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Araştırma Makalesi / Research Article

Hiperbolik Denklem İçeren Bir Optimal Kontrol Probleminin Nümerik Çözümü Üzerine

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Öz
Bu makalede, hiperbolik denklem içeren optimal kontrol problemlerinin bir sınıfını çözmek için bir nümerik algoritma sunulmaktadır. Bir regüler uzayda optimal çözümün var ve tek olduğu gösterilmektedir. Eşlenik problemi elde ettikten ve amaç fonksiyonelinin türevini hesapladıktan sonra, Gradyen metoduyla nümerik yaklaşımlar elde edilmektedir. Hesaplanan sonuçlar, önerilen metodun optimal kontrol problemleri için iyi nümerik yaklaşımlar üretebildiğini göstermektedir.

Anahtar kelimeler: Optimal kontrol, hiperbolik denklemler, Frechet diferansiyellenebilirlik.

On Numerical Solution of an Optimal Control Problem Including Hyperbolic Equation

Abstract
In this study, a numerical algorithm for solving a class of optimal control problems with hyperbolic equation is offered. It has been showed that the optimal solution is exist and unique in a regular space. After obtaining adjoint problem and calculating derivative of the cost functional, numerical approximations are obtained via Gradient Method. Computational results show that the considered method is able to generate good numerical approximations for optimal control problems.

Keywords: Optimal control, hyperbolic equations, Frechet differentiability.

1. Introduction and Statement of the Problem

As it is known, hyperbolic partial differential equations state many physical phenomena like heat conduction, vibration, sound waves, diffusion and many more. However, studies related with optimal control problems including hyperbolic equation have been studied from various aspects by many researchers in recent years. When these studies are examined, it has been found too many researches where the control function can be the right hand side of equation, in the coefficient or on boundary for hyperbolic equation in literature [1-14].

In this study, we present numerical results in view of making the contribution on obtaining the initial control.

We consider the following initial boundary value problem defined on the domain $\Omega = (0,l) \times (0,T)$

$$\psi_{tt} - \psi_{xx} + v(x)\psi = F(x,t), \quad (x,t) \in \Omega \tag{1}$$

$$\psi(x,0) = \varphi_1(x), \quad \psi_t(x,0) = \varphi_2(x), \quad x \in (0,l) \tag{2}$$

$$\psi(0,t) = 0, \quad \psi(l,t) = 0, \quad t \in (0,T). \tag{3}$$

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For this problem, the function $\psi(x,t)$ describes displacement of the string. The term $v(x)\psi$ is known as the restoring force. Also, the function $F(x,t)$ is the source function, the function $\varphi_1(x)$ is the initial status which is our control function for this problem and the function $\varphi_2(x)$ is the initial velocity.

The object of this study is to find the control function minimizing the cost functionals

$$J_\alpha(\varphi_1) = \iint_\Omega \left[\psi(x,t) - y_1(x,t)\right]^2 dxdt + \int_0^l \left[\psi_t(x,T) - y_2(x)\right]^2 dx + \alpha \|\varphi_1 - \kappa\|^2_{H^1(0,l)} \tag{4}$$

on the set

$$V = \left\{\varphi_1(x) : \varphi_1(x) \in H^1(0,l), \|\varphi_1\|^2_{H^1(0,l)} \le \tilde{\varphi}\right\}$$

which is closed, bounded and convex subset of $H^1(0,l)$. In other words, we deal with the problem of controlling the initial status under the conditions

$$F(x,t) \in H^1(\Omega), \varphi_1(x) \in H^1(0,l), \varphi_2(x) \in L_2(0,l)$$

In the cost functional $J_\alpha(\varphi_1)$, $y_1(x,t)$ and $y_2(x)$ are the desired target functions to which must be close enough $\psi(x,t)$ and $\psi_t(x,T)$, respectively.

The parameter α is called regularization parameter and plays vital role in the minimization process.

Now, we give some properties that will be used in this work.

The space $L_2(\Omega)$ describes space of square integrable functions. The space $H^1(\Omega)$ consists of all elements $L_2(\Omega)$ with generalized derivatives of first.

The norm and inner product on these spaces are given by

$$\langle \psi_1, \psi_2 \rangle_{L_2(\Omega)} = \iint_\Omega (\psi_1.\psi_2)\, dxdt,$$
$$\|\psi\|_{L_2(\Omega)} = \sqrt{\langle \psi, \psi \rangle_{L_2(\Omega)}}.$$

and

$$\langle \psi_1, \psi_2 \rangle_{H^1(\Omega)} = \iint_\Omega \left(\psi_1.\psi_2 + \frac{\partial \psi_1}{\partial x} \cdot \frac{\partial \psi_2}{\partial x} + \frac{\partial \psi_1}{\partial t} \cdot \frac{\partial \psi_2}{\partial t}\right) dxdt,$$
$$\|\psi\|_{H^1(\Omega)} = \sqrt{\langle \psi, \psi \rangle_{H^1(\Omega)}}.$$

Organization of this study is presented as follows. Firstly, we introduce some definitions and preliminary results. Firstly we show that the weak solution and optimal solution is exist and unique. Then, we obtain the adjoint problem for the considered problem and calculate Frechet derivative of the functional. Later, we offer necessary optimality condition for the considered problem. Finally, we give two illustrations with regard to obtaining the control function.

