Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (4) (2020), 1315 – 1333
DOI : 10.15672/hujms.532964

Research Article

Some results on higher orders quasi-isometries

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Abstract

The purpose of the present paper is to pursue further study of a class of linear bounded operators, known as n-quasi-m-isometric operators acting on an infinite complex separable Hilbert space $\mathcal{H}$. We give an equivalent condition for any T to be n-quasi-m-isometric operator. Using this result we prove that any power of an n-quasi-m-isometric operator is also an n-quasi-m-isometric operator. In general the converse is not true. However, we prove that if T^r and T^{r+1} are n-quasi-m-isometries for a positive integer r, then T is an n-quasi-m-isometric operator. We study the sum of an n-quasi-m-isometric operator with a nilpotent operator. We also study the product and tensor product of two n-quasi-m-isometries. Further, we define n-quasi strict m-isometric operators and prove their basic properties.

Mathematics Subject Classification (2010). 47B99, 47A05

Keywords. m-isometry, strict m-isometry, n-quasi-m-isometry

1. Introduction

Throughout this paper, $\mathbb{N}$ denotes the set of non negative integers, $\mathcal{H}$ stands for an infinite separable complex Hilbert space with inner product $\langle . \mid . \rangle$, $\mathcal{L}(\mathcal{H})$ is the Banach algebra of all bounded linear operators on $\mathcal{H}$ and $I = I_{\mathcal{H}}$ the identity operator. For every $T \in \mathcal{L}(\mathcal{H})$ we denote by $\mathcal{R}(T)$, $\mathcal{N}(T)$ and T^* the range, the null space and the adjoint of T, respectively. A closed subspace $\mathcal{M} \subset \mathcal{H}$ is invariant for T (or T-invariant) if $T\mathcal{M} \subset \mathcal{M}$. As usual, the orthogonal complement and the closure of $\mathcal{M}$ are denoted $\mathcal{M}^{\perp}$ and $\overline{\mathcal{M}}$, respectively. We denote by $P_{\mathcal{M}}$ the orthogonal projection on $\mathcal{M}$.

Some of the most important subclasses of the algebra of all bounded linear operators acting on a Hilbert space, are the classes of partial isometries and quasi-isometries. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an isometry if $T^*T = I$, a partial isometry if $TT^*T = T$ and quasi-isometry if $T^{*2}T^2 = T^*T$.

In recent years these classes has been generalized, in some sense, to the larger sets of operators so-called m-isometries, m-partial isometries and n-quasi-isometries. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be

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Received: 27.02.2019; Accepted: 12.09.2019

(1) m-isometric operator for some integer $m \geq 1$ if it satisfies the operator equation

$$\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}T^{*k}T^{k}=0, \tag{1.1}$$

It is immediate that T is m-isometric operator if and only if

$$\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\|T^{k}x\|^{2}=0 \quad \forall\, x\in\mathcal{H}. \tag{1.2}$$

(2) m-partial-isometry for some integer $m \geq 1$ if

$$T\Big(\sum_{0\leq k\leq m}(-1)^{k}\binom{m}{k}T^{*m-k}T^{m-k}\Big)=0, \tag{1.3}$$

(3) (m,q)-partial isometry (or q-partial-m-isometry) if

$$T^{q}\Big(\sum_{0\leq k\leq m}(-1)^{k}\binom{m}{k}T^{*m-k}T^{m-k}\Big)=0, \tag{1.4}$$

(4) m-quasi-isometry for some integer $n \geq 1$ if

$$T^{*m+1}T^{m+1}-T^{*m}T^{m}=0. \tag{1.5}$$

It is immediate that T is m-quasi-isometric if and only if

$$T^{*m}\Big(T^{*}T-I\Big)T^{m}=0.$$

Here $\binom{m}{k}$ is the binomial coefficient. In [1], J. Agler and M. Stankus initiated the study of operators T that satisfy the identity (1.1). In [24], A. Saddi and O. A. M. Sid Ahmed studied an operator T which satisfies (1.3). This concept was later generalized to the operators satisfying (1.4), was defined by O. A. M. Sid Ahmed [17]. The study of operators satisfying (1.5) was introduced and study by L. Suciu in [25]. The 1-quasi-isometries are shortly called quasi-isometries, such operators being firstly studied in [20] and [22].

Recently, S. Mecheri and T. Prasad [19] introduced the class of n-quasi-m-isometric operators which generalizes the class of m-isometric operators and n-quasi-isometries. For positive integers m and n, an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an n-quasi-m-isometric operator if

$$T^{*n}\Big(\sum_{0\leq k\leq m}(-1)^{k}\binom{m}{k}T^{*m-k}T^{m-k}\Big)T^{n}=0, \tag{1.6}$$

After an introduction on the subject and some connection with known facts in this context, the results of the paper are briefly described. In section two, we give a matrix characterization of n-quasi-m-isometries by using the decomposition $\mathcal{H}=\overline{T^{n}(\mathcal{H})}\oplus T^{*-n}(0)$. Several properties are proved by exploiting the special kind of operator matrix representation associated with such operators. In the course of our investigation, we find some properties of m-isometries which are retained by n-quasi-m-isometries. In particular, we show that if $T\in\mathcal{L}(\mathcal{H})$ is an n-quasi-isometry then its power is an n-quasi-isometry. If T and S are doubly commuting such that T is an n_1-quasi-m-isometry and S is an n_2-quasi-l-isometry, then TS is a $n_0=\max\{n_1,n_2\}$-quasi-$(m+l-1)$-isometry. It has also been proved that the sum of an n-quasi-m-isometry and a commuting nilpotent operator of degree p is a $2\max\{n,p\}$-quasi-$(m+2p-2)$-isometry. In section three, we recall the definition of n-quasi strict-m-isometries and we give some of their properties which are similar to those of n-quasi-m-isometries.

2. Some properties of n-quasi-m-isometric operators

In this section, we study some further properties of n-quasi-m-isometries. First, we will start with the following notations.

For $T \in \mathcal{L}(\mathcal{H})$, we set

$$\beta_m(T) := \sum_{0\leq k\leq m} (-1)^{m-k}\binom{m}{k} T^{*k}T^k, \tag{2.1}$$

$$\beta_{m,\,n}(T) := T^{*n}\Big(\sum_{0\leq k\leq m} (-1)^{m-k}\binom{m}{k} T^{*k}T^k\Big)T^n \tag{2.2}$$

and

$$\Delta_{m,\,n}(T,x) := \sum_{0\leq k\leq m} (-1)^{m-k}\binom{m}{k} \|T^{k+n}x\|^2, \qquad x \in \mathcal{H}. \tag{2.3}$$

Observe that T is an n-quasi-m-isometric operator if and only if $\beta_{m,\,n}(T) = 0$ or equivalently, if

$$\Delta_{m,\,n}(T,x) = 0, \quad \forall\ x \in \mathcal{H}.$$

Lemma 2.1. *Let $T \in \mathcal{L}(\mathcal{H})$. Then T is an n-quasi-m-isometric operator if and only if*

$$\sum_{0\leq k\leq m} (-1)^{m-k}\binom{m}{k} \|T^k x\|^2 = 0 \quad \forall\ x \in \overline{\mathcal{R}(T^n)}, \tag{2.4}$$

i.e., T is an m-isometric operator on $\overline{\mathcal{R}(T^n)}$.

Proof. The proof is obvious. □

Let $\mathbb{Z}$ denote the set of integers and $\mathbb{Z}_+$ denote the set of nonnegative integers.

Lemma 2.2 ([13, Lemma 5.4])**.** *Let $(a_j)_{j\in\mathbb{Z}_+}$ be a sequence of real numbers. Then*

$$\sum_{0\leq k\leq m} (-1)^{m-k}\binom{m}{k} a_{j+k} = 0 \quad \text{for } j \geq 0$$

if and only if there exists a polynomial P of degree less than or equal to $m-1$ such that $a_j = P(j)$. In this case P is the unique polynomial interpolating $\{(j, a_j)\}$, $0 \leq j \leq m-1$.

