_n = \{k \leq n : k \in K\}$. Then the natural density of K is defined by $\delta(K) = \lim_n n^{-1}|K_n|$ if the limit exists, where $|K_n|$ denotes the cardinality of K_n. A sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$, the set $K_\varepsilon := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has natural density zero, i.e., for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n}|\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0,$$

which is denoted by $st - \lim x = L$. Over the years, generalizations and applications of this notion have been investigated by various researchers [3]-[14].

Aktuglu [14] introduced $\alpha\beta-$statistical convergence as follows. Let $\alpha(n)$ and $\beta(n)$ be two sequences of positive numbers satisfying the following conditions:

$P_1 : \alpha$ and β are both non $-$ decreasing,
$P_2 : \beta(n) \geq \alpha(n)$,
$P_3 : \beta(n) - \alpha(n) \to \infty$ as $n \to \infty$.

Let Λ denote the set of pairs (α, β) satisfying P_1, P_2 and P_3.
For each pair $(\alpha,\beta) \in \Lambda$, $0 < \gamma \leq 1$ and $K \subset \mathbb{N}$, we define

$$\delta^{\alpha,\beta}(K,\gamma) = \lim_{n\to\infty} \frac{\left|K \cap P_n^{\alpha,\beta}\right|}{(\beta(n) - \alpha(n) + 1)^\gamma}$$

where $P_n^{\alpha,\beta}$ is the closed interval $[\alpha(n), \beta(n)]$ and $|S|$ represents the cardinality of S.

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Definition 1.1. *[14] A sequence $x=(x_k)$ is said to be $\alpha\beta-$statistically convergent of order γ to L, if for every $\varepsilon>0$*

$$\delta^{\alpha,\beta}\left(\{k:|x_k-L|\geq\varepsilon\},\gamma\right)=\lim_{n\to\infty}\frac{\left|\left\{k\in P_n^{\alpha,\beta}:|x_k-L|\geq\varepsilon\right\}\right|}{\left(\beta(n)-\alpha(n)+1\right)^{\gamma}}=0$$

which is denoted by $st^{\gamma}_{\alpha\beta}-\lim x=L$. For $\gamma=1$, we say that x is $\alpha\beta-$statistically convergent to L, and this is denoted by $st_{\alpha\beta}-\lim x=L$.

Definition 1.2. *[15] A sequence $x=(x_k)$ is said to be $[N^{\gamma},\alpha\beta]_q-$summable to a number L, $0<q<\infty$, if*

$$\lim_{n\to\infty}\frac{1}{\left(\beta(n)-\alpha(n)+1\right)^{\gamma}}\sum_{k\in P_n^{\alpha,\beta}}|x_k-L|^q=0,$$

which is denoted by $x_k\to L[N^{\gamma},\alpha\beta]_q$. Similarly, for $\gamma=1$ the sequence $x=(x_k)$ is said to be $[N,\alpha\beta]_q-$summable to L.

By the convergence of a double sequence we mean the convergence in the Pringsheim sense, that is, a double sequence $x=\left(x_{jk}\right)$ is said to be convergent to L in the Pringsheim sense, if for every $\varepsilon>0$ there exists $N\in\mathbb{N}$ such that $\left|x_{jk}-L\right|<\varepsilon$ whenever $j,k>N$. In this case we write $P-\lim x=L$ [16].
A double sequence $x=\left(x_{jk}\right)$ is bounded if there exists positive number M such that $\left|x_{jk}\right|<M$ for all $j,k\in\mathbb{N}$. We denote the set of all bounded double sequence by l^2_{∞}.
Let $K\subseteq\mathbb{N}\times\mathbb{N}$ and $K(m,n)=\{(j,k):j\leq m,k\leq n\}$. The double natural density of K is defined by

$$\delta_2(K)=P-\lim_{m,n}|K(m,n)|,$$

if the limit exists.
A double sequence $x=\left(x_{jk}\right)$ is said to be statistically convergent to a number L, if for every $\varepsilon>0$ the set $\left\{(j,k),\ j\leq m \text{ and } k\leq n:\left|x_{jk}-L\right|\geq\varepsilon\right\}$ has double natural density zero, i.e. for every $\varepsilon>0$,

