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Global Behavior of Two Rational Third Order Difference Equations

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Abstract

In this paper, we solve and study the global behavior of all admissible solutions of the two difference equations

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1} - x_{n-2}}, \quad n = 0, 1, \ldots,$$

and

$$x_{n+1} = \frac{x_n x_{n-2}}{-x_{n-1} + x_{n-2}}, \quad n = 0, 1, \ldots,$$

where the initial values x_{-2}, x_{-1}, x_0 are real numbers.
We show that every admissible solution for the first equation converges to zero. For the other equation, we show that every admissible solution is periodic with prime period six. Finally we give some illustrative examples.

1. Introduction

In [11], the author determined the forbidden sets and discussed the global behaviors of solutions of the two difference equations

$$x_{n+1} = \frac{x_n x_{n-1}}{x_n - x_{n-2}}, \quad n = 0, 1, \ldots,$$

and

$$x_{n+1} = \frac{x_n x_{n-1}}{-x_n + x_{n-2}}, \quad n = 0, 1, \ldots,$$

where the initial values x_{-2}, x_{-1}, x_0 are real numbers.
In [2], the author determined the forbidden sets and discussed the global behaviors of solutions of the two difference equations

$$x_{n+1} = \frac{a x_n x_{n-1}}{\pm b x_{n-1} + c x_{n-2}}, \quad n = 0, 1, \ldots,$$

where a, b, c are positive real numbers and the initial conditions x_{-2}, x_{-1}, x_0 are real numbers.
Elsayed in [19] studied the behavior of solutions of the nonlinear difference equation

$$x_{n+1} = a x_{n-1} + \frac{b x_n x_{n-1}}{c x_n + d x_{n-2}}, \quad n = 0, 1, \ldots,$$

where a, b, c, d are positive real constants and the initial conditions x_{-2}, x_{-1}, x_0 are arbitrary positive real numbers. For more on difference equations (See [1, 3–10, 12–18, 20–28]) and the references therein.
In this paper, we study the two difference equations

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1} - x_{n-2}}, \quad n = 0, 1, \ldots, \tag{1.1}$$

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and

$$x_{n+1}=\frac{x_n x_{n-2}}{-x_{n-1}+x_{n-2}},\quad n=0,1,..., \tag{1.2}$$

where the initial values x_{-2}, x_{-1}, x_0 are real numbers.

2. The difference equation $x_{n+1}=\frac{x_n x_{n-2}}{x_{n-1}-x_{n-2}}$

During this section, we suppose that

$$\lambda_- = \frac{1}{2}-\frac{\sqrt{5}}{2} \text{ and } \lambda_+ = \frac{1}{2}+\frac{\sqrt{5}}{2}.$$

2.1. Solution of Equation (1.1)

The transformation

$$y_n=\frac{x_{n-1}}{x_n},\ \text{with } y_{-1}=\frac{x_{-2}}{x_{-1}},\ y_0=\frac{x_{-1}}{x_0} \tag{2.1}$$

reduces Equation (1.1) into the difference equation

$$y_{n+1}=\frac{1}{y_{n-1}}-1,\ n=0,1,.... \tag{2.2}$$

By solving Equation (2.2) and after some calculations, the solution of Equation (1.1) can be obtained.

Theorem 2.1. *Let $\{x_n\}_{n=-2}^{\infty}$ be an admissible solution of Equation (1.1). Then*

$$x_n=\begin{cases} -\frac{v}{(x_0 f_{\frac{n-1}{2}}-x_{-1}f_{\frac{n+1}{2}})(x_{-1}f_{\frac{n+1}{2}}-x_{-2}f_{\frac{n+3}{2}})}, & n=1,3,...,\\ \frac{v}{(x_0 f_{\frac{n}{2}}-x_{-1}f_{\frac{n}{2}+1})(x_{-1}f_{\frac{n}{2}}-x_{-2}f_{\frac{n}{2}+1})}, & n=2,4,..., \end{cases} \tag{2.3}$$

where $v=x_0x_{-1}x_{-2}$ and f_n is the solution of the difference equation

$$f_{n+2}=f_n+f_{n+1},\ f_0=0,\ f_1=1,\ n=0,1,....$$

Proof. We can write the solution formula (2.3) as

$$x_{2m+1}=-\frac{v}{(x_0 f_m-x_{-1}f_{m+1})(x_{-1}f_{m+1}-x_{-2}f_{m+2})}$$

and (2.4)

$$x_{2m+2}=\frac{v}{(x_0 f_{m+1}-x_{-1}f_{m+2})(x_{-1}f_{m+1}-x_{-2}f_{m+2})}.$$

When $m=0$,

$$\begin{aligned} x_1 &= -\frac{v}{(x_0 f_0-x_{-1}f_1)(x_{-1}f_1-x_{-2}f_2)}\\ &=\frac{v}{x_{-1}(x_{-1}-x_{-2})}=\frac{x_0x_{-2}}{x_{-1}-x_{-2}}. \end{aligned}$$