Proposition 2.3. *Let $T \in \mathcal{L}(\mathcal{H})$ and $x \in \mathcal{H}$. Then T is an n-quasi-m-isometry if and only if for each $x \in \mathcal{H}$, there exists a polynomial P of degree less than or equal to $m-1$, such that $P(j) = \|T^{n+j}x\|^2$ for $j \in \mathbb{Z}_+$.*

Proof. The proof is a consequence of Lemma 2.1, Lemma 2.2 and [13, Theorem 5.5]. □

In [19], S. Mecheri and T. Prasad studied the matrix representation of n-quasi-m-isometric operator with respect to the direct sum of $\overline{\mathcal{R}(T^n)}$ and its orthogonal complement. In the following, we give an equivalent condition for T to be n-quasi-m-isometric operator. Using this result we obtained several important properties of this class of operators.

Theorem 2.4. *Let $T \in \mathcal{L}(\mathcal{H})$ such that $\overline{\mathcal{R}(T^n)} \neq \mathcal{H}$, then the following statements are equivalent.*

(1) *T is an n-quasi-m-isometric operator.*

(2) $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ *on $\mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}(T^{*n})$, where T_1 is an m-isometric operator and $T_3^n = 0$.*

Proof. (1) $\Rightarrow$ (2) It follows from [19, Lemma 2.1].

(2) $\Rightarrow$ (1) Suppose that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}(T^{*n})$, with

$$\beta_m(T_1) = 0 \text{ and } T_3^n = 0.$$

Since $T^k = \begin{pmatrix} T_1^k & \sum\limits_{0 \le j \le k-1} T_1^j T_2 T_3^{k-1-j} \\ 0 & T_3^k \end{pmatrix}$ for all $k \ge 1$, we have

$$\begin{aligned}
\beta_{m,n}(T) =& T^{*n}\Big(\sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k}T^{*k}T^k\Big)T^n \\
=& \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^{*n}\Big\{(-1)^m I + \sum_{1\le k\le m}(-1)^{m-k}\binom{m}{k}\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^{*k}\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^{k}\Big\} \\
&\times \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^{n} \\
=& \begin{pmatrix} T_1^{*n} & 0 \\ \sum\limits_{0\le j\le n-1} T_3^{*n-1-j}T_2^*T_1^{*j} & T_3^{*n} \end{pmatrix} \\
&\times \Big\{(-1)^m I + \sum_{1\le k\le m}(-1)^k\binom{m}{k}\begin{pmatrix} T_1^{*k} & 0 \\ \sum\limits_{0\le j\le k-1} T_3^{*k-1-j}T_2^*T_1^{*j} & T_3^{*k} \end{pmatrix} \\
&\times \begin{pmatrix} T_1^k & \sum\limits_{j=0}^{k-1} T_1^jT_2T_3^{k-1-j} \\ 0 & T_3^k \end{pmatrix}\Big\} \\
&\times \begin{pmatrix} T_1^n & \sum\limits_{0\le j\le n-1} T_1^jT_2T_3^{n-1-j} \\ 0 & T_3^n \end{pmatrix} \\
=& \begin{pmatrix} T_1^{*n} & 0 \\ \sum\limits_{0\le j\le n-1} T_3^{*n-1-j}T_2^*T_1^{*j} & 0 \end{pmatrix} \\
&\times \Big\{\begin{pmatrix} \beta_m(T_1) & C \\ D & B \end{pmatrix}\Big\} \times \begin{pmatrix} T_1^n & \sum\limits_{0\le j\le n-1} T_1^jT_2T_3^{n-1-j} \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

where $B, C, D \in \mathcal{L}(\mathcal{H})$. Moreover

$$\beta_{m,n}(T) = \begin{pmatrix} T_1^{*n}\beta_m(T_1)T_1^n & T_1^{*n}\beta_m(T_1)\sum\limits_{0\le j\le n-1} T_1^jT_2T_3^{n-1-j} \\ \sum\limits_{0\le j\le n-1} T_3^{*n-1-j}T_2^*T_1^{*j}\beta_m(T_1)T_1^n & \sum\limits_{0\le j\le n-1} T_3^{*n-1-j}T_2^*T_1^{*j}\beta_m(T_1)\sum\limits_{0\le j\le n-1} T_1^jT_2T_3^{n-1-j} \end{pmatrix}.$$

Since $\beta_m(T_1) = 0$, it follows that $\beta_{m,\,n}(T) = 0$. Thus T is an n-quasi-m-isometric operator. □

For $T \in \mathcal{L}(\mathcal{H})$, we denote by $\sigma(T)$, $\sigma_{ap}(T)$ and $\sigma_p(T)$ the spectrum, the approximate point spectrum and the point spectrum of T, respectively.

Corollary 2.5. *Let $T \in \mathcal{L}(\mathcal{H})$ be an n-quasi-m-isometric operator. The following statements hold.*

(i) $\sigma(T) = \sigma(T_1) \cup \{0\}$ *where* $T_1 = T|_{\overline{\mathcal{R}(T^n)}}$.

(ii) T_1 *is bounded below.*

(iii) *If* $\mu \in \sigma_{ap}(T) \setminus \{0\}$ *then* $\overline{\mu} \in \sigma_{ap}(T^*)$. *In particular, if* $\mu \in \sigma_p(T) \setminus \{0\}$ *then* $\overline{\mu} \in \sigma_p(T^*)$.

Proof. (i) Since T is an n-quasi-m-isometric operator, it follows from Theorem 2.4 that

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } \mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}(T^{*n}),$$

where T_1 is an m-isometric operator and $T_3^n = 0$. From [15, Corollary 7], it follows that $\sigma(T) \cup W = \sigma(T_1) \cup \sigma(T_3)$, where W is the union of certain of the holes in $\sigma(T)$ which is a subset of $\sigma(T_1) \cap \sigma(T_3)$. Further $\sigma(T_3) = \{0\}$ and $\sigma(T_1) \cap \sigma(T_3)$ has no interior points. So we have by [15, Corollary 8]

$$\sigma(T) = \sigma(T_1) \cup \sigma(T_3) = \sigma(T_1) \cup \{0\}.$$

(ii) By [1, Lemma 1.21], it is well known that the approximate spectrum of T_1 lies in unit circle. Hence $0 \notin \sigma_{ap}(T_1)$. Consequently, T_1 is bounded from below.

(iii) The proof follows from [1, Theorem 2.2]. □

Recall that two operators $T \in \mathcal{L}(\mathcal{H})$ and $S \in \mathcal{L}(\mathcal{H})$ are similar if there exists an invertible operator $X \in \mathcal{L}(\mathcal{H})$ such that $XT = SX$ (i.e., $T = X^{-1}SX$ or $S = XTX^{-1}$).

Corollary 2.6. *Let $T \in \mathcal{L}(\mathcal{H})$ be an n-quasi-m-isometric operator. If $T_1 = T|_{\overline{\mathcal{R}(T^n)}}$ is invertible, then T is similar to a direct sum of an m-isometric operator and a nilpotent operator.*

Proof. By Theorem 2.4 we write the matrix representation of T on $\mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}(T^{*n})$ as follows $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ where $T_1 = T|_{\overline{\mathcal{R}(T^n)}}$ is an m-isometric operator and $T_3^n = 0$. Since T_1 is invertible, we have $\sigma(T_1) \cap \sigma(T_3) = \emptyset$. Then there exists an operator A such that $T_1A - AT_3 = T_2$ by [23]. Hence

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix}.$$

The desired result follows from Theorem 2.4. □

Clearly, every n-quasi-m-isometric operator is an $(n+1)$-quasi-m-isometric operator. In [18, Theorem 2.4], S. Mechri and S. M. Patel proved that if T is a quasi-2-isometry, then T is quasi-m-isometry for all $m \geq 2$. In the following corollary, we give a generalization that every n-quasi-m-isometric operator is an n-quasi-k-isometric operator for $k \geq m$.