$$P-\lim_{m,n}\frac{1}{mn}\left|\left\{(j,k),\ j\leq m \text{ and } k\leq n:\left|x_{jk}-L\right|\geq\varepsilon\right\}\right|=0,$$

which is denoted by $st_2-\lim_{j,k}x_{jk}=L$ [17]. We denote the set of all statistically convergent double sequences by st_2. Note that if $x=\left(x_{jk}\right)$ is $P-$convergent then it is statisically convergent, but not conversely. Also a statistically convergent double sequence need not be bounded. For this, consider a sequence $x=\left(x_{jk}\right)$ defined by

$$x_{jk}=\begin{cases} jk, & \text{if } j \text{ and } k \text{ are square,}\\ 1, & \text{otherwise.}\end{cases}$$

Then, $st_2-\lim x=1$. But x is neither $P-$convergent nor bounded.
Our purpose is to extend the concepts of $\alpha\beta-$statistical convergence and strong $\alpha\beta-$summability from ordinary (i.e. single) sequences to double sequences. This paper organized as follows: In section 2, we introduce the concepts of $\alpha\beta-$statistical convergence and strong $\alpha\beta-$summability of double sequences, and also establish the some relations these new concepts. Moreover, statistical convergence and $\alpha\beta-$statistical convergence of double sequences are compared under some certain assumptions. In section 3, we prove Korovkin type approximation theorem through $\alpha\beta-$statistical convergence for functions of two variables.

2. Main results

We now begin defining the our new concepts of $\alpha\beta-$statistical convergence and strong $\alpha\beta-$summability for double sequences. Throughout the paper, let $(\alpha_1,\beta_1)\in\Lambda$ and $(\alpha_2,\beta_2)\in\Lambda$.

Definition 2.1. *A double sequence $x=\left(x_{jk}\right)$ is said to be $\alpha\beta-$statistically convergent to a number L, if for every $\varepsilon>0$*

$$\lim_{m,n\to\infty}\frac{1}{\left|P_m^{\alpha_1,\beta_1}\right|\left|P_n^{\alpha_2,\beta_2}\right|}\left|\left\{(j,k),\ j\in P_m^{\alpha_1,\beta_1} \text{ and } k\in P_n^{\alpha_2,\beta_2}:\left|x_{jk}-L\right|\geq\varepsilon\right\}\right|=0,$$

which is denoted by $st_2(\alpha\beta)-\lim x_{jk}=L$, where $P_m^{\alpha_1,\beta_1}$ and $P_n^{\alpha_2,\beta_2}$ are the closed intervals $[\alpha_1(m),\beta_1(m)]$ and $[\alpha_2(n),\beta_2(n)]$, respectively.

This definition also includes the following special cases:
i) If we take $\alpha_1(m)=1$, $\beta_1(m)=m$ for all $m\in\mathbb{N}$ and $\alpha_2(n)=1$, $\beta_2(n)=n$ for all $n\in\mathbb{N}$, then $\alpha\beta-$statistical convergence of double sequence is reduced to statistical convergence of double sequences introduced in [17].
ii) Let $\lambda=(\lambda_m)$ and $\mu=\mu(n)$ be two non-decreasing sequences of positive numbers tending to ∞ such that
$\lambda_{m+1}\leq\lambda_m+1,\quad \lambda_1=1,$
and
$\mu_{n+1}\leq\mu_n+1,\quad \mu_1=1.$
Then in the case of $\alpha_1(m)=m-\lambda_m+1$, $\beta_1(m)=m$ for all $m\in\mathbb{N}$ and $\alpha_2(n)=n-\mu_n+1$, $\beta_2(n)=n$ for all $n\in\mathbb{N}$, $\alpha\beta-$statistical convergence of double sequence is reduced to $(\lambda,\mu)-$statistical convergence of double sequence introduced in [18].
iii) Recall that a double lacunary sequence $\theta_{r,s}=\{(k_r,l_s)\}$, which means there exist two increasing of integers such that
$k_0=0,\qquad h_r=k_r-k_{r-1}\to\infty\quad \text{as}\quad r\to\infty,$
and
$l_0=0,\qquad \bar{h}_s=l_s-l_{s-1}\to\infty\quad \text{as}\quad s\to\infty.$
If we take $\alpha_1(m)=k_{m-1}+1$, $\beta_1(m)=k_m$ for all $m\in\mathbb{N}$ and $\alpha_2(n)=l_{n-1}+1$, $\beta_2(n)=l_n$ for all $n\in\mathbb{N}$, then $\alpha\beta-$statistical convergence of double sequence is reduced to lacunary statistical convergence of double sequence introduced in [19].

