A new subclass of analytic functions

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Abstract

In the present paper, we introduce a class $\mathcal{B}_\theta(\alpha, \beta)$ of functions, analytic in $|z| < 1$, such that $f(0) = 0$, $f'(0) = 1$ and

$$\alpha < \mathrm{Re}\left(f'(z) + \frac{1 + e^{i\theta}}{2} z f''(z) \right) < \beta \quad (|z| < 1),$$

where $\theta \in (-\pi, \pi]$, $0 \leq \alpha < 1$ and $\beta > 1$. Integral representation, differential subordination results and coefficient estimates are considered. Also Fekete-Szegö coefficient functional associated with the k–th root transform $[f(z^k)]^{1/k}$ for functions in the class $\mathcal{B}_\theta(\alpha, \beta)$ is investigated.

1. Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. The most familiar subclass of $\mathcal{A}$ consists of univalent functions and it is denoted by $\mathcal{S}$. For a univalent function f of the form (1.1), the k–th root transform is defined by

$$F(z) = [f(z^k)]^{1/k} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1} \quad (z \in \Delta). \tag{1.2}$$

Let $\mathcal{P}$ denote the class of functions q of the form

$$q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n, \quad (z \in \Delta) \tag{1.3}$$

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such that $\text{Re}\{q(z)\} > 0$, for $z \in \Delta$. Functions in $\mathcal{P}$ are sometimes called Carathéodory functions. We also denote by $\mathcal{B}$ the class of analytic functions of the form

$$w(z) = \sum_{n=0}^{\infty} c_n z^n, \quad (z \in \Delta) \tag{1.4}$$

such that $|w(z)| < 1$ for $z \in \Delta$. If f and g are two functions in $\mathcal{A}$, we say that f is subordinate to g, written $f(z) \prec g(z)$, if there exists a $w \in \mathcal{B}$ such that $f(z) = g(w(z))$ for all $z \in \Delta$. Furthermore, if the function g belongs to $\mathcal{S}$, then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow (f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta)).$$

For functions f given by (1.1) and $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (z \in \Delta), \tag{1.5}$$

their Hadamard product (or convolution), denoted by $f * g$, is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z) \quad (z \in \Delta).$$

We now recall that the function class $\mathcal{R}$ is given by

$$\mathcal{R} = \left\{ f \in \mathcal{A} : \quad \text{Re}\left(f'(z) + zf''(z)\right) > 0, \quad z \in \Delta \right\}.$$

This class was investigated by Chichra [2] and also by Singh and Singh [14]. Another function class $\mathcal{R}_\alpha$, $0 \le \alpha < 1$, given by

$$\mathcal{R}_\alpha = \left\{ f \in \mathcal{A} : \quad \text{Re}\left(f'(z) + zf''(z)\right) > \alpha, \quad z \in \Delta \right\},$$

was considered by Silverman [12]. Recently, Silverman and Silvia [13] considered the following classes of functions:

$$\mathfrak{L}_\theta := \left\{ f \in \mathcal{A} : \quad \text{Re}\left(f'(z) + \frac{1 + e^{i\theta}}{2} z f''(z) \right) > 0, \quad z \in \Delta \right\}$$

and

$$\mathfrak{L}_\theta(b) := \left\{ f \in \mathcal{A} : \quad \left| f'(z) + \frac{1 + e^{i\theta}}{2} z f''(z) - b \right| < b, \quad z \in \Delta \right\},$$

for some and $b > 1/2$ and $-\pi < \theta \le \pi$. Clearly, if $b \to \infty$, then $\mathfrak{L}_\theta(b) \to \mathfrak{L}_\theta$.