Similarly

$$\begin{aligned} x_2 &= \frac{v}{(x_0 f_1-x_{-1}f_2)(x_{-1}f_1-x_{-2}f_2)}\\ &=\frac{v}{(x_0-x_{-1})(x_{-1}-x_{-2})}=\frac{x_1x_{-1}}{x_0-x_{-1}}. \end{aligned}$$

Suppose that the solution formula (2.4) is true for $m>0$. Then

$$\begin{aligned} \frac{x_{2m+1}x_{2m-1}}{x_{2m}-x_{2m-1}} &= \frac{\left(\frac{v}{(x_0f_m-x_{-1}f_{m+1})(x_{-1}f_{m+1}-x_{-2}f_{m+2})}\right)\left(\frac{v}{(x_0f_{m-1}-x_{-1}f_m)(x_{-1}f_m-x_{-2}f_{m+1})}\right)}{\frac{v}{(x_0f_m-x_{-1}f_{m+1})(x_{-1}f_m-x_{-2}f_{m+1})}+\frac{v}{(x_0f_{m-1}-x_{-1}f_m)(x_{-1}f_m-x_{-2}f_{m+1})}}\\ &=\frac{v}{(x_0f_{m-1}-x_{-1}f_m)(x_{-1}f_{m+1}-x_{-2}f_{m+2})+(x_0f_m-x_{-1}f_{m+1})(x_{-1}f_{m+1}-x_{-2}f_{m+2})}\\ &=\frac{v}{(x_{-1}f_{m+1}-x_{-2}f_{m+2})(x_0(f_{m-1}+f_m)-x_{-1}(f_m+f_{m+1}))}\\ &=\frac{v}{(x_{-1}f_{m+1}-x_{-2}f_{m+2})(x_0f_{m+1}-x_{-1}f_{m+2})}\\ &=x_{2m+2}. \end{aligned}$$

Similarly we can show that

$$\frac{x_{2m+2}x_{2m}}{ax_{2m+1}+bx_{2m}}=x_{2m+3}.$$

This completes the proof. □

It is clear for Equation (1.1) that if we start with the point $(x_0, x_{-1}, x_{-2}) \in \mathbb{R}^3$, we have the following:
If $x_0 = 0$ and $x_{-1}x_{-2} \neq 0$, then x_3 is undefined.
If $x_{-1} = 0$ and $x_0x_{-2} \neq 0$, then x_5 is undefined.
If $x_{-2} = 0$ and $x_0x_{-1} \neq 0$, then x_4 is undefined.
Therefore, any point $(x_0, x_{-1}, x_{-2}) \in \mathbb{R}^3$ with $x_0x_{-1}x_{-2} = 0$ belongs to the forbidden set of Equation (1.1).
The following result provides the forbidden set of Equation (1.1).

Theorem 2.2. *The forbidden set of equation (1.1) is*

$$F = \bigcup_{i=0}^{2}\{(u_0,u_{-1},u_{-2}) \in \mathbb{R}^3 : u_{-i} = 0\} \cup \bigcup_{m=1}^{\infty}\{(u_0,u_{-1},u_{-2}) \in \mathbb{R}^3 : u_0 = u_{-1}\frac{f_{m+1}}{f_m}\} \cup$$
$$\bigcup_{m=1}^{\infty}\{(u_0,u_{-1},u_{-2}) \in \mathbb{R}^3 : u_{-1} = u_{-2}\frac{f_{m+1}}{f_m}\}.$$

Proof. The proof is clear using the arguments after Theorem (2.1) and formula (2.3). □

2.2. Global behavior of equation (1.1)

In this section, we shall give two invariant sets for Equation (1.1) and a result concerns the global behavior of the solutions of Equation (1.1).
Consider the set

$$D_1 = \{(x,y,z) \in \mathbb{R}^3 : \frac{x}{1/\lambda_-^2} = -\frac{y}{1/\lambda_-} = z\}$$

and

$$D_2 = \{(x,y,z) \in \mathbb{R}^3 : \frac{x}{1/\lambda_+^2} = -\frac{y}{1/\lambda_+} = z\}.$$