Corollary 2.7. *Let $T \in \mathcal{L}(\mathcal{H})$. If T is an n-quasi-m-isometric operator, then T is an n-quasi-k-isometric operator for every positive integer $k \geq m$.*

Proof. If $\mathcal{R}(T^n)$ is dense, then T is an m-isometric operator. Hence T is a k-isometric operator for every positive integer $k \geq m$.

If $\mathcal{R}(T^n)$ is not dense, by Theorem 2.4 we write the matrix representation of T on $\mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}(T^{*n})$ as follows $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ where $T_1 = T|_{\overline{\mathcal{R}(T^n)}}$ is an m-isometric operator and $T_3^n = 0$. Obviously, T_1 is a k-isometric operator for every integer $k \geq m$. The conclusion follows from Theorem 2.4(2). □

We consider the following example of n-quasi-m-isometric operator, which is not a quasi-m-isometry.

Example 2.8. Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. Define $T \in \mathcal{L}(\mathcal{H})$ as follows

$$Te_1 = 2e_2,\ Te_2 = 3e_3 \quad \text{and} \quad Te_k = e_{k+1} \quad \text{for} \ \ k \geq 3.$$

Then by a straightforward calculation, one can show

$$\|T^2e_k\|^2 - 2\|T^3e_k\|^2 + \|T^4e_k\|^2 = 0 \ \ \forall\, k = 1, 2, \cdots,$$

and

$$\|Te_1\|^2 - 2\|T^2e_1\|^2 + \|T^3e_1\|^2 \neq 0.$$

Therefore T is a 2-quasi-2-isometry but it is not a quasi-2-isometry.

In the following theorem, we give a sufficient condition such that n-quasi-m-isometric operator for $n \geq 2$ to be a quasi-m-isometric operator.

Theorem 2.9. *Let $T \in \mathcal{L}(\mathcal{H})$ be an n-quasi-m-isometry for $n \geq 2$. If $\mathcal{N}(T^{*p}) = \mathcal{N}(T^{*(p+1)})$ for some $1 \leq p \leq n-1$, then T is a p-quasi-m-isometry.*

Proof. From the assumption $\mathcal{N}(T^{*p}) = \mathcal{N}(T^{*(p+1)})$, it follows that $\mathcal{N}(T^{*p}) = \mathcal{N}(T^{*n})$. Since T is an n-quasi-m-isometry, we have

$$T^{*n}\left(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}T^{*k}T^k\right)T^n = 0,$$

we deduce that

$$T^{*p}\left(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}T^{*k}T^k\right)T^n = 0.$$

This means that

$$\begin{aligned} 0 &= \left(T^{*p}\left(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}T^{*k}T^k\right)T^n\right)^* \\ &= T^{*n}\left(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}\left(T^{*k}T^k\right)^*\right)T^p \\ &= 0. \end{aligned}$$

Using again the condition $\mathcal{N}(T^{*p}) = \mathcal{N}(T^{*(p+1)})$, we obtain

$$T^{*p}\left(\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}T^{*k}T^k\right)T^p = 0.$$

Hence T is a p-quasi-m-isometric operator. □

Remark 2.10. The following example shows that Theorem 2.9 is not necessarily true if $\mathcal{N}(T^{*p}) \neq \mathcal{N}(T^{*(p+1)})$.

Example 2.11. Consider the operator $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ acting on the two dimensional Hilbert space $\mathbb{C}^2$. Then by a straightforward calculation, one can show that T is a 2-quasi-isometry but it is not a quasi-isometry. However $\mathcal{N}(T^*) \neq \mathcal{N}(T^{*2})$.

Patel [21, Theorem 2.1], proved that any power of a 2-isometry is again a 2-isometry. T. Bermúdez et al. [6, Theorem 3.1] proved that any power of (m,p)-isometry is an (m,p)-isometry. Later S. Mechri and S. M. Patel [18] gave a partial generalization to quasi-2-isometry. The following theorem shows that any power of an n-quasi-m-isometry is an n-quasi-m-isometry.

Theorem 2.12. *Let $T \in \mathcal{L}(\mathcal{H})$. Suppose T is an n-quasi-m-isometric operator. Then T^k is n-quasi-m-isometric operator for any $k \in \mathbb{N}$.*

Proof. Two different proofs of this statement will be given.

First proof. Suppose that T is n-quasi-m-isometric operator. By Lemma 2.1, T is m-isometric on $\overline{\mathcal{R}(T^n)}$. Therefore, in view of [6, Theorem 3.1], the operator T^k is m-isometry on $\overline{\mathcal{R}(T^n)}$. Thus

$$< \beta_m(T^k)x \,|\, x > = 0, \quad \forall x \in \overline{\mathcal{R}(T^n)}.$$

Using the inclusion

$$\overline{\mathcal{R}((T^k)^n)} \subset \overline{\mathcal{R}(T^n)},$$

we get

$$< \beta_m(T^k)x \,|\, x > = 0, \quad \forall x \in \overline{\mathcal{R}((T^k)^n)}.$$

Hence T^k is m-isometry on $\overline{\mathcal{R}((T^k)^n)}$. This shows, by Lemma 2.1, that T^k is n-quasi-m-isometric.

Second proof. If $\mathcal{R}(T^n)$ is dense, then T is an m-isometric operator and hence T^k is m-isometry (by [6, Theorem 3.1]). If $\mathcal{R}(T^n)$ is not dense, by Theorem 2.4 we write the matrix representation of T on $\mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}(T^{*n})$ as follows $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ where $T_1 = T_{|\overline{\mathcal{R}(T^n)}}$ is an m-isometric and $T_3^n = 0$. We notice that

$$T^k = \begin{pmatrix} T_1^k & \sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j} \\ 0 & T_3^k \end{pmatrix},$$

where T_1^k is m-isometry and $(T_3^k)^n = 0$. Hence T^k is an n-quasi-m-isometric operator by Theorem 2.4. □

Remark 2.13. The converse of Theorem 2.12 is not true in general as shown in the following example.

Example 2.14. It is not difficult to prove that the operator $T := \begin{pmatrix} -1 & -1 \\ 3 & 2 \end{pmatrix}$ defined in $\mathbb{C}^2$ with the Euclidean norm satisfies T^3 is a quasi-3-isometry but T is not a quasi-3-isometry.

The following theorem generalizes [6, Theorem 3.6].

Theorem 2.15. *Let $T \in \mathcal{L}(\mathcal{H})$ and r, s, m, n_1, n_2, l be positive integers. If T^r is an n_1-quasi-m-isometry and T^s is an n_2-quasi-l-isometry, then T^q is an n-quasi-p-isometry, where q is the greatest common divisor of r and s, $n = \frac{1}{q}\min(n_1 r,\ n_2 s)$ and $p = \min(m,\ l)$.*

Proof. Consider the matrix representation of T with respect to the decomposition $\mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}(T^{*n})$ as follows $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ where $T_1 = T|_{\overline{\mathcal{R}(T^n)}}$ is m-isometric and $T_3^n = 0$.

We have $T^r = \begin{pmatrix} T_1^r & \sum_{j=0}^{r-1} T_1^j T_2 T_3^{r-1-j} \\ 0 & T_3^r \end{pmatrix}$. Since T^r is an n-quasi-m-isometry, we need to prove that T_1^r is an m-isometry and $(T_3^r)^n = 0$.

In fact, let $P = P_{\overline{\mathcal{R}(T^n)}}$ be the projection on $\overline{\mathcal{R}(T^n)}$. Then

$$\begin{pmatrix} T_1^r & 0 \\ 0 & 0 \end{pmatrix} = T^r P = PT^r P.$$

Since T^r is an n-quasi-m-isometry, we have

$$P\Big(\sum_{0\le k\le m} (-1)^{m-k} \binom{m}{k} (T^r)^{*k} (T^r)^k \Big) P = 0,$$

that is,

$$\sum_{0\le k\le m} (-1)^{m-k} \binom{m}{k} (T_1^r)^{*k} (T_1^r)^k = 0.$$

Hence, T_1^r is an m-isometry.