Definition 2.2. *A double sequence $x=(x_{jk})$ is said to be strongly $\alpha\beta-$summable or briefly $[N_2,\alpha\beta]-$summable to a number L, if*

$$\lim_{m,n\to\infty}\frac{1}{\left|P_m^{\alpha_1,\beta_1}\right|\left|P_n^{\alpha_2,\beta_2}\right|}\sum_{j\in P_m^{\alpha_1,\beta_1}}\sum_{k\in P_n^{\alpha_2,\beta_2}}\left|x_{jk}-L\right|=0,$$

and we denote it by $x_{jk}\to L[N_2,\alpha\beta]$.

We shall denote the set of all $\alpha\beta-$statistically convergent double sequences by $st_2(\alpha\beta)$, and the set of all $[N_2,\alpha\beta]-$summable double sequences by $[N_2,\alpha\beta]$.

Then, we get the following results.

Theorem 2.3. *If a double sequence $x=(x_{jk})$ is $[N_2,\alpha\beta]-$summable to L, then it is $\alpha\beta-$statistically convergent to L, that is, $[N_2,\alpha\beta]\subseteq st_2(\alpha\beta)$ and also the inclusion is strict.*

Proof. Let $x_{jk}\to L[N_2,\alpha\beta]$ and given $\varepsilon>0$. Then, we have

$$\frac{1}{\left|P_m^{\alpha_1,\beta_1}\right|\left|P_n^{\alpha_2,\beta_2}\right|}\sum_{j\in P_m^{\alpha_1,\beta_1}}\sum_{k\in P_n^{\alpha_2,\beta_2}}\left|x_{jk}-L\right|=\frac{1}{\left|P_m^{\alpha_1,\beta_1}\right|\left|P_n^{\alpha_2,\beta_2}\right|}\sum_{\substack{j\in P_m^{\alpha_1,\beta_1}\\ |x_{jk}-L|\geq\varepsilon}}\sum_{k\in P_n^{\alpha_2,\beta_2}}\left|x_{jk}-L\right|+\frac{1}{\left|P_m^{\alpha_1,\beta_1}\right|\left|P_n^{\alpha_2,\beta_2}\right|}\sum_{\substack{j\in P_m^{\alpha_1,\beta_1}\\ |x_{jk}-L|<\varepsilon}}\sum_{k\in P_n^{\alpha_2,\beta_2}}\left|x_{jk}-L\right|$$

$$\geq\varepsilon\frac{1}{\left|P_m^{\alpha_1,\beta_1}\right|\left|P_n^{\alpha_2,\beta_2}\right|}\left|\left\{(j,k),\ j\in P_m^{\alpha_1,\beta_1}\text{ and }k\in P_n^{\alpha_2,\beta_2}:\left|x_{jk}-L\right|\geq\varepsilon\right\}\right|,$$

which means that $st_2(\alpha\beta)-\lim x_{jk}=L$.
To show that the inclusion is strict, we consider the following example: Let $\alpha_1(m)\leq 1\leq\beta_1(m)$ and $\alpha_2(n)\leq 1\leq\beta_2(n)$ for all $m,n\in\mathbb{N}$, and the sequence $x=(x_{jk})$ be defined by

$$x_{jk}=\begin{cases} jk, & 1\leq j\leq\left[\sqrt{\beta_1(m)-\alpha_1(m)+1}\right]\text{ and }1\leq j\leq\left[\sqrt{\beta_2(n)-\alpha_2(n)+1}\right],\\ 0, & \text{otherwise.}\end{cases}$$

Then, we have

$$\frac{1}{\left|P_m^{\alpha_1,\beta_1}\right|\left|P_n^{\alpha_2,\beta_2}\right|}\left|\left\{(j,k),\ j\in P_m^{\alpha_1,\beta_1}\text{ and }k\in P_n^{\alpha_2,\beta_2}:\left|x_{jk}-L\right|\geq\varepsilon\right\}\right|$$

$$=\frac{\left[\sqrt{\beta_1(m)-\alpha_1(m)+1}\right]\left[\sqrt{\beta_2(n)-\alpha_2(n)+1}\right]}{(\beta_1(m)-\alpha_1(m)+1)(\beta_2(n)-\alpha_2(n)+1)}\to 0\quad\text{as}\quad m,n\to\infty.$$