2. Some Theoretical Properties

In this section, we present some theoretical properties in order to prove existence and uniqueness of the optimal solution. We show that the weak solution of the hyperbolic problem and optimal solution for optimal control problem is exist and unique. Also gradient of the functional is calculated by adjoint problem. With these operations, we make sure that necessary optimality conditions are hold. One can find detailed information about these results in [15]. Thus we can say that we can investigate the control function minimizing our problem taking into account these conditions.

2.1. Weak solution

The weak solution of hyperbolic problem is the function which holds the following integral equality

$$\iint_{\Omega}[-\psi_t\zeta_t+\psi_x\zeta_x+v(x)\psi\zeta]dxdt=\iint_{\Omega}F(x,t)\zeta dxdt+\int_0^l\varphi_2(x)\zeta(x,0)dx \tag{5}$$

for $\forall\ \zeta\in \overset{o}{H}{}^1(\Omega),\ \ \zeta(x,T)=0.$

According to (Ladyzhenskaya, 1985), the weak solution is exist, unique and continuous dependence according to initial data under some conditions [16]. Namely the following priori is valid for this solution;

$$\|\psi\|^2_{H^1(\Omega)}\le c_0\left(\|\varphi_1\|^2_{H^1(0,l)}+\|\varphi_2\|^2_{L_2(0,l)}+\|F\|^2_{L_2(0,l)}\right). \tag{6}$$

2.2. Optimal solution

According to Goebel's theorem, we can say that optimal solution for minimization problem (1)-(4) is exist and unique [17].

Now, we will show that the conditions in the theorem are hold for our problem.

- The set V is a closed, bounded and convex subset of $H^1(0,l)$ which is a uniformly convex Banach space [18].
- On the other hand, for the increment of the functional $J(\varphi_1)$, the following inequality is valid;
-

$$|\delta J(\varphi_1)|\le c_2\left(\|\delta\varphi_1\|_{H^1(0,l)}+\|\delta\varphi_1\|^2_{H^1(0,l)}\right). \tag{7}$$

This inequality can easily be obtained by giving increment $\varphi_1\in H^1(0,l)$ to the control function φ_1 for the functional $J(\varphi_1)$. It follows that the functional $J(\varphi_1)$ is also lower semi continuous and bounded from below on the set V from the inequality (7). Thus, optimal solution is exist.

- Finally, for our problem since we choose $\omega=2$, then the optimal solution is unique.

2.3. Adjoint problem and derivative of the cost functional

Here, we aim to achieve an adjoint problem and the derivative of the functional $J_\alpha(\varphi_1)$ by via of the adjoint problem on the set V. Later we will give a necessary optimality condition in variational form. To do this, we shall offer difference problem which plays an important role for obtaining adjoint problem and calculating the derivative of the functional. So, we give an increment $\Delta\varphi_1\in H^1(0,l)$ to the control function $\varphi_1(x)$ such as $\varphi_1+\Delta\varphi_1\in V$.

The difference $\Delta\psi=\Delta\psi(x,t)=\psi(x,t;\varphi_1+\Delta\varphi_1)-\psi(x,t;\varphi_1)$ is the solution of the following difference initial-boundary value problem;

$$\Delta\psi_{tt}-\Delta\psi_{xx}+v(x)\Delta\psi=0 \tag{8}$$

$$\Delta\psi(x,0)=\Delta\varphi_1,\ \ \Delta\psi_t(x,0)=0 \tag{9}$$

$$\Delta\psi(0,t)=0,\ \ \Delta\psi(l,t)=0. \tag{10}$$

Using the inequality (6), we have the following inequality;

$$\left\|\Delta\psi(\cdot,t)\right\|^2_{L_2(0,l)} \le c\left\|\Delta\varphi_1\right\|^2_{H^1(0,l)}. \tag{11}$$

Here the solution of above problem.

Now, we shall calculate the derivative of the functional to find the control function which is satisfy the following problem

$$\inf \tilde{J}_\alpha(\psi,\varphi_1,\eta) \tag{12}$$

where

$$\begin{aligned}\tilde{J}_\alpha(\psi,\varphi_1,\eta) &= \iint_\Omega \left[\psi(x,t)-y_1(x,t)\right]^2 dxdt + \int_0^l \left[\psi_t(x,T)-y_2(x)\right]^2 dxdt + \alpha\left\|\varphi-\kappa\right\|^2_{H^1(0,l)} \\ &+ \int_0^T\int_0^l \left[\psi_{tt}-\psi_{xx}+v(x)\psi-F(x,t)\right]\eta_t dxdt.\end{aligned} \tag{13}$$

Here $\tilde{J}_\alpha(\psi,\varphi_1,\eta)$ is also known as augmented functional.