Theorem 2.3. *The two sets D_1 and D_2 are invariant sets for Equation (1.1).*

Proof. Let $(x_0, x_{-1}, x_{-2}) \in D_1$. We show that $(x_n, x_{n-1}, x_{n-2}) \in D_1$ for each $n \in \mathbb{N}$. The proof is by induction on n. The point $(x_0, x_{-1}, x_{-2}) \in D_1$ implies

$$\frac{x_0}{1/\lambda_-^2} = -\frac{x_{-1}}{1/\lambda_-} = x_{-2}.$$

Now for $n = 1$, we have

$$x_1 = \frac{x_0x_{-2}}{x_{-1} - x_{-2}} = \frac{(1/\lambda_-)x_{-1}\lambda_- x_{-1}}{x_{-1} + \lambda_- x_{-1}} = \frac{x_{-1}}{\lambda_-^2}.$$

Then we have

$$\frac{x_1}{1/\lambda_-^2} = -\frac{x_0}{1/\lambda_-} = x_{-1}.$$

This implies that $(x_1, x_0, x_{-1}) \in D_1$.
Suppose now that $(x_n, x_{n-1}, x_{n-2}) \in D_1$. This means that

$$\frac{x_n}{1/\lambda_-^2} = -\frac{x_{n-1}}{1/\lambda_-} = x_{n-2}.$$

Then

$$x_{n+1} = \frac{x_nx_{n-2}}{x_{n-1} - x_{n-2}} = \frac{(1/\lambda_-)x_{n-1}\lambda_- x_{n-1}}{x_{n-1} + \lambda_- x_{n-1}} = \frac{x_{n-1}}{\lambda_-^2}.$$

This implies that $(x_{n+1}, x_n, x_{n-1}) \in D_1$. Therefore, D_1 is an invariant set for Equation (1.1).
By similar way, we can show that D_2 is an invariant set for Equation (1.1).
This completes the proof. □

Theorem 2.4. *Every admissible solution of Equation (1.1) converges to zero.*

Proof. Suppose that $\{x_n\}_{n=-2}^{\infty}$ is an admissible solution of Equation (1.1).
Using Formula (2.4), we can write

$$\begin{aligned} x_{2m+1} &= -\frac{\nu}{(x_0f_m - x_{-1}f_{m+1})(x_{-1}f_{m+1} - x_{-2}f_{m+2})} \\ &= -\frac{\nu}{f_mf_{m+1}(x_0 - x_{-1}\frac{f_{m+1}}{f_m})(x_{-1} - x_{-2}\frac{f_{m+2}}{f_{m+1}})}. \end{aligned} \tag{2.5}$$

But

$$\frac{f_{m+1}}{f_m} \to \lambda_+ \text{ and } f_m \to \infty \text{ as } m \to \infty.$$

This implies that

$$x_{2m+1} \to 0 \text{ as } m \to \infty.$$

Similarly, we can show that $x_{2m+2} \to 0$, as $m \to \infty$.
Therefore, $x_n \to 0$ as $n \to \infty$. This completes the proof. □

Example (1)
Figure (2.1) shows that a solution $\{x_n\}_{n=-2}^{\infty}$ of equation (1.1) with $x_{-2} = 2$, $x_{-1} = -0.2$ and $x_0 = 1$ converges to zero.

Example (2)
Figure (2.2) shows that a solution $\{x_n\}_{n=-2}^{\infty}$ of equation (1.1) with $x_{-2} = -1$, $x_{-1} = -0.2$ and $x_0 = -1.8$ converges to zero.

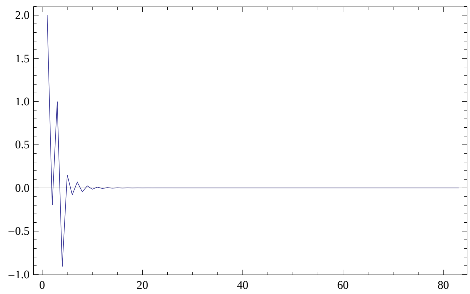

Figure 2.1: $x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1} - x_{n-2}}$

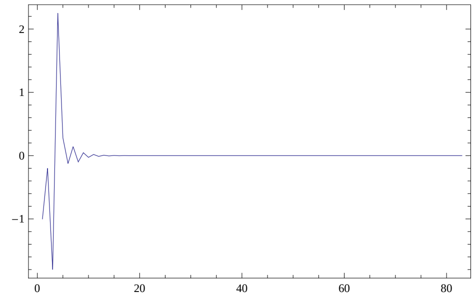

Figure 2.2: $x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1} - x_{n-2}}$

3. The difference equation $x_{n+1} = \frac{x_n x_{n-2}}{-x_{n-1} + x_{n-2}}$

In this section, we study the difference equation (1.2).