That is, $st_2(\alpha\beta)-\lim x_{jk}=0$. But

$$\frac{1}{\left|P_m^{\alpha_1,\beta_1}\right|\left|P_n^{\alpha_2,\beta_2}\right|}\sum_{j\in P_m^{\alpha_1,\beta_1}}\sum_{k\in P_n^{\alpha_2,\beta_2}}\left|x_{jk}-0\right|$$

$$=\frac{\left[\sqrt{\beta_1(m)-\alpha_1(m)+1}\right]\left(\left[\sqrt{\beta_1(m)-\alpha_1(m)+1}\right]+1\right)\left[\sqrt{\beta_2(n)-\alpha_2(n)+1}\right]\left(\left[\sqrt{\beta_2(n)-\alpha_2(n)+1}\right]+1\right)}{4(\beta_1(m)-\alpha_1(m)+1)(\beta_2(n)-\alpha_2(n)+1)}\to\frac{1}{4}$$

which means $x_{jk}\not\to 0[N_2,\alpha\beta]$. □

Theorem 2.4. *If a double sequence $x=(x_{jk})$ bounded and $\alpha\beta-$statistically convergent to L, then $x_{jk}\to L[N_2,\alpha\beta]$.*

Proof. Assume that $x=(x_{jk})$ is bounded and $\alpha\beta-$statistically convergent to L. Since $x=(x_{jk})$ is bounded, there exists $M>0$ such that $\left|x_{jk}-L\right|\leq M$ for $j,k\in\mathbb{N}$. Then we can see that

$$\frac{1}{\left|P_m^{\alpha_1,\beta_1}\right|\left|P_n^{\alpha_2,\beta_2}\right|}\sum_{j\in P_m^{\alpha_1,\beta_1}}\sum_{k\in P_n^{\alpha_2,\beta_2}}\left|x_{jk}-L\right|=\frac{1}{\left|P_m^{\alpha_1,\beta_1}\right|\left|P_n^{\alpha_2,\beta_2}\right|}\sum_{\substack{j\in P_m^{\alpha_1,\beta_1}\\ |x_{jk}-L|\geq\varepsilon}}\sum_{k\in P_n^{\alpha_2,\beta_2}}\left|x_{jk}-L\right|+\frac{1}{\left|P_m^{\alpha_1,\beta_1}\right|\left|P_n^{\alpha_2,\beta_2}\right|}\sum_{\substack{j\in P_m^{\alpha_1,\beta_1}\\ |x_{jk}-L|<\varepsilon}}\sum_{k\in P_n^{\alpha_2,\beta_2}}\left|x_{jk}-L\right|$$

$$\leq\frac{M}{\left|P_m^{\alpha_1,\beta_1}\right|\left|P_n^{\alpha_2,\beta_2}\right|}\left|\left\{(j,k),\ j\in P_m^{\alpha_1,\beta_1}\text{ and }k\in P_n^{\alpha_2,\beta_2}:\left|x_{jk}-L\right|\geq\varepsilon\right\}\right|+\varepsilon.$$

Taking limit as $m,n\to\infty$ on the both sides of last inequality and also using the hypothesis, we obtain that

$$\lim_{m,n}\frac{1}{\left|P_m^{\alpha_1,\beta_1}\right|\left|P_n^{\alpha_2,\beta_2}\right|}\sum_{j\in P_m^{\alpha_1,\beta_1}}\sum_{k\in P_n^{\alpha_2,\beta_2}}\left|x_{jk}-L\right|=0,$$

which completes the proof. □

Theorem 2.5. *If $\liminf_m\frac{\beta_1(m)}{\alpha_1(m)}>1$ and $\liminf_n\frac{\beta_2(n)}{\alpha_2(n)}>1$, then $st_2-\lim x_{jk}=L$ implies $st_2(\alpha\beta)-\lim x_{jk}=L$.*