On the other hand, let $x = x_1 + x_2 \in \mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}(T^{*n})$. A simple computation shows that

$$\langle (T_3^r)^n x_2, x_2 \rangle = \langle (T^r)^n (I-P)x, (I-P)x \rangle = \langle (I-P)x, (T^r)^{*n} (I-P)x \rangle = 0.$$

So, $(T_3^r)^n = 0$.

Analogously, as $T^s = \begin{pmatrix} T_1^s & \sum_{j=0}^{s-1} T_1^j T_2 T_3^{s-1-j} \\ 0 & T_3^s \end{pmatrix}$ is an n-quasi-l-isometry by similar arguments, we can conclude that T_1^s is an l-isometry and $(T_3^s)^n = 0$.

Now, we have obtained that T_1^r is an m-isometry and T_1^s is an l-isometry. By [6, Theorem 3.6], it follows that T_1^q is a $p = \min(m, l)$-isometry. Moreover we have

$$(T_3^r)^{n_1} = 0 = (T_3^q)^{\frac{r}{q} n_1} \quad \text{and} \quad (T_3^s)^{n_2} = 0 = (T_3^q)^{\frac{s}{q} n_2}.$$

Since $n = \frac{1}{q} \min(n_1 r,\ n_2 s)$, we obtain $(T_3^q)^n = 0$.

Consequently, $T^q = \begin{pmatrix} T_1^q & \sum_{j=0}^{q-1} T_1^j T_2 T_3^{q-1-j} \\ 0 & T_3^q \end{pmatrix}$ is an n-quasi-p-isometry by Theorem 2.4.

The proof is completed. □

The following corollary shows that if two suitable different powers of T are n-quasi-m-isometries, then T is a n-quasi-m-isometry.

Corollary 2.16. *Let $T \in \mathcal{L}(\mathcal{H})$ and r, s, m, n, l be positive integers. The following properties hold.*

(1) *If T is an n-quasi-m-isometry such that T^s is an n-quasi-isometry, then T is an n-quasi-isometry.*

(2) *If T^r and T^{r+1} are n-quasi-m-isometries, then T is a nr-quasi-m-isometry.*

(3) *If T^r is an n-quasi-m-isometry and T^{r+1} is an n-quasi-l-isometry with $m < l$, then T is an nr-quasi-m-isometry.*

Proof. The proof is an immediate consequence of Theorem 2.15. □

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be power bounded, if $\sup_k \|T^k\| < \infty$ or equivalently, there exists $C > 0$ such that for every k and every $\xi \in \mathcal{H}$, one has

$$\|T^k \xi\| \leq C\|\xi\|.$$

In [8, Theorem 2], it was proved that every power bounded m-isometry operator is an isometry. The following theorem extends this result to n-quasi-m-isometry.

Theorem 2.17. *If $T \in \mathcal{L}(\mathcal{H})$ is an n-quasi-m-isometric operator which is power bounded, then T is an n-quasi-isometry.*

Proof. We consider the following two cases:

Case 1: If $\overline{\mathcal{R}(T^n)}$ is dense, then T is an m-isometric operator which is power bounded, thus T is an isometry by [8, Theorem 2]. It follows that T is an n-quasi-isometry.

Case 2: If $\overline{\mathcal{R}(T^n)}$ is not dense, by Theorem 2.4 we write the matrix representation of T on $\mathcal{H} = \overline{\mathcal{R}(T^n)} \oplus \mathcal{N}(T^{*n})$ as follows $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ where $T_1 = T|_{\overline{\mathcal{R}(T^n)}}$ is an m-isometric operator and $T_3^n = 0$. By taking into account that T is power bounded, it is easily checked that T_1 is power bounded, from which we deduce that T_1 is an isometry. The result follows by applying Theorem 2.4(2). □

Recall that for two operators T, S in $\mathcal{L}(\mathcal{H})$, the commutator $[T, S]$ is defined to be

$$[T, S] = TS - ST.$$

A pair of operators $(T, S) \in \mathcal{L}(\mathcal{H})^2$ is said to be a doubly commuting pair if (T, S) satisfies $TS = ST$ and $T^*S = ST^*$ or equivalently $[T, S] = [T, S^*] = 0$.

In [16, Theorem 2.2], it was proved that if T and S are commuting bounded linear operators on a Banach space such that T is a 2-isometry and S is an m-isometry, then ST is an $(m + 1)$-isometry. This result was improved in [3, Theorem 3.3] as follows: if $TS = ST$, T is an (m, p)-isometry and S is an (l, p)-isometry, then ST is an $(m+l-1, p)$-isometry. It is natural to ask whether the product of two n-quasi-m-isometries is also n-quasi-m-isometry. The following theorem gives an affirmative answer under suitable conditions.

Theorem 2.18. *Let S and T in $\mathcal{L}(\mathcal{H})$ be doubly commuting operators and let m, l, n_1, n_2 be positive integers. If T is an n_1-quasi-m-isometry and S is an n_2-quasi-l-isometry, then TS is a $n_0 = \max\{n_1, n_2\}$-quasi-$(m + l - 1)$-isometry.*

Proof. Since T and S are doubly commuting, it follows that $[T^*, S^*] = [T, S] = [T, S^*] = 0$. By taking into account [11, Lemma 12] we obtain that

$$\begin{aligned}
&\beta_{m+l-1,\, n_0}(TS) \\
&\quad = (TS)^{*(n_0)} \beta_{m+l-1}(TS)(TS)^{n_0} \\
&\quad = (T^*)^{n_0} (S^*)^{n_0}\, \beta_{m+l-1}(TS)\, (T)^{n_0} (S)^{n_0} \\
&\quad = (T^*)^{n_0} (S^*)^{n_0} \Big(\sum_{0 \leq j \leq m+l-1} \binom{m+l-1}{j} T^{*j} \beta_{m+l-1-j}(T)\, T^j \beta_j(S) \Big) (T)^{n_0} (S)^{n_0} \\
&\quad = \Big(\sum_{0 \leq j \leq m+l-1} \binom{m+l-1}{j} T^{*j}\, \underbrace{(T^*)^{n_0} \beta_{m+l-1-j}(T) T^{n_0}}\, T^j\, \underbrace{(S^*)^{n_0} \beta_j(S) S^{n_0}} \Big).
\end{aligned}$$

Since S is an n_2-quasi-l-isometry, it follows from Corollary 2.7 that $(S^*)^{n_0}\beta_j(S)S^{n_0} = 0$ for $j \geq l$. On the other hand, if $j \leq l-1$, then $m+l-1-j \geq m+l-1-l+1 = m$, and so $(T^*)^{n_0}\beta_{k+m-1-j}(T)T^{n_0} = 0$ by the fact that T is an n_1-quasi-m-isometry. This completes the proof. $\square$

The following example shows that Theorem 2.18 is not necessarily true if S, T are not doubly commuting.

Example 2.19. We consider the operators $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ on the two dimensional Hilbert space $\mathbb{C}^2$. Note that $ST \neq TS$. Moreover, by a direct computation, we show that T is a quasi-3-isometry and S is a 2-quasi-3-isometry. However neither TS nor ST is a 2-quasi-5-isometry.