3.1. Solution of Equation (1.2)

The transformation (2.1) reduces Equation (1.2) into the difference equation

$$y_{n+1} = -\frac{1}{y_{n-1}} + 1, \; n = 0, 1, ... \tag{3.1}$$

By solving Equation (3.1) and after some calculations, the solution of Equation (1.2) can be obtained.

Theorem 3.1. *Let $\{x_n\}_{n=-2}^{\infty}$ be an admissible solution of Equation (1.2). Then*

$$x_n = \begin{cases} \dfrac{\mu}{(\alpha_0 \cos\frac{(n-3)\pi}{6} - \beta_0 \sin\frac{(n-3)\pi}{6})(\alpha_{-1}\cos\frac{(n-1)\pi}{6} - \beta_{-1}\sin\frac{(n-1)\pi}{6})}, & n = 1, 3, ..., \\ \dfrac{\mu}{(\alpha_0 \cos\frac{(n-2)\pi}{6} - \beta_0 \sin\frac{(n-2)\pi}{6})(\alpha_{-1}\cos\frac{(n-2)\pi}{6} - \beta_{-1}\sin\frac{(n-2)\pi}{6})}, & n = 2, 4, ..., \end{cases} \tag{3.2}$$

where $\mu = x_0 x_{-1} x_{-2}$, $\alpha_0 = -x_0 + x_{-1}$, $\beta_0 = \frac{1}{\sqrt{3}}(x_0 + x_{-1})$, $\alpha_{-1} = -x_{-1} + x_{-2}$ and $\beta_{-1} = \frac{1}{\sqrt{3}}(x_{-1} + x_{-2})$.

Proof. We can write the given solution (3.2) as

$$x_{2m+1} = \frac{\mu}{\gamma_0(m-1)\gamma_{-1}(m)}$$

and (3.3)

$$x_{2m+2} = \frac{\mu}{\gamma_0(m)\gamma_{-1}(m)},$$

where

$$\gamma_0(m) = \alpha_0 \cos\frac{m\pi}{3} - \beta_0 \sin\frac{m\pi}{3}$$

and

$$\gamma_{-1}(m) = \alpha_{-1}\cos\frac{m\pi}{3} - \beta_{-1}\sin\frac{m\pi}{3}.$$

When $m = 0$,

$$\begin{aligned} x_1 &= \frac{\mu}{\gamma_0(-1)\gamma_{-1}(0)} = \frac{\mu}{(\alpha_0\cos\frac{-\pi}{3} - \beta_0 \sin\frac{-\pi}{3})(\alpha_{-1})} \\ &= \frac{\mu}{\frac{1}{2}(\alpha_0 + \sqrt{3}\beta_0)(\alpha_{-1})} = \frac{\mu}{x_{-1}(-x_{-1} + x_{-2})} \\ &= \frac{x_0 x_{-2}}{-x_{-1} + x_{-2}}. \end{aligned}$$

Similarly

$$\begin{aligned} x_2 &= \frac{\mu}{\gamma_0(0)\gamma_{-1}(0)} = \frac{\mu}{\alpha_0\alpha_{-1}} \\ &= \frac{x_0x_{-1}x_{-2}}{(-x_0+x_{-1})(-x_{-1}+x_{-2})} \\ &= \frac{x_1x_{-1}}{-x_0+x_{-1}}. \end{aligned}$$

Suppose that the solution (3.3) is true for $m > 0$.
Then

$$\begin{aligned} \frac{x_{2m+1}x_{2m-1}}{-x_{2m}+x_{2m-1}} &= \frac{\left(\frac{\mu}{\gamma_0(m-1)\gamma_{-1}(m)}\right)\left(\frac{\mu}{\gamma_0(m-2)\gamma_{-1}(m-1)}\right)}{-\frac{\mu}{\gamma_0(m-1)\gamma_{-1}(m-1)}+\frac{\mu}{\gamma_0(m-2)\gamma_{-1}(m-1)}} \\ &= \frac{\mu}{\gamma_{-1}(m)(-\gamma_0(m-2)+\gamma_0(m-1))}. \end{aligned}$$

But we can show that

$$\gamma_0(m-1)-\gamma_0(m-2)=\gamma_0(m),\ m=0,1,....$$

This implies that

$$\begin{aligned} \frac{x_{2m+1}x_{2m-1}}{-x_{2m}+x_{2m-1}} &= \frac{\mu}{\gamma_0(m)\gamma_{-1}(m)} \\ &= x_{2m+2}. \end{aligned}$$