Proof. Suppose that $\liminf_m \frac{\beta_1(m)}{\alpha_1(m)} > 1$ and $\liminf_n \frac{\beta_2(n)}{\alpha_2(n)} > 1$. Then, there exists $\delta > 0$ such that $\frac{\beta_1(m)}{\alpha_1(m)} \geq 1+\delta$ and $\frac{\beta_2(n)}{\alpha_2(n)} \geq 1+\delta$, hence we obtain that $\frac{\beta_1(m)-\alpha_1(m)+1}{\beta_1(m)} \geq \frac{\delta}{1+\delta}$ and $\frac{\beta_2(n)-\alpha_2(n)+1}{\beta_2(n)} \geq \frac{\delta}{1+\delta}$. Now let $st_2-\lim x_{jk}=L$. Then, for a given $\varepsilon>0$, we may write that

$$\begin{aligned}
&\frac{1}{\beta_1(m)\beta_2(n)}\left|\left\{(j,k),\ j\leq\beta_1(m) \text{ and } k\leq\beta_2(n): \left|x_{jk}-L\right|\geq\varepsilon\right\}\right| \\
&\geq \frac{1}{\beta_1(m)\beta_2(n)}\left|\left\{(j,k),\ j\in P_m^{\alpha_1,\beta_1} \text{ and } k\in P_n^{\alpha_2,\beta_2}: \left|x_{jk}-L\right|\geq\varepsilon\right\}\right| \\
&= \frac{(\beta_1(m)-\alpha_1(m)+1)(\beta_2(n)-\alpha_2(n)+1)}{\beta_1(m)\beta_2(n)}\frac{\left|\left\{(j,k),\ j\in P_m^{\alpha_1,\beta_1} \text{ and } k\in P_n^{\alpha_2,\beta_2}: \left|x_{jk}-L\right|\geq\varepsilon\right\}\right|}{(\beta_1(m)-\alpha_1(m)+1)(\beta_2(n)-\alpha_2(n)+1)} \\
&\geq \left(\frac{\delta}{1+\delta}\right)^2\frac{\left|\left\{(j,k),\ j\in P_m^{\alpha_1,\beta_1} \text{ and } k\in P_n^{\alpha_2,\beta_2}: \left|x_{jk}-L\right|\geq\varepsilon\right\}\right|}{(\beta_1(m)-\alpha_1(m)+1)(\beta_2(n)-\alpha_2(n)+1)}.
\end{aligned}$$

Since $st_2-\lim x_{jk}=L$, the left hand side of the last inequality tends to zero as $m,n\to\infty$, which yields that

$$\lim_{m,n\to\infty}\frac{\left|\left\{(j,k),\ j\in P_m^{\alpha_1,\beta_1} \text{ and } k\in P_n^{\alpha_2,\beta_2}: \left|x_{jk}-L\right|\geq\varepsilon\right\}\right|}{(\beta_1(m)-\alpha_1(m)+1)(\beta_2(n)-\alpha_2(n)+1)}=0.$$

This completes the proof of the theorem. □

Theorem 2.6. *If* $\lim_{m,n\to\infty}\frac{\alpha_1(m)\alpha_2(n)}{\beta_1(m)\beta_2(n)}=0$, $\alpha_1(m)\geq 1$ *and* $\alpha_2(n)\geq 1$ *for all* $m,n\in\mathbb{N}$, *then* $st_2(\alpha\beta)-\lim x_{jk}=L$ *implies* $st_2-\lim x_{jk}=L$.

Proof. Suppose that $\lim_{m,n\to\infty}\frac{\alpha_1(m)\alpha_2(n)}{\beta_1(m)\beta_2(n)}=0$, $\alpha_1(m)\geq 1$ and $\alpha_2(n)\geq 1$ for all $m,n\in\mathbb{N}$. Then, for a given $\varepsilon>0$, we can write

$$\begin{aligned}
&\frac{1}{\beta_1(m)\beta_2(n)}\left|\left\{(j,k),\ j\leq\beta_1(m) \text{ and } k\leq\beta_2(n): \left|x_{jk}-L\right|\geq\varepsilon\right\}\right| \\
&= \frac{1}{\beta_1(m)\beta_2(n)}\left|\left\{(j,k),\ j<\alpha_1(m) \text{ and } k<\alpha_2(n): \left|x_{jk}-L\right|\geq\varepsilon\right\}\right| \\
&\quad+\frac{1}{\beta_1(m)\beta_2(n)}\left|\left\{(j,k),\ j\in P_m^{\alpha_1,\beta_1} \text{ and } k\in P_n^{\alpha_2,\beta_2}: \left|x_{jk}-L\right|\geq\varepsilon\right\}\right| \\
&\leq \frac{\alpha_1(m)\alpha_2(n)}{\beta_1(m)\beta_2(n)}+\frac{1}{\left|P_m^{\alpha_1,\beta_1}\right|\left|P_n^{\alpha_2,\beta_2}\right|}\left|\left\{(j,k),\ j\in P_m^{\alpha_1,\beta_1} \text{ and } k\in P_n^{\alpha_2,\beta_2}: \left|x_{jk}-L\right|\geq\varepsilon\right\}\right|.
\end{aligned}$$