In [8] Kuroki and Owa introduced the class $\mathcal{S}(\alpha, \beta)$, $(\alpha < 1 < \beta)$, of all functions $f \in \mathcal{A}$ satisfying the following two–sided inequality

$$\alpha < \text{Re}\left(\frac{zf'(z)}{f(z)} \right) < \beta \quad (z \in \Delta).$$

In [4], Kargar *et al.* introduced the class $\mathcal{V}(\alpha, \beta)$ as follows:

$$\mathcal{V}(\alpha, \beta) := \left\{ f \in \mathcal{A} : \alpha < \text{Re}\left\{ \left(\frac{z}{f(z)} \right)^2 f'(z) \right\} < \beta, z \in \Delta \right\},$$

for some $0 \le \alpha < 1$ and $\beta > 1$. Also, they (see [5]) introduced the class $\mathcal{M}(\delta)$ including all functions $f \in \mathcal{A}$ such that

$$1 + \frac{\delta - \pi}{2 \sin \delta} < \text{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} < 1 + \frac{\delta}{2 \sin \delta} \quad (z \in \Delta),$$

where $\pi/2 \le \delta < \pi$.

In this work, motivated by the above definitions, we define a new subclass of analytic functions related to functions of positive real part.

Definition 1.1. Assume that $\theta \in (-\pi, \pi]$, $0 \leq \alpha < 1$ and $\beta > 1$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{B}_\theta(\alpha, \beta)$ if it satisfies the following two-sided inequality

$$\alpha < \operatorname{Re}\left(f'(z) + \frac{1 + e^{i\theta}}{2} z f''(z)\right) < \beta, \quad (z \in \Delta). \tag{1.6}$$

We put $\mathcal{B}_0(\alpha, \beta) = \mathcal{B}(\alpha, \beta)$. We remark that the class $\mathcal{B}_0(0, \beta \to \infty) \equiv \mathcal{R}$, $\mathcal{B}_0(\alpha, \beta \to \infty) \equiv \mathcal{R}_\alpha$. Also $\mathcal{B}_\theta(0, \beta \to \infty) \equiv \mathcal{L}_\theta$ where $\theta \in (-\pi, \pi]$.

In order to prove our results, we need the following lemmas.

Lemma 1.2 ([11]). *Let $q(z) = \sum_{n=1}^\infty C_n z^n$ be analytic and univalent in Δ, and suppose that q maps Δ onto a convex domain. If $p(z) = \sum_{n=1}^\infty A_n z^n$ is analytic in Δ and satisfies the following subordination*

$$p(z) \prec q(z) \quad (z \in \Delta),$$

then

$$|A_n| \leq |C_1| \quad n \geq 1.$$

The next lemma is due to Hallenbeck and Ruscheweyh.

Lemma 1.3 ([3]). *Let h be a convex univalent function with $h(0) = a$ and let $\gamma \in \mathbb{C}$ with $\operatorname{Re}\{\gamma\} \geq 0$. If the function $p(z)$ given by $p(z) = a + p_n z^n + p_{n+1} z^{n+1} + \cdots$ is analytic in Δ and*

$$p(z) + \frac{1}{\gamma} z p'(z) \prec h(z) \quad (z \in \Delta), \tag{1.7}$$

then

$$p(z) \prec q(z) \prec h(z) \quad (z \in \Delta), \tag{1.8}$$

where

$$q(z) = \frac{\gamma}{n z^{\gamma/n}} \int_0^z h(\xi) \xi^{\gamma/n - 1} d\xi. \tag{1.9}$$

The result is best possible, in the sense that if p satisfies (1.7) and $p \prec q_1$, then $q \prec q_1$.

Finally, we need the following Lemma 1.4 due to Keogh and Merkes (see [7]).

Lemma 1.4. *Let the function g given by*

$$g(z) = 1 + c_1 z + c_2 z^2 + \cdots,$$

be in the class $\mathcal{P}$. Then, for any complex number μ

$$|c_2 - \mu c_1^2| \leq 2 \max\{1, |2\mu - 1|\}.$$

The result is sharp for the function g given by

$$g(z) = \frac{1 + z}{1 - z} \qquad or \qquad g(z) = \frac{1 + z^2}{1 - z^2}.$$

In this paper some properties of the function class $\mathcal{B}_\theta(\alpha, \beta)$ including, integral representation, differential subordination results and coefficient estimates are investigated. Furthermore, Fekete-Szegö coefficient functional associated with the k–th root transform $[f(z^k)]^{1/k}$ is considered.

2. Main results

We begin this section with the following lemma that will be useful. Indeed, it presents a necessary and sufficient condition for functions to be in the class $\mathcal{B}_\theta(\alpha, \beta)$.