Similarly we can show that

$$\frac{x_{2m+2}x_{2m}}{ax_{2m+1}+bx_{2m}} = x_{2m+3}.$$

This completes the proof. □

It is clear for Equation (1.2) that if we start with the point $(x_0,x_{-1},x_{-2}) \in \mathbb{R}^3$, we have the following:
If $x_0 = 0$ and $x_{-1}x_{-2} \neq 0$, then x_3 is undefined.
If $x_{-1} = 0$ and $x_0x_{-2} \neq 0$, then x_5 is undefined.
If $x_{-2} = 0$ and $x_0x_{-1} \neq 0$, then x_4 is undefined.
Therefore, any point $(x_0,x_{-1},x_{-2}) \in \mathbb{R}^3$ with $x_0x_{-1}x_{-2} = 0$ belongs to the forbidden set of Equation (1.2).
The following result provides the forbidden set of Equation (1.2).

Theorem 3.2. *The forbidden set of equation* (1.2) *is*

$$\begin{aligned} F = &\bigcup_{i=0}^{2}\{(u_0,u_{-1},u_{-2}) \in \mathbb{R}^3 : u_{-i}=0\} \cup \{(u_0,u_{-1},u_{-2}) \in \mathbb{R}^3 : u_0 = u_{-1}\} \cup \\ &\{(u_0,u_{-1},u_{-2}) \in \mathbb{R}^3 : u_{-1} = u_{-2}\}. \end{aligned}$$

3.2. Global Behavior of Equation (1.2)

Theorem 3.3. *Every admissible solution for Equation* (1.2) *is periodic with prime period six.*

Proof. Suppose that $\{x_n\}_{n=-2}^{\infty}$ is an admissible solution for Equation (1.2).
It is clear that both the functions $\gamma_{-1}(m)$ and $\gamma_0(m)$ satisfy

$$\gamma_{-1}(m+3) = -\gamma_{-1}(m) \text{ and } \gamma_0(m+3) = -\gamma_0(m).$$

Then

$$\begin{aligned} x_{2(m+3)+1} &= \frac{\mu}{\gamma_0(m+2)\gamma_{-1}(m+3)} \\ &= \frac{\mu}{\gamma_0(m-1)\gamma_{-1}(m)} \\ &= x_{2m+1},\ m=-1,0,.... \end{aligned}$$

Similarly

$$\begin{aligned} x_{2(m+3)+2} &= \frac{\mu}{\gamma_0(m+3)\gamma_{-1}(m+3)} \\ &= \frac{\mu}{\gamma_0(m)\gamma_{-1}(m)} \\ &= x_{2m+2},\ m=-2,-1,..... \end{aligned}$$

Therefore, the solution $\{x_n\}_{n=-2}^{\infty}$ is periodic with prime period six. This completes the proof. □

Example (3)
Figure (3.1) shows that a solution $\{x_n\}_{n=-2}^{\infty}$ of equation (1.2) with $x_{-2} = -3.2$, $x_{-1} = 2.8$ and $x_0 = 0.9$ is periodic with prime period six.

Example (4)
Figure (3.2) shows that a solution $\{x_n\}_{n=-2}^{\infty}$ of equation (1.2) with $x_{-2} = 1.2$, $x_{-1} = 1.7$ and $x_0 = -0.2$ is periodic with prime period six.

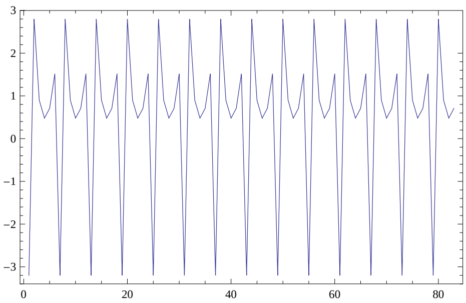

Figure 3.1: $x_{n+1} = \frac{x_n x_{n-2}}{-x_{n-1}+x_{n-2}}$

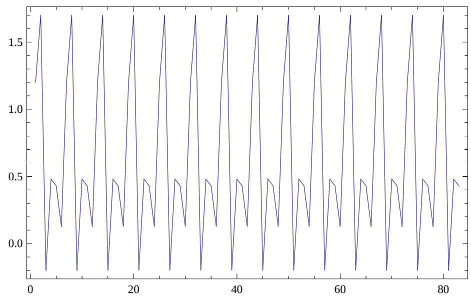

Figure 3.2: $x_{n+1} = \frac{x_n x_{n-2}}{-x_{n-1}+x_{n-2}}$

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