Taking limit as $m,n\to\infty$ on the both sides of last inequality, since $st_2(\alpha\beta)-\lim x_{jk}=L$, we obtain that

$$\frac{1}{\beta_1(m)\beta_2(n)}\left|\left\{(j,k),\ j\leq\beta_1(m) \text{ and } k\leq\beta_2(n): \left|x_{jk}-L\right|\geq\varepsilon\right\}\right|=0,$$

which completes the proof. □

3. Application to Korovkin type approximation theorem

Let $C[a,b]$ be the linear space of all real valued continuous functions f on $[a,b]$. It is well known that $C[a,b]$ is a Banach space with the norm

$$\|f\|_\infty=\sup_{x\in[a,b]}|f(x)|,\qquad f\in C[a,b].$$

Suppose that T be a linear operator from $C[a,b]$ into $C[a,b]$. We write $T_n(f,x)$ for $T_n(f(t),x)$ and we say that T is a positive linear operator if $T(f,x)\geq 0$ for all $f(x)\geq 0$. The classical Korovkin theorem states as follows [20]:
Suppose that (T_n) be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$. Then

$$\lim_n\|T_n(f,x)-f(x)\|_\infty=0,\qquad \text{for all}\quad f\in C[a,b],$$

if and only if

$$\lim_n\|T_n(f_i,x)-f_i(x)\|_\infty=0,\qquad \text{for}\quad i=0,1,2,$$

where $f_0(x)=1$, $f_1(x)=x$ and $f_2(x)=x^2$.
Recently, Korovkin type approximation theorem have been studied for functions of one or two variables by using different summability methods, see for instance [21]-[33] and etc.
By $C(K)$, we denote the space of all continuous real valued functions on any compact subset of the real two-dimensional space. This space is equipped with the supremum norm

$$\|f\|_{C(K)}=\sup_{(x,y)\in K}|f(x,y)|,\qquad f\in C(K).$$

Before proceeding further, we recall here the classical Korovkin type approximation theorem for a function of two variables in Pringsheim sense given in [21].

Theorem 3.1. *[21] Let (T_{jk}) be a double sequence of positive linear operators from $C(K)$ into $C(K)$. Then for all $f \in C(K)$,*

$$P - \lim \left\| T_{jk} f - f \right\|_{C(K)} = 0$$

if and only if

$$P - \lim \left\| T_{jk} f_i - f_i \right\|_{C(K)} = 0, \qquad (i = 0,1,2,3)$$

where $f_0(x,y) = 1$, $f_1(x,y) = x$, $f_2(x,y) = y$ and $f_3(x,y) = x^2 + y^2$.

Now, we give the main result of this section.

Theorem 3.2. *Let (T_{jk}) be a double sequence of positive linear operators from $C(K)$ into $C(K)$. Then for all $f \in C(K)$,*

$$st_2(\alpha\beta) - \lim \left\| T_{jk} f - f \right\|_{C(K)} = 0 \tag{3.1}$$

if and only if

$$st_2(\alpha\beta) - \lim \left\| T_{jk} f_i - f_i \right\|_{C(K)} = 0, \qquad (i = 0,1,2,3) \tag{3.2}$$

where $f_0(x,y) = 1$, $f_1(x,y) = x$, $f_2(x,y) = y$ and $f_3(x,y) = x^2 + y^2$.