Corollary 2.20. *Let $T, S \in \mathcal{L}(\mathcal{H})$ be doubly commuting operators such that T is an n_1-quasi-m-isometry and S is an n_2-quasi-l-isometry, then T^pS^q is a $\max\{n_1,\ n_2\}$-quasi-$(m+l-1)$-isometry for all positive integers p and q.*

Proof. Since T and S are doubly commuting, then T^p and S^q are doubly commuting. By Theorem 2.12 we know that T^p is an n_1-quasi-m-isometry and S^q is an n_2-quasi-l-isometry. Now by applying Theorem 2.18, we get that T^pS^q is a $\max\{n_1,\ n_2\}$-quasi-$(m+l-1)$-isometry. This completes the proof. $\square$

Let $\mathcal{H}\overline{\otimes}\mathcal{H}$ denote the completion, endowed with a reasonable uniform cross-norm, of the algebraic tensor product $\mathcal{H}\otimes\mathcal{H}$ of $\mathcal{H}$ and $\mathcal{H}$. For $T \in \mathcal{L}(\mathcal{H})$ and $S \in \mathcal{L}(\mathcal{H})$, $T\otimes S \in \mathcal{L}(\mathcal{H}\overline{\otimes}\mathcal{H})$ denote the tensor product operator defined by T and S.

In the following proposition we prove that the tensor product of an n_1-quasi-m-isometric operator with an n_2-quasi-l-isometric operator is a $\max\{n_1, n_2\}$-quasi-$(m+l-1)$-isometric operator. This proposition generalizes [9, Theorem 2.10].

Proposition 2.21. *If $T \in \mathcal{L}(\mathcal{H})$ is an n_1-quasi-m-isometry and $S \in \mathcal{L}(\mathcal{H})$ is an n_2-quasi-l-isometry, then $T\otimes S$ is a $\max\{n_1, n_2\}$-quasi-$(m+l-1)$-isometric operator.*

Proof. Observe that an operator $T \in \mathcal{L}(\mathcal{H})$ is n-quasi m-isometric if and only if $T\otimes I$ and $I\otimes T$ are n-quasi-m-isometry. In view of the fact that

$$T\otimes S = (T\otimes I)(I\otimes S) = (I\otimes S)(T\otimes I)$$

it follows that

$$[T\otimes I, I\otimes S] = [T\otimes I, (I\otimes S)^*] = 0.$$

Now $T\otimes I$ is an n_1-quasi-m-isometry and $I\otimes S$ is an n_2-quasi-m-isometry such that $T\otimes I$ and $I\otimes S$ are doubly commuting operators. By applying Theorem 2.18 we obtain that $(T\otimes I)(I\otimes S)$ is a $\max\{n_1, n_2\}$-quasi-$(m+l-1)$-isometric operator. Hence $T\otimes S$ is a $\max\{n_1, n_2\}$-quasi-$(m+l-1)$-isometric as required. $\square$

The following corollary is an immediate consequence of Theorem 2.12 and Proposition 2.21. We omitted its proof.

Corollary 2.22. *If $T \in \mathcal{L}(\mathcal{H})$ is an n_1-quasi-m-isometry and $S \in \mathcal{L}(\mathcal{H})$ is an n_2-quasi-l-isometry, then $T^p\otimes S^q$ is a $\max\{n_1, n_2\}$-quasi-$(m+l-1)$-isometry.*

It was proved in [4, Thoerem 2.2] that if $T \in \mathcal{L}(\mathcal{H})$ is an isometry and $Q \in \mathcal{L}(\mathcal{H})$ is a nilpotent operator of order p such that $TQ = QT$, then $T+Q$ is a strict $(2p-1)$-isometry. Later T. Bermúdez et al. [2] gave a partial generalization to m-isometry, that is if T is an

m-isometry with $m > 1$, Q is a nilpotent operator with order p, and $TQ = QT$, then $T+Q$ is an $(m+2p-2)$-isometry. Recently, C. Gu. and M. Stankus [14] gave a generalization to m-isometry, that is, if T is an m-isometry with $m > 1$, Q is a nilpotent operator with order p, and $TQ = QT$, then $T+Q$ is a strict $(m+2p-2)$-isometry. The following theorem states the corresponding partial generalization to the sum of an n-quasi-m-isometry and a nilpotent operator.

Theorem 2.23. *Let $T, Q \in \mathcal{L}(\mathcal{H})$ such that T commutes with Q. If T is an n-quasi-m-isometry and Q is a nilpotent operator of order p, then $T+Q$ is a $(n+p)$-quasi-$(m+2p-2)$-isometry.*

Proof. We need to show that $\beta_{m+2p-2,\,\alpha}(T+Q) = 0$. Set $q = m+2p-2$ and $\alpha = n+p$, by [14, Lemma 1] we have

$$\beta_q(T+Q) = \sum_{0\le k\le q}\sum_{0\le j\le q-k}\binom{q}{k}\binom{q-k}{j}(T^*+Q^*)^k Q^{*j}\beta_{q-k-j}(T)T^jQ^k.$$

In fact, note that

$$\begin{aligned}
&\beta_{m+2p-2,\,\alpha}(T+Q)\\
&= (T+Q)^{*\alpha}\beta_{m+2p-2}(T+Q)(T+Q)^{\alpha}\\
&= \Big(\sum_{0\le r\le\alpha}\binom{\alpha}{r}T^{*(\alpha-r)}Q^{*r}\Big)\Big(\sum_{0\le k\le q}\sum_{0\le j\le q-k}\binom{q}{k}\binom{q-k}{j}(T^*+Q^*)^k Q^{*j}\beta_{q-k-j}(T)T^jQ^k\Big)\\
&\quad\times\Big(\sum_{0\le r\le\alpha}\binom{\alpha}{r}T^{\alpha-r}Q^r\Big).
\end{aligned}$$

Now observe that if $k \ge p$ or $j \ge p$ then $Q^k = 0$ or $Q^{*j} = 0$ and hence

$$(T^*+Q^*)^k Q^{*j}\beta_{q-k-j}(T)T^jQ^k = 0.$$

However , if $k < p$ and $j < p$, we obtain

$$q-k-j = m+2p-2-k-j \ge m+2p-2-(p-1)-(p-1) = m$$

and using the fact that T is an n-quasi-m-isometry, we get

$$T^{*(n+p-r)}\beta_{q-k-j}(T)T^{n+p-r} = 0 \quad \text{for } \ r = 0,\cdots,p$$

and

$$T^{*(n+p-r)}Q^{*r}\beta_{q-k-j}(T)T^{n+p-r}Q^r = 0 \quad \text{for } \ r = p+1,\cdots,n+p.$$

Combining the above arguments we obtain $\beta_{m+2p-2,\,n+p}(T+Q) = 0$. □

Remark 2.24. A simple example shows that the commuting condition of T and Q can not be removed from the above theorem.

Example 2.25. Let $T = \begin{pmatrix} -5 & 0 \\ 0 & -1 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then T is a quasi-3-isometry and $Q^2 = 0$. Set $S = T+Q$, by direct calculation we show that S is not 5-quasi-5-isometry.

Corollary 2.26. *Let $T \in \mathcal{L}(\mathcal{H})$ be an n-quasi-m-isometry and $Q \in \mathcal{L}(\mathcal{H})$ be a nilpotent operator of order p. Then $T\otimes I + I\otimes Q$ is a $(n+p)$-quasi-$(m+2p-2)$-isometry.*

Proof. We note that $T\otimes I \in \mathcal{L}(\mathcal{H}\overline{\otimes}\mathcal{H})$ is an n-quasi-m-isometry and $I\otimes Q \in \mathcal{L}(\mathcal{H}\overline{\otimes}\mathcal{H})$ is nilpotent of order p. Moreover $(T\otimes I)(I\otimes Q) = (I\otimes Q)(T\otimes I)$. □

The following theorem shows that the class of n-quasi-m-isometry is a closed subset of $\mathcal{L}(\mathcal{H})$ equipped with the uniform operator (norm) topology.