Lemma 2.1. *Assume that $f \in \mathcal{A}$, $0 \le \alpha < 1$, $\beta > 1$ and $\theta \in (-\pi, \pi]$. Then $f \in \mathcal{B}_\theta(\alpha, \beta)$, if and only if*

$$f'(z) + \frac{1 + e^{i\theta}}{2} z f''(z) \prec P_{\alpha,\beta}(z) \quad (z \in \Delta), \tag{2.1}$$

where

$$P_{\alpha,\beta}(z) := 1 + \frac{\beta - \alpha}{\pi} i \log\left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1 - z} \right). \tag{2.2}$$

Proof. Note that the function $P_{\alpha,\beta}$ defined by (2.2) is convex univalent function in Δ and maps Δ onto a convex domain

$$\Omega_{\alpha,\beta} = \{ w \in \mathbb{C} : \alpha < \mathrm{Re}\{w\} < \beta \}, \tag{2.3}$$

conformally (see [8]). By (1.6), $\{f'(z) + [(1 + e^{i\theta})/2] z f''(z)\}$ lies in the strip $\Omega_{\alpha,\beta}$ and it is known that $P_{\alpha,\beta}(\Delta) = \Omega_{\alpha,\beta}$. Because $P_{\alpha,\beta}$ is univalent then by the subordination principle, we get (2.1). □

It is easy to see the function $P_{\alpha,\beta}$ has the form

$$P_{\alpha,\beta}(z) = 1 + \sum_{n=1}^\infty B_n z^n, \tag{2.4}$$

where

$$B_n = \frac{\beta - \alpha}{n\pi} i \left(1 - e^{2n\pi i \frac{1-\alpha}{\beta-\alpha}} \right) \quad (n = 1, 2, \ldots). \tag{2.5}$$

Lemma 2.2. *Let $P_{\alpha,\beta}$ be defined as (2.2). Then the function*

$$\lambda(z) = \int_0^1 P_{\alpha,\beta}(z t^{1/\gamma}) dt \quad (\mathrm{Re}\{\gamma\} > 0, z \in \Delta), \tag{2.6}$$

is convex in Δ.

Proof. Define

$$\psi_\gamma(z) := \int_0^1 \frac{1}{1 - z t^{1/\gamma}} dt = \sum_{n=0}^\infty \frac{\gamma}{n + \gamma} z^n. \tag{2.7}$$

The function $\psi_\gamma(z)$ is convex in Δ when $\mathrm{Re}(\gamma) > 0$ (see [9]). From (2.7) we obtain

$$P_{\alpha,\beta}(z) * \psi_\gamma(z) = P_{\alpha,\beta}(z) * \int_0^1 \frac{1}{1 - z t^{1/\gamma}} dt$$

$$= \int_0^1 P_{\alpha,\beta}(z t^{1/\gamma}) dt =: \lambda(z).$$

On the other hand, since $P_{\alpha,\beta}$ and ψ_γ are convex univalent functions, by the Pòlya–Schoenberg conjecture (this conjecture states that the class of convex univalent functions is preserved under the convolution) that is proved by Ruscheweyh and Sheil–Small (see [10]), the function $\lambda(z)$ is convex univalent in the open unit disk Δ. This completes the proof. □

In the sequel, applying the Lemma 2.1, we obtain an integral representation for functions which belong to the class $\mathcal{B}(\alpha, \beta)$.

Theorem 2.3. *Let $\theta \in (-\pi, \pi)$, $0 \le \alpha < 1$ and $\beta > 1$. Then $f \in \mathcal{B}_\theta(\alpha, \beta)$ if, and only if there exists $w \in \mathcal{B}$ such that*

$$f(z) = \int_0^z \frac{\gamma}{\eta^\gamma} \left(\int_0^\eta \xi^{\gamma-1} P_{\alpha,\beta}(w(\xi)) d\xi \right) d\eta \quad (z \in \Delta, \gamma = 2/(1 + e^{i\theta})), \tag{2.8}$$

where $P_{\alpha,\beta}(z)$ defined by (2.2).