Proof. Since each $f_i \in C(K)$ for $(i = 0,1,2,3)$, condition (3.2) follows immediately from (3.1). Suppose now that the condition (3.2) holds and $f \in C(K)$. By the continuity of f on compact set K, we can write $|f(x,y)| \leq M$ where $M := \|f\|_{C(K)}$. Also, since f is continuous on K, for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|f(u,v) - f(x,y)| < \varepsilon$ for all $(u,v) \in K$ satisfying $|u - x| < \delta$ and $|v - y| < \delta$. Hence, we get

$$|f(u,v) - f(x,y)| < \varepsilon + \frac{2M}{\delta^2} \left\{ (u-x)^2 + (v-y)^2 \right\} \tag{3.3}$$

Since T_{jk} is linear and positive, from (3.3), we obtain that

$$\begin{aligned}
&\left| T_{jk}(f;x,y) - f(x,y) \right| \\
&= \left| T_{jk}(f(u,v) - f(x,y);x,y) - f(x,y)\left(T_{jk}(f_0;x,y) - f_0(x,y)\right) \right| \\
&\leq T_{jk}(|f(u,v) - f(x,y)|;x,y) + M \left| T_{jk}(f_0;x,y) - f_0(x,y) \right| \\
&\leq \left| T_{jk}\left(\varepsilon + \tfrac{2M}{\delta^2} \left\{ (u-x)^2 + (v-y)^2 \right\};x,y \right) \right| + M \left| T_{jk}(f_0;x,y) - f_0(x,y) \right| \\
&\leq \left(\varepsilon + M + \tfrac{2M}{\delta^2}\left(A^2 + B^2\right) \right) \left| T_{jk}(f_0;x,y) - f_0(x,y) \right| \\
&\quad + \tfrac{4M}{\delta^2} A \left| T_{jk}(f_1;x,y) - f_1(x,y) \right| + \tfrac{4M}{\delta^2} B \left| T_{jk}(f_2;x,y) - f_2(x,y) \right| \\
&\quad + \tfrac{2M}{\delta^2} \left| T_{jk}(f_3;x,y) - f_3(x,y) \right| + \varepsilon,
\end{aligned}$$

where $A := \max |x|$ and $B := \max |y|$. Taking supremum over $(x,y) \in K$, we get

$$\left\| T_{jk} f - f \right\|_{C(K)} \leq R \left\{ \left\| T_{jk} f_0 - f_0 \right\|_{C(K)} + \left\| T_{jk} f_1 - f_1 \right\|_{C(K)} + \left\| T_{jk} f_2 - f_2 \right\|_{C(K)} + \left\| T_{jk} f_3 - f_3 \right\|_{C(K)} \right\} + \varepsilon,$$

where $R = \max \left\{ \varepsilon + M + \frac{2M}{\delta^2}\left(A^2 + B^2\right), \frac{4M}{\delta^2} A, \frac{4M}{\delta^2} B, \frac{2M}{\delta^2} \right\}$.

Now, for a given $r > 0$, choose $\varepsilon' > 0$ such that $\varepsilon' < r$. Define the following sets

$$D = \left\{ (j,k),\ j \in P_m^{\alpha_1,\beta_1} \text{ and } k \in P_n^{\alpha_2,\beta_2} : \left\| T_{jk} f - f \right\|_{C(K)} \geq r \right\},$$

$$D_i = \left\{ (j,k),\ j \in P_m^{\alpha_1,\beta_1} \text{ and } k \in P_n^{\alpha_2,\beta_2} : \left\| T_{jk} f_i - f_i \right\|_{C(K)} \geq \tfrac{r - \varepsilon'}{4R} \right\},$$