Theorem 2.27. *Let $T \in \mathcal{L}(\mathcal{H})$. If $(T_k)_k$ is a sequence of n-quasi-m-isometry such that $\lim\limits_{k\to\infty} \|T_k - T\| = 0$, then T is also n-quasi-m-isometry.*

Proof. Suppose that $(T_k)_k$ is a sequence of n-quasi-m-isometric operators such that

$$\lim_{k\to\infty} \|T_k - T\| = 0.$$

Since for every positive integer k, T_k is an n-quasi-m-isometry, we have $\beta_{m,n}(T_k) = 0$. It follows that

$$\begin{aligned}
&\|\beta_{m,\,n}(T)\| = \|\beta_{m,\,n}(T_k) - \beta_{m,\,n}(T)\| \\
&= \|T_k^{*n}\Big(\sum_{0\le j\le m}(-1)^{m-j}\binom{m}{j}T_k^{*j}T_k^j\Big)T_k^n - T^{*n}\Big(\sum_{0\le j\le m}(-1)^{m-j}\binom{m}{j}T^{*j}T^j\Big)T^n\| \\
&\le \|T_k^{*n}\Big(\sum_{0\le j\le m}(-1)^{m-j}\binom{m}{j}T_k^{*j}T_k^j\Big)T_k^n - T_k^{*n}\Big(\sum_{0\le j\le m}(-1)^{m-j}\binom{m}{j}T_k^{*j}T^j\Big)T^n\| \\
&\quad + \|T_k^{*n}\Big(\sum_{0\le j\le m}(-1)^{m-j}\binom{m}{j}T_k^{*j}T^j\Big)T^n - T^{*n}\Big(\sum_{0\le j\le m}(-1)^{m-j}\binom{m}{j}T^{*j}T^j\Big)T^n\| \\
&\le \|\sum_{0\le j\le m}(-1)^{m-j}\binom{m}{j}T_k^{*n+j}\Big(T_k^jT_k^n - T^jT^n\Big)\| \\
&\quad + \|\sum_{0\le j\le m}(-1)^{m-j}\binom{m}{j}\Big(T_k^{*n+j} - T^{*n+j}\Big)T^jT^n\| \\
&\le \sum_{0\le j\le m}\binom{m}{j}\|T_k^*\|^{n+j}\|T_k^jT_k^n - T^jT^n\| + \sum_{0\le j\le m}\binom{m}{j}\|T_k^{*n+j} - T^{n+j}\|\|T\|^{j+n}. \qquad (2.5)
\end{aligned}$$

Since the product of operators is sequentially continuous in the strong topology, one concludes that $T_k^jT_k^n$, T_k^{n+j} converges strongly to T^jT^n and T^{n+j} respectively for $j = 0, 1, ..., m$. Hence the limiting case of (2.5) shows that T belongs to the class of n-quasi-m-isometric operators. ☐

3. n-quasi strict-m-isometries

In this section we introduce and study some properties of the class of n-quasi strict-m-isometric operators.

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be a strict m-isometry if T is an m-isometry but it is not an $(m-1)$-isometry.

Definition 3.1. We say that $T \in \mathcal{L}(\mathcal{H})$ is a n-quasi strict m-isometry if T is an n-quasi-m-isometry, but T is not an n-quasi-$(m-1)$-isometry.

Example 3.2 ([18]). Let $(e_k)_{k\ge1}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$. Consider an operator $T \in \mathcal{L}(\mathcal{H})$ defined by:

$$\begin{cases} Te_1 = ae_2 & a \ne \sqrt{2} \\ Te_p = \sqrt{\frac{p+1}{p}}e_{p+1} & (p = 2, 3, ...). \end{cases}$$

A direct calculation shows that T is a quasi-2-isometric operator, but not a quasi-isometric operator. Therefore T is a quasi strict-2-isometry.

Example 3.3. Consider the operator $T \in \mathcal{L}(\mathbb{C}^2)$ given by $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ which is quasi-3-isometric operator but is not quasi-2-isometric operator. Hence T is a quasi strict-3-isometry.

Remark 3.4. It is proved in Corollary 2.7 that an n-quasi-m-isometric operator is n-quasi-k-isometric operator for all integers $k \geq m$. Hence if an $T \in \mathcal{L}(\mathcal{H})$ is a strict n-quasi-m-isometry, then it is not a n-quasi-k-isometry for all integers $1 \leq k < m$.

Recall that the multinomial coefficient is given by $\begin{pmatrix} k \\ p_1, \cdots, p_k \end{pmatrix} = \dfrac{k!}{p_1!p_2! \cdots p_k!}$ where k and $p_1, \cdots, p_k$ are nonnegative integers subject to $k = p_1 + p_2 + \cdots + p_k$.

We will use the following formula for commuting variables $z = (z_1, ..., z_q)$:

$$(z_1 + \cdots + z_q)^k = \sum_{p_1+p_2+\cdots+p_q=k} \begin{pmatrix} k \\ p_1, p_2, \cdots, p_q \end{pmatrix} z_1^{p_1} z_2^{p_2} \cdot \cdots z_q^{p_q}.$$

In particular, if $z_1 = \cdots = z_q = 1$, we have

$$\sum_{p_1+p_2+\cdots+p_q=k} \begin{pmatrix} k \\ p_1, p_2, \cdots, p_q \end{pmatrix} = q^k.$$

Proposition 3.5. *Let $T \in \mathcal{L}(\mathcal{H})$. Then the following statements hold.*

(1) *If m is a positive integer and $x \in \mathcal{H}$, then*

$$\Delta_{m,\,n}(T, x) = \Delta_{m-1,\,n}(T, Tx) - \Delta_{m-1,\,n}(T, x). \tag{3.1}$$

In particular, if T is an n-quasi-m-isometry, then for every positive integer k one has

$$\Delta_{m-1,\,n}(T, T^k x) = \Delta_{m-1,\,n}(T, x). \tag{3.2}$$

(2) *If k is a positive integer and $x \in \mathcal{H}$, then*

$$\Delta_{m,\,n}(T^k, x) = \sum_{p_1+\cdots+p_k=m} \begin{pmatrix} m \\ p_1, \cdots, p_k \end{pmatrix} \Delta_{m,\,n}(T, T^{(0.p_1+1.p_2+\cdots+(k-1)p_k)+(k-1)n} x). \tag{3.3}$$

Proof. For $m \geq 1$ and $x \in \mathcal{H}$ we have

$$\begin{aligned}
\Delta_{m,\,n}(T, x) &= \sum_{0\leq k\leq m} (-1)^{m-k} \binom{m}{k} \|T^{k+n}x\|^2 \\
&= (-1)^m \|T^n x\|^2 + \sum_{1\leq k\leq m} (-1)^{m-1-k} \binom{m}{k} \|T^{k+n}x\|^2 + \|T^{m+n}x\|^2 \\
&= (-1)^m \|T^n x\|^2 + \sum_{1\leq k\leq m} (-1)^{m-1-k} \left(\binom{m-1}{k} + \binom{m-1}{k-1} \right) \|T^{k+n}x\|^2 + \|T^{m+n}x\|^2 \\
&= \sum_{0\leq k\leq m-1} (-1)^{m-1-k} \binom{m-1}{k} \|T^{k+n}Tx\|^2 - \sum_{0\leq k\leq m-1} (-1)^{m-1-k} \binom{m-1}{k} \|T^{k+n}x\|^2 \\
&= \Delta_{m-1,\,n}(T, Tx) - \Delta_{m-1,n}(T, x).
\end{aligned}$$

Hence (3.1) is proved.