Proof. Let $f \in \mathcal{B}_\theta(\alpha, \beta)$. By definition of subordination and by the Lemma 2.1, if $f \in \mathcal{B}_\theta(\alpha, \beta)$, then there exists a function $w \in \mathcal{B}$ such that

$$f'(z) + \frac{1 + e^{i\theta}}{2} z f''(z) = P_{\alpha,\beta}(w(z)) \quad (z \in \Delta). \tag{2.9}$$

From the following equality

$$f'(z) + \frac{1 + e^{i\theta}}{2} z f''(z) = \frac{1 - e^{i\theta}}{2} f'(z) + \frac{1 + e^{i\theta}}{2} (z f'(z))' \quad (z \in \Delta), \tag{2.10}$$

we get that (2.9) is equivalent to

$$\left(\frac{1 - e^{i\theta}}{1 + e^{i\theta}} \right) f'(z) + (z f'(z))' = \frac{2}{1 + e^{i\theta}} P_{\alpha,\beta}(w(z)) \quad (z \in \Delta). \tag{2.11}$$

Assuming

$$\gamma := \frac{2}{1 + e^{i\theta}} \quad (-\pi < \theta < \pi), \tag{2.12}$$

the above relation (2.11) implies that

$$(\gamma - 1) z^{\gamma-1} f'(z) + z^{\gamma-1} (z f'(z))' = \gamma z^{\gamma-1} P_{\alpha,\beta}(w(z)).$$

Therefore, we find that

$$\left[z^{\gamma-1} \left(z f'(z) \right) \right]' = \gamma z^{\gamma-1} P_{\alpha,\beta}(w(z)),$$

which readily yields

$$z^\gamma f'(z) = \gamma \int_0^z \xi^{\gamma-1} P_{\alpha,\beta}(w(\xi)) d\xi. \tag{2.13}$$

Integrating once more the equality (2.13), we get (2.8) and concluding the proof. $\qquad\square$

Let $t \in [0, 1]$ and $\phi \in [0, 2\pi)$. By using Theorem 2.3, the function

$$f(z, \phi, t) = \int_0^z \frac{\gamma}{\eta^\gamma} \left(\int_0^\eta \xi^{\gamma-1} P_{\alpha,\beta} \left(\frac{e^{i\phi} \xi(\xi + t)}{1 + \xi t} \right) d\xi \right) dt \quad (z \in \Delta), \tag{2.14}$$

belongs to the class $\mathcal{B}_\theta(\alpha, \beta)$.

Theorem 2.4. *Let $\theta \in (-\pi, \pi)$, $0 \le \alpha < 1$ and $\beta > 1$. If $f \in \mathcal{B}_\theta(\alpha, \beta)$, then*

$$f'(z) \prec \int_0^1 P_{\alpha,\beta}(z t^{1/\gamma}) dt \prec P_{\alpha,\beta}(z) \quad (z \in \Delta, \gamma = 2/(1 + e^{i\theta})), \tag{2.15}$$

where $P_{\alpha,\beta}$ defined by (2.2). The result is sharp.

Proof. Since $f \in \mathcal{B}_\theta(\alpha, \beta)$, from Lemma 2.1 it follows that (2.1) holds true. If we take $p(z) = f'(z)$, then

$$p(z) + \frac{1}{\gamma} z p'(z) = f'(z) + \frac{1 + e^{i\theta}}{2} z f''(z) \prec P_{\alpha,\beta}(z) \quad (z \in \Delta),$$

where γ is as defined in (2.12). Therefore, applying the Lemma 1.3 and since $P_{\alpha,\beta}$ is a convex univalent function in the open unit disk Δ, we get

$$p(z) \prec \frac{\gamma}{z^\gamma} \int_0^z \xi^{\gamma-1} P_{\alpha,\beta}(\xi) d\xi \prec P_{\alpha,\beta}(z) \quad (z \in \Delta). \tag{2.16}$$

Now by letting $\xi = zt^{1/\gamma}$ in the above integral (2.16) the differential chain (2.16) implies that (2.15) holds true for all $z \in \Delta$. $\square$

Theorem 2.5. *Let $\theta \in (-\pi, \pi)$, $0 \leq \alpha < 1$ and $\beta > 1$. If $f \in \mathcal{B}_\theta(\alpha, \beta)$, then*

$$\frac{f(z)}{z} \prec \int_0^1 \int_0^1 P_{\alpha,\beta}(zrt^{1/\gamma})drdt \quad (z \in \Delta), \tag{2.17}$$

where $P_{\alpha,\beta}$ defined by (2.2).