for $i = 0,1,2,3$. Then, $D \subset \bigcup_{i=0}^{3} D_i$ and so we also get

$$\begin{aligned}
&\frac{1}{\left|P_m^{\alpha_1,\beta_1}\right| \left|P_n^{\alpha_2,\beta_2}\right|} \left\{ (j,k),\ j \in P_m^{\alpha_1,\beta_1} \text{ and } k \in P_n^{\alpha_2,\beta_2} : \left\| T_{jk} f - f \right\|_{C(K)} \geq r \right\} \\
&\leq \frac{1}{\left|P_m^{\alpha_1,\beta_1}\right| \left|P_n^{\alpha_2,\beta_2}\right|} \left\{ (j,k),\ j \in P_m^{\alpha_1,\beta_1} \text{ and } k \in P_n^{\alpha_2,\beta_2} : \left\| T_{jk} f_0 - f_0 \right\|_{C(K)} \geq \tfrac{r - \varepsilon'}{4R} \right\} \\
&\quad + \frac{1}{\left|P_m^{\alpha_1,\beta_1}\right| \left|P_n^{\alpha_2,\beta_2}\right|} \left\{ (j,k),\ j \in P_m^{\alpha_1,\beta_1} \text{ and } k \in P_n^{\alpha_2,\beta_2} : \left\| T_{jk} f_1 - f_1 \right\|_{C(K)} \geq \tfrac{r - \varepsilon'}{4R} \right\} \\
&\quad + \frac{1}{\left|P_m^{\alpha_1,\beta_1}\right| \left|P_n^{\alpha_2,\beta_2}\right|} \left\{ (j,k),\ j \in P_m^{\alpha_1,\beta_1} \text{ and } k \in P_n^{\alpha_2,\beta_2} : \left\| T_{jk} f_2 - f_2 \right\|_{C(K)} \geq \tfrac{r - \varepsilon'}{4R} \right\} \\
&\quad + \frac{1}{\left|P_m^{\alpha_1,\beta_1}\right| \left|P_n^{\alpha_2,\beta_2}\right|} \left\{ (j,k),\ j \in P_m^{\alpha_1,\beta_1} \text{ and } k \in P_n^{\alpha_2,\beta_2} : \left\| T_{jk} f_3 - f_3 \right\|_{C(K)} \geq \tfrac{r - \varepsilon'}{4R} \right\}.
\end{aligned}$$

Hence, using condition (3.2), we obtain

$$st_2(\alpha\beta) - \lim \left\| T_{jk}f - f \right\|_{C(K)} = 0.$$

This completes the proof of theorem. □

Remark 3.3. *We now construct an example of sequence of positive linear operators of two variables satisfying the conditions of Theorem 3.2, but does not satisfy the conditions of the Korovkin theorem given in Theorem 3.1. For this, we consider the following Bernstein operators given by*

$$B_{mn}(f;x,y) = \sum_{k=0}^{m} \sum_{j=0}^{n} f\left(\frac{k}{m}, \frac{j}{n}\right) C_m^k x^k (1-x)^{m-k} C_n^j y^j (1-y)^{n-j},$$

where $(x,y) \in K = [0,1] \times [0,1]$; $f \in C(K)$. *Also, observe that*

$$B_{mn}(f_0;x,y) = 1,$$

$$B_{mn}(f_1;x,y) = x,$$

$$B_{mn}(f_2;x,y) = y,$$

$$B_{mn}(f_3;x,y) = x^2 + y^2 + \tfrac{x-x^2}{m} + \tfrac{y-y^2}{n},$$

where $f_0(x,y) = 1$, $f_1(x,y) = x$, $f_2(x,y) = y$ *and* $f_3(x,y) = x^2 + y^2$. *Then, by Theorem 3.1, we know that, for any* $f \in C(K)$,

$$P - \lim \| B_{mn}f - f \|_{C(K)} = 0.$$

Now, we define the sequence of linear operators as $T_{mn} : C(K) \to C(K)$ *with* $T_{mn}(f;x,y) = (1 + x_{mn}) B_{mn}(f;x,y)$, *where* $x = (x_{mn})$ *is defined in Theorem 2.3. Note that the sequence* $x = (x_{mn})$ *is* $\alpha\beta-$*statistically convergent to zero, but not* $P-$ *convergent. Then the double sequence* T_{mn} *satisfies condition* (3.2) *for* $i = 0,1,2,3$, *hence, by Theorem 3.2, we get*

$$st_2(\alpha\beta) - \lim \| T_{mn}f - f \|_{C(K)} = 0.$$

On the other hand, we have $T_{mn}(f;0,0) = (1 + x_{mn}) f(0,0)$ *since* $B_{mn}(f;0,0) = f(0,0)$, *and hence we obtain*

$$\| T_{mn}(f;x,y) - (f;x,y) \|_{C(K)} \geq |T_{mn}(f;0,0) - (f;0,0)| \geq x_{mn} |(f;0,0)|.$$

One can see that (T_{mn}) *does not satisfy the Korovkin theorem for positive linear operators of two variables in the Pringsheim's sense, since* $P - \lim x_{mn}$ *does not exists. That is, Theorem 3.1 does not work for our operators* T_{mn}. *Hence, our Theorem 3.2 is stronger than Theorem 3.1. This proves our claim.*

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