If T is an n-quasi-m-isometry, then $\Delta_{m,\,n}(T, x) = 0$ and so that

$$\Delta_{m-1,\,n}(T, Tx) = \Delta_{m-1,\,n}(T, x).$$

Hence,

$$\Delta_{m-1,\,n}(T,T^kx)=\Delta_{m-1,\,n}(T,T^{k-1}x)=\cdots=\Delta_{m-1,\,n}(T,x).$$

(2) From (2.3) we have

$$\Delta_{m,\,n}(T^k,x)=\sum_{0\le j\le m}(-1)^{m-j}\binom{m}{j}\|(T^k)^{j+n}x\|^2 \tag{3.4}$$

and

$$\sum_{p_1+\cdots+p_k=m}\binom{m}{p_1,\cdots,p_k}\Delta_{m,\,n}(T,T^{(0.p_1+1.p_2+\cdots+(k-1)p_k)+(k-1)n}x)=$$
$$\sum_{p_1+\cdots+p_k=m}\binom{m}{p_1,\cdots,p_k}\sum_{0\le j\le m}(-1)^{m-j}\binom{m}{j}\|T^{j+kn}T^{(0.p_1+1.p_2+\cdots+(k-1)p_k)}x\|^2. \tag{3.5}$$

Thus, for proving (3.3) it suffices to show that the coefficients of the various powers of $\|T^jx\|^2$ for $nk\le j\le km+kn$ in (3.4) and (3.5) are the same. To do so, we merely need to check the following combinatorial identity:

$$\begin{aligned}&\sum_{0\le j\le m}(-1)^{m-j}\binom{m}{j}(z^k)^{j+n}\\&\quad=\sum_{p_1+\cdots+p_k=m}\binom{m}{p_1,\cdots,p_k}\Big(\sum_{0\le j\le m}(-1)^{m-j}\binom{m}{j}z^{j+kn}z^{(0.p_1+1.p_2+\cdots+(k-1)p_k)}\Big).\end{aligned}$$

In fact, observe that

$$\sum_{0\le j\le m}(-1)^{m-j}\binom{m}{j}(z^k)^{j+n}=\left(z^k-1\right)^mz^{kn},$$

and

$$\begin{aligned}&\sum_{p_1+\cdots+p_k=m}\binom{m}{p_1,\cdots,p_k}\Big(\sum_{0\le j\le m}(-1)^{m-j}\binom{m}{j}z^{j+kn}z^{(0.p_1+1.p_2+\cdots+(k-1)p_k)}\Big)\\&\quad=z^{kn}\sum_{p_1+\cdots+p_k=m}\binom{m}{p_1,\cdots,p_k}\Big(\sum_{0\le j\le m}(-1)^{m-j}\binom{m}{j}z^j\Big)z^{(0.p_1+1.p_2+\cdots+(k-1)p_k)}.\end{aligned}$$

By applying the multinomial formula in reverse order, we have

$$\begin{aligned}&\sum_{p_1+\cdots+p_k=m}\binom{m}{p_1,\cdots,p_k}\Big(\sum_{0\le j\le m}(-1)^{m-j}\binom{m}{j}z^j\Big)z^{(0.p_1+1.p_2+\cdots+(k-1)p_k)}\\&\quad=\left(z-1\right)^m\sum_{p_1+\cdots+p_k=m}\binom{m}{p_1,\cdots,p_k}z^{(0.p_1+1.p_2+\cdots+(k-1)p_k)}\\&\quad=\left(z-1\right)^m(z^{k-1}+z^{k-2}+\cdots+z+1)^m=(z^k-1)^m.\end{aligned}$$

□

Theorem 3.6. *If $T\in\mathcal{L}(\mathcal{H})$ is a n-quasi strict m-isometry, then for any positive integer k, T^k is a n-quasi strict m-isometry. Furthermore*

$$\Delta_{m-1,\,n}(T^k,x)=k^{m-1}\Delta_{m-1,\,n}(T,x).$$

Proof. Since T is an n-quasi strict m-isometry, by Theorem 2.12, T^k is an n-quasi-m-isometry. Furthermore, by (3.2) and (3.3) we get

$$\begin{aligned}\Delta_{m-1,n}(T^k,x) &= \sum_{p_1+\cdots+p_k=m-1}\binom{m-1}{p_1,\cdots,p_k}\Delta_{m-1,n}(T,T^{(0.p_1+1.p_2+\cdots+(k-1)p_k)+(k-1)n}x)\\ &= \sum_{p_1+\cdots+p_k=m-1}\binom{m-1}{p_1,\cdots,p_k}\Delta_{m-1,n}(T,x)\\ &= k^{m-1}\Delta_{m-1,\,n}(T,x).\end{aligned}$$

Consequently, T^k is a n-quasi strict-m-isometry. □

Remark 3.7. The converse of the above theorem is not true in general. In fact, by Theorem 2.15 if T^r and T^s are n-quasi-m-isometries for two coprime positive integers r and s, then T is an n-quasi-m-isometry.

Recall that a sequence $(a_j)_{j\geq 0}$ in a group G is an arithmetic progression of order h if

$$\sum_{0\leq k\leq h+1}(-1)^{h+1-k}a_{k+j}=0$$

for any $j\geq 0$. An arithmetic progression of order h is of strict order h if $h=0$ or if $h\geq 1$ and it is not of order $h-1$. We refer the interested reader to [5] for complete details.

Lemma 3.8 ([5, Theorem 4.1])**.** *Let $a=(a_j)_{j\geq 0}$ be a numerical sequence. Suppose that $(a_{cj})_{j\geq 0}$ is an arithmetic progression of strict order h and $(a_{dj})_{j\geq 0}$ is an arithmetic progressions of strict order $k\geq 0$, for $c,d\geq 1$ and $h,k\geq 0$. Then $(a_{ej})_{j\geq 0}$ is an arithmetic progression of strict order l, where e is the greatest common divisor of c and d, and l the minimum of h and k.*

Theorem 3.9. *Let $T\in\mathcal{L}(\mathcal{H})$ and r,s,n,m,l be positive integers. If T^r is a n-quasi strict m-isometry and T^s is an n-quasi strict l-isometry, then T^q is an n-quasi strict p-isometry, where q is the greatest common divisor of r and s, and p is the minimum of m and l.*

Proof. Without loss of generality, we will suppose that $r\geq s$. Fix $x\in\mathcal{H}$ and set $a_j:=\|T^{j+nr}x\|^2$ for $j\geq 0$. Since T^r is a n-quasi strict m-isometry, it follows that $(a_{rj})_{j\geq 0}$ is an arithmetic progression of strict order $m-1$ and satisfies the recursive equation

$$\sum_{0\leq k\leq m}(-1)^{m-k}\binom{m}{k}a_{rk+j}=0 \qquad \text{for all } \ j\geq 0.$$

Since T^s is a n-quasi strict l-isometry, it follows that

$$\sum_{0\leq k\leq l}(-1)^{l-k}\binom{l}{k}\|T^{s(n+k)}x\|^2=0=\sum_{0\leq k\leq l}(-1)^{l-k}\binom{l}{k}\|T^{s(n+k)+n(r-s)}x\|^2.$$

Then $(a_{sj})_{j\geq 0}$ is an arithmetic progression of strict order $l-1$ satisfies the recursive equation

$$\sum_{0\leq k\leq l}(-1)^{l-k}\binom{l}{k}a_{sk+j}=0 \qquad \text{for all } \ j\geq 0.$$

Applying Lemma 3.8 it results that $(a_{qj})_{j\geq 0}$ is an arithmetic progression of strict order $p-1$, so T^q is an n-quasi-strict p-isometry, where $q=pgcd(r,s)$ and $p=\min\{m,l\}$. □

The following corollary shows that if two suitable different powers of T are n-quasi strict-m-isometries, then T is a n-quasi strict-m-isometry.