Proof. Let that $p(z) = f(z)/z$. Then by using of (2.15) and applying Lemma 2.2, and with a simple computation we have

$$p(z) + zp'(z) = f'(z) \prec \int_0^1 P_{\alpha,\beta}(zt^{1/\gamma})dt \quad (z \in \Delta).$$

If we apply Lemma 1.3 once more with $\gamma = 1$, we conclude that

$$p(z) \prec \frac{1}{z}\int_0^z \lambda(\xi)d\xi \prec \lambda(z) \quad (z \in \Delta), \tag{2.18}$$

where λ defined in (2.6). Now, it is sufficient that we put $\xi = rz$ in the integral in (2.18). In this case, if we take into account (2.6), then the first differential subordination in (2.18) implies that (2.17) holds true and concluding the proof. $\square$

Remark 2.6. Since the result of Lemma 1.3 is best possible, it follows that the differential subordinations in (2.15) and (2.17) are also best possible.

3. On coefficients

First, we start this section by estimating the coefficients of members of the family $\mathcal{B}_\theta(\alpha, \beta)$.

Theorem 3.1. *Let $\theta \in (-\pi, \pi]$, $0 \leq \alpha < 1$ and $\beta > 1$. Also, let $f(z) = z + \sum_{n=2}^\infty a_n z^n$ belongs to the class $\mathcal{B}_\theta(\alpha, \beta)$. Then*

$$|a_n| \leq \frac{2\sqrt{2}(\beta - \alpha)}{n\pi\sqrt{n^2 + 1 + (n^2 - 1)\cos\theta}}\sin\frac{\pi(1 - \alpha)}{\beta - \alpha} \quad (n \geq 2). \tag{3.1}$$

Proof. Let f be of the form (1.1) belongs to the class $\mathcal{B}_\theta(\alpha, \beta)$. Then by Lemma 2.1, we have

$$f'(z) + \frac{1 + e^{i\theta}}{2}zf''(z) \prec P_{\alpha,\beta}(z) \quad (z \in \Delta) \tag{3.2}$$

or equivalently

$$1 + \sum_{n=1}^\infty p_n z^n \prec 1 + \sum_{n=1}^\infty B_n z^n.$$

From Lemma 1.2, the last differential subordination implies that

$$|p_n| \leq |B_1| \quad n \geq 1.$$

On the other hand, by equating the coefficients of z^n on both sides of (3.2), the following relation between the coefficients holds true:

$$\frac{n}{2}\left[2 + (n - 1)(1 + e^{i\theta})\right]a_n = p_{n-1}. \tag{3.3}$$

Thus, from (3.3), we obtain

$$|a_n| \leq \left|\frac{B_1}{\frac{n}{2}\left[2 + (n - 1)(1 + e^{i\theta})\right]}\right| = \frac{\sqrt{2}|B_1|}{n\sqrt{n^2 + 1 + (n^2 - 1)\cos\theta}},$$

where

$$|B_1| = \frac{2(\beta - \alpha)}{\pi}\sin\frac{\pi(1 - \alpha)}{\beta - \alpha}.$$

This completes the proof. $\square$

With suitable choices of α, β and θ, we get the following interesting results.

Corollary 3.2. *If $f \in \mathcal{A}$ of the form* (1.1) *satisfying the following two–sided inequality*

$$\frac{1}{2} < \mathrm{Re}\,(f'(z) + zf''(z)) < \frac{3}{2} \quad (z \in \Delta),$$

then

$$|a_n| \leq \frac{2}{\pi n^2} \quad (n \geq 2).$$

In particular, $|a_2| \leq 1/2\pi$.