Corollary 3.10. *Let $T \in \mathcal{L}(\mathcal{H})$ and r, s, m, n, l be positive integers. The following properties hold.*

(1) *If T is an n-quasi strict-m-isometry such that T^s is an n-quasi strict-isometry, then T is an n-quasi strict-isometry.*

(2) *If T^r and T^{r+1} are n-quasi strict-m-isometries, then so is T.*

(3) *If T^r is an n-quasi strict-m-isometry and T^{r+1} is an n-quasi strict-l-isometry with $m < l$, then T is an n-quasi strict-m-isometry.*

Proof. The proof is an immediate consequence of Theorem 3.9. □

Theorem 3.11. *Let $T \in \mathcal{L}(\mathcal{H})$ and $S \in \mathcal{L}(\mathcal{H})$ be doubly commuting operators. If T is an n-quasi strict m-isometry and S is an n-quasi strict l-isometry, then TS is an n-quasi strict $(m+l-1)$-isometry if and only if*

$$(T^*)^{n+l-1}\beta_{m-1}(T)T^{n+l-1}S^{*n}\beta_{l-1}(S)S^n = 0.$$

Proof. In view of Theorem 2.18, it is obvious that $\beta_{m+l-1,\, n}(TS) = 0$. On the other hand, since T and S are doubly commuting operators, it follows from [14, Corollary 3.9] that

$$\beta_q(TS) = \sum_{0\le k\le q} \binom{q}{k} T^{*k}\beta_{q-k}(T)T^k\beta_k(S),$$

from which it follows that

$$\begin{aligned}
&\beta_{m+l-2,\, n}(TS)\\
&\quad= (TS)^{*n}\beta_{m+l-2}(TS)(TS)^n\\
&\quad= (T^*)^n(S^*)^n\Big(\sum_{0\le k\le m+l-2}\binom{m+l-2}{k}T^{*k}\beta_{m+l-2-k}(T)T^k\beta_k(S)\Big)T^nS^n\\
&\quad= \sum_{0\le k\le m+l-2}\binom{m+l-2}{k}T^{*k}(T^*)^n\beta_{m+l-2-k}(T)T^nT^k(S^*)^n\beta_k(S)S^n\\
&\quad= \sum_{0\le k\le l-2}\binom{m+l-2}{k}T^{*k}(T^*)^n\beta_{m+l-2-k}(T)T^nT^k(S^*)^n\beta_k(S)S^n\\
&\qquad+\binom{m+l-2}{l-1}(T^*)^{n+l-l}\beta_{m-1}(T)T^{n+l-1}(S^*)^n\beta_{l-1}(S)S^n\\
&\qquad+\sum_{l\le k\le m+l-2}\binom{m+l-2}{k}T^{*k}(T^*)^n\beta_{m+l-2-k}(T)T^nT^k(S^*)^n\beta_k(S)S^n.
\end{aligned}$$

If $k \in [0, l-2]$, then $(m+l-2-k) \in [m, m+l-2]$ and hence $T^{*n}\beta_{m+l-2-k}(T)T^n = 0$ by Corollary 2.7. If $k \ge l$, then $S^n\beta_k(S)S^n = 0$ also in view of Corollary 2.7.

Consequently, TS is a n-quasi strict-$(m+l-1)$-isometry if and only if

$$(T^*)^{n+l-1}\beta_{m-1}(T)T^{n+l-1}(S^*)^n\beta_{l-1}(S)S^n = 0.$$

Hence the proof is finished. □

Theorem 3.12. *Let $T \in \mathcal{L}(\mathcal{H})$ and $S \in \mathcal{L}(\mathcal{H})$. If T is an n-quasi strict-m-isometry and S is an n-quasi strict-l-isometry, then $T \otimes S$ on $\mathcal{H}\overline{\otimes}\mathcal{H}$ is an n-quasi strict-$(m+l-1)$-isometry.*

Proof. In view of [12, Corollary 3.10], it follows that

$$\beta_q(T\otimes S)=\sum_{0\le k\le q}\binom{q}{k}T^{*k}\beta_{q-k}(T)T^k\otimes\beta_k(S).$$

By calculations we have

$$\begin{aligned}&\beta_{m+l-2,\,n}(T\otimes S)\\ =\;&(T\otimes S)^{*n}\beta_{m+l-2}(T\otimes S)(T\otimes S)^n\\ =\;&(T^{*n}\otimes S^{*n})\Big(\sum_{0\le k\le m+l-2}\binom{m+l-2}{k}T^{*k}\beta_{m+l-2-k}(T)T^k\otimes\beta_k(S)\Big)(T^n\otimes S^n)\\ =\;&\sum_{0\le k\le m+l-2}\binom{m+l-2}{k}T^{*k}T^{*n}\beta_{m+l-2-k}(T)T^nT^k\otimes S^{*n}\beta_k(S)S^n.\end{aligned}$$

Similar arguments as in the proof of Theorem 3.11 give

$$\beta_{m+l-2,\,n}(T\otimes S)=(T^*)^{n+l-1}\beta_{m-1}(T)T^{n+l-1}\otimes(S^*)^n\beta_{l-1}(S)S^n.$$

This means that $T\otimes S$ is a n-quasi strict $(m+l-1)$-isometry as required. □

In [7, Theorem 3.1] it has been proved that if $T\in\mathcal{L}(\mathcal{H})$ is a strict m-isometry, then the list of operators $\{\,T^{*k}T^k, k=0,1,\,\cdots m-1\,\}$ is linearly independent which is equivalent to the fact that $\{\,\beta_k(T), k=0,1,\,\cdots,m-1\,\}$ is linearly independent.

In the following proposition we extend this result to n-quasi strict m-isometries as follows.

Proposition 3.13. *If $T\in\mathcal{L}(\mathcal{H})$ is an n-quasi strict m-isometry, then the list of operators*

$$\{\,\beta_{k,\,n}(T), k=0,1,\,\cdots,m-1\,\}$$

is linearly independent.

Proof. The outline of the proof is inspired from [10].

It was observed in [12] that $\beta_k(T)=T^*\beta_{k-1}(T)T-\beta_{k-1}(T)$ for all $k\ge 1$.

We will just write

$$\begin{aligned}\beta_{m,\,n}(T)=T^{*n}\beta_m(T)T^n&=T^{*n}\Big(T^*\beta_{m-1}(T)T-\beta_{m-1}(T)\Big)T^n\\ &=T^{*n+1}\beta_{m-1}(T)T^{n+1}-T^{*n}\beta_{m-1}(T)T^n.\end{aligned}$$

Now assume that for some complex numbers λ_k,

$$\sum_{0\le k\le m-1}\lambda_k\beta_{k,\,n}(T)=0$$

or equivalently

$$\sum_{0\le k\le m-1}\lambda_kT^{*n}\beta_k(T)T^n=0.$$

Multiplying the above equation on the left and right by T^* and T and subtracting two equations, we have

$$\sum_{0\le k\le m-1}\lambda_k\Big(T^{*n+1}\beta_k(T)T^{n+1}-T^{*n}\beta_k(T)T^n\Big)=\sum_{0\le k\le m-1}\lambda_k\beta_{k+1,\,n}(T)=0.$$

By applying the same procedure to the equation $\sum_{0\le k\le m-1}\lambda_k\beta_{k+1,\,n}(T)=0$ we get

$$\sum_{0\le k\le m-1}\lambda_k\beta_{k+2,\,n}(T)=0.$$

By continuing this process we obtain

$$\sum_{0\leq k\leq m-1} \lambda_k \beta_{k+j,\, n}(T) = 0 \quad \text{for all} \;\; j \in \mathbb{N}.$$

Since every n-quasi-m-isometric operator is an n-quasi-k-isometric operator for all $k \geq m$ (Corollary 2.7) we have the following cases:

For $j = m-1$, $\sum_{0\leq k\leq m-1} \lambda_k \beta_{k+j,\, n}(T) = 0 \Rightarrow \lambda_0 \beta_{m-1,\, n}(T) = 0$, so $\lambda_0 = 0$.

For $j = m-2$, $\sum_{0\leq k\leq m-1} \lambda_k \beta_{k+j,\, n}(T) = 0 \Rightarrow \lambda_1 \beta_{m-1,\, n}(T) = 0$, so $\lambda_1 = 0$.

Continuing this process we see that all $\lambda_k = 0$ for $k = 2, \cdots, m-1$. Hence the result is proved. □

Acknowledgment. The authors wish to thank the referee for suggestions which led to a significant improvement of this paper.

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