Corollary 3.3. *If $f \in \mathcal{A}$ of the form* (1.1) *satisfying the following two–sided inequality*

$$0 < \mathrm{Re}\,(f'(z) + zf''(z)) < 2 \quad (z \in \Delta),$$

then

$$|a_n| \leq \frac{4}{\pi n^2} \quad (n \geq 2).$$

In particular, $|a_2| \leq 1/\pi$.

Recently, many researchers (see e.g. [1], [6]) have considered the Fekete-Szegö functional associated with the k–th root transform for several subclasses of analytic functions. In the next result, we consider this problem for functions in the class $\mathcal{B}_\theta(\alpha, \beta)$.

Theorem 3.4. *Let $\theta \in (-\pi, \pi]$, $0 \leq \alpha < 1$ and $\beta > 1$. Also let $f \in \mathcal{B}_\theta(\alpha, \beta)$ and F be the k–th root transform of f defined by* (1.2). *Then, for any complex number μ, the following sharp estimates hold true.*

$$\left| b_{2k+1} - \mu b_{2k+1}^2 \right| \leq \frac{2(\beta - \alpha)}{3k\pi\sqrt{5 + 4\cos\theta}} \sin\frac{\pi(1-\alpha)}{\beta - \alpha} \tag{3.4}$$

$$\times \max\left\{ 1, \left| \frac{1+w}{2} - (2\mu + k - 1)\frac{3(2 + e^{i\theta})(\beta - \alpha)i}{2k\pi(3 + e^{i\theta})^2}(1 - w) \right| \right\},$$

where

$$w = \exp\left\{ \frac{2\pi i(1-\alpha)}{\beta - \alpha} \right\}.$$

Proof. Since $f \in \mathcal{B}_\theta(\alpha, \beta)$, from definition of subordination, there exists a function $w \in \mathcal{B}$ such that

$$f'(z) + \frac{1 + e^{i\theta}}{2}zf''(z) = 1 + \sum_{n=2}^{\infty} \frac{n}{2}\left[2 + (n-1)(1 + e^{i\theta}) \right] a_n z^n = P_{\alpha,\beta}(w(z)). \tag{3.5}$$

Define

$$\mathfrak{p}(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + p_1 z + p_2 z^2 + \cdots. \tag{3.6}$$

Since $w \in \mathcal{B}$, it follows that $\mathfrak{p} \in \mathcal{P}$. From (3.6) and (2.4) we have

$$P_{\alpha,\beta}(w(z)) = 1 + \frac{1}{2}B_1 p_1 z + \left(\frac{1}{4}B_2 p_1^2 + \frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2 \right) \right)z^2 + \cdots. \tag{3.7}$$

Equating the coefficients of z and z^2 on both sides of (3.5), we get

$$\left(3 + e^{i\theta} \right)a_2 = \frac{1}{2}B_1 p_1, \tag{3.8}$$

and

$$\left(6 + 3e^{i\theta} \right)a_3 = \left(\frac{1}{4}B_2 p_1^2 + \frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2 \right) \right). \tag{3.9}$$

On the other hand, a simple computation shows that

$$F(z) = [f(z^{1/k})]^{1/k} = z + \frac{1}{k}a_2 z^{k+1} + \left(\frac{1}{k}a_3 - \frac{1}{2}\frac{k-1}{k^2}a_2^2 \right)z^{2k+1} + \cdots. \tag{3.10}$$

From equations (1.2) and (3.10), we have

$$b_{k+1} = \frac{1}{k}a_2 \quad \text{and} \quad b_{2k+1} = \frac{1}{k}a_3 - \frac{1}{2}\frac{k-1}{k^2}a_2^2. \tag{3.11}$$

Substituting from (3.8) and (3.9) into (3.11), we obtain

$$b_{k+1} = \frac{B_1 p_1}{2k(3 + e^{i\theta})},$$

and

$$b_{2k+1} = \frac{1}{k(6 + 3e^{i\theta})}\left(\frac{1}{4}B_2 p_1^2 + \frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right)\right) - \frac{(k-1)B_1^2 p_1^2}{8k^2(3 + e^{i\theta})^2},$$

so that

$$b_{2k+1} - \mu b_{k+1}^2 = \frac{B_1}{6k(2 + e^{i\theta})}\left[p_2 - \frac{1}{2}\left(1 - \frac{B_2}{B_1} + (2\mu + k - 1)\frac{\left(6 + 3e^{i\theta}\right)B_1}{2k(3 + e^{i\theta})^2}\right)p_1^2\right]. \tag{3.12}$$

From (2.5), we have

$$B_1 = \frac{\beta - \alpha}{\pi}i\left(1 - e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\right), \tag{3.13}$$

and

$$B_2 = \frac{\beta - \alpha}{2\pi}i\left(1 - e^{4\pi i\frac{1-\alpha}{\beta-\alpha}}\right). \tag{3.14}$$

Putting (3.13) and (3.14) into (3.12) and letting

$$\mu = \frac{1}{2}\left(1 - \frac{B_2}{B_1} + (2\mu + k - 1)\frac{\left(6 + 3e^{i\theta}\right)B_1}{2k(3 + e^{i\theta})^2}\right),$$

the inequality (3.4) now follows by an application of Lemma 1.4. It is easy to check that the result is sharp for the k–th root transforms of the functions $f(z, \phi, 1)$ and $f(z, \phi, 0)$ defined by (2.14). $\qquad\qquad\square$

The problem of finding sharp upper bounds for the coefficient functional $|a_3 - \mu a_2^2|$ for different subclasses of the normalized analytic function class $\mathcal{A}$ is known as the Fekete–Szegö problem. If we take $k = 1$ in Theorem 3.4, we have.

Corollary 3.5. *Let $\theta \in (-\pi, \pi]$, $0 \le \alpha < 1$ and $\beta > 1$. Suppose also that $f \in \mathcal{B}_\theta(\alpha, \beta)$. Then, for any complex number μ,*

$$\left|a_3 - \mu a_2^2\right| \le \frac{2(\beta - \alpha)}{3\pi\sqrt{5 + 4\cos\theta}}\sin\frac{\pi(1 - \alpha)}{\beta - \alpha} \tag{3.15}$$

$$\times \max\left\{1, \left|\frac{1 + \exp\left\{\frac{2\pi i(1-\alpha)}{\beta-\alpha}\right\}}{2} - \mu\frac{3(2 + e^{i\theta})(\beta - \alpha)i}{\pi(3 + e^{i\theta})^2}\left(1 - \exp\left\{\frac{2\pi i(1-\alpha)}{\beta - \alpha}\right\}\right)\right|\right\}.$$

The result is sharp.

Corollary 3.6. *Let $0 \le \alpha < 1$ and $\beta > 1$. Suppose also that $f \in \mathcal{B}(\alpha, \beta)$. Then, for any complex number μ,*

$$\left|a_3 - \mu a_2^2\right| \le \frac{2(\beta - \alpha)}{9\pi}\sin\frac{\pi(1 - \alpha)}{\beta - \alpha} \tag{3.16}$$

$$\times \max\left\{1, \left|\frac{1 + \exp\left\{\frac{2\pi i(1-\alpha)}{\beta-\alpha}\right\}}{2} - \mu\frac{9(\beta - \alpha)i}{16\pi}\left(1 - \exp\left\{\frac{2\pi i(1-\alpha)}{\beta - \alpha}\right\}\right)\right|\right\}.$$

The result is sharp.

Corollary 3.7. *If* $f \in \mathcal{A}$ *of the form* (1.1) *satisfying the following two–sided inequality*

$$\frac{1}{2} < \operatorname{Re}\left(f'(z) + zf''(z)\right) < \frac{3}{2} \quad (z \in \Delta),$$

then

$$|a_3 - \mu a_2^2| \leq \frac{2}{9\pi} \max\left\{1, \frac{9|\mu|}{8\pi}\right\} \quad (\mu \in \mathbb{C}).$$

Corollary 3.8. *If* $f \in \mathcal{A}$ *of the form* (1.1) *satisfying the following two–sided inequality*

$$0 < \operatorname{Re}\left(f'(z) + zf''(z)\right) < 2 \quad (z \in \Delta),$$

then

$$|a_3 - \mu a_2^2| \leq \frac{4}{9\pi} \max\left\{1, \frac{9|\mu|}{4\pi}\right\} \quad (\mu \in \mathbb{C}).$$

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