Giving an increment $\Delta\varphi_1 \in H^1(0,l)$ to the augmented functional such as $\varphi_1+\Delta\varphi_1 \in V$, the first variation of this functional is given by

$$\begin{aligned}\Delta\tilde{J}_\alpha(\varphi_1) &= \int_0^l \left(-\eta(x,0)\Delta\varphi_1 - \eta_x(x,0)\Delta\varphi_{1x} + 2\alpha(\varphi_1-\kappa)\Delta\varphi_1 + 2\alpha(\varphi_1-\kappa)\Delta\varphi_{1x}\right)dx + o\left(\left\|\delta\varphi_1\right\|^2_{H^1(0,l)}\right) \\ &= \left\langle -\eta(x,0)+2\alpha(\varphi_1-\kappa),\Delta\varphi_1\right\rangle_{H^1(0,l)} + o\left(\left\|\delta\varphi_1\right\|^2_{H^1(0,l)}\right)\end{aligned} \tag{14}$$

where $\Delta\tilde{J}_\alpha(\varphi_1)=\tilde{J}_\alpha(\varphi_1+\Delta\varphi_1)-\tilde{J}_\alpha(\varphi_1)$.

To obtain derivative of the functional, we achieve the following problem

$$\eta_{tt}-\eta_{xx}+v(x)\eta = 2\left[\psi(x,t)-y_1(x,t)\right] \tag{15}$$

$$\eta_x(x,T)=0,\ \ \eta_t(x,T)=2\left[\psi_t(x,T)-y_2(x)\right] \tag{16}$$

$$\eta(0,t)=0,\ \ \eta(l,t)=0. \tag{17}$$

Thus the gradient of the cost functional is in the form of;

$$\tilde{J}'_\alpha(\varphi_1) = -\eta(x,0)+2\alpha(\varphi_1-\kappa). \tag{18}$$

2.4. Necessary condition for optimal solution

Calculating the gradient of the functional, it can be seen that the derivative $\tilde{J}'_\alpha(\varphi_1)$ is continuous on the set V. The fact that the functional $\tilde{J}'_\alpha(\varphi_1)$ is continuously differentiable on the set V and the set V is convex, in that case the following inequality is valid according to theorem given by (Vasilyev, 1981) in [19];

$$\left\langle \tilde{J}'_\alpha(\varphi_1^*),\varphi_1-\varphi_1^*\right\rangle_{H^1(0,l)} \ge 0, \quad \forall\varphi_1 \in V. \tag{19}$$

Therefore, necessary condition for optimal solution is given by the following inequality;

$$\left\langle -\eta(x,0)+2\alpha\varphi_1^*,\varphi_1-\varphi_1^*\right\rangle_{H^1(0,l)}\geq 0,\quad \forall\varphi_1\in V. \tag{20}$$

3. Results

In this section, we give main results about obtaining optimal solution for our problem. To do this, we use well-known Galerkin method for solving the considered problem. Also we will benefit from Gradient method to get the optimal solution.

3.1. Numerical approximation: Galerkin method

Now, we will use the Galerkin method for solving hyperbolic problem. Galerkin method is useful and efficient to solve various types of hyperbolic equations. In literature, there have been many researches about Galerkin method which is an important tool in for numerical solution of hyperbolic problems [20, 21].

Now we give the following approximate solution of our problem;

$$\psi^N(x,t)=\sum_{s=1}^{N}c_s^N(t)\lambda_s(x) \tag{21}$$

where the coefficients $c_s^N(t)$ are the functions such that

$$c_s^N(t)=\left\langle \psi^N(x,t),\lambda_s(x)\right\rangle_{L_2(0,l)} \tag{22}$$

for $s=1,2,...N.$ Also, the functions $\lambda_s(x)$ are basis functions and they are given by

$$\left\langle \lambda_s(x),\lambda_t(x)\right\rangle_{L_2(0,l)}=\delta_s^t \tag{23}$$

where δ_s^t states Kronecker delta.

To obtain the coefficients $c_s^N(t)$, we replace the approximate solution $\psi^N(x,t)$ into our equation,

$$\sum_{s=1}^{N}\frac{d^2c_s^N(t)}{dt^2}\lambda_s(x)-\sum_{s=1}^{N}c_s^N(t)\frac{d^2\lambda_s(x)}{dx^2}+v(x)\sum_{s=1}^{N}c_s^N(t)\lambda_s(x)=F(x,t). \tag{24}$$

Later, we integrate on the domain $(0,l)$ after multiplying the equation with the function $\lambda_s(x)$. Thus we have the following,

$$\int_0^l\left[\sum_{s=1}^{N}\frac{d^2c_s^N(t)}{dt^2}\lambda_s(x)-\sum_{s=1}^{N}c_s^N(t)\frac{d^2\lambda_s(x)}{dx^2}+v(x)\sum_{s=1}^{N}c_s^N(t)\lambda_s(x)\right]\lambda_t(x)dx=\int_0^l F(x,t)\lambda_t(x)dx \tag{25}$$

where $\left\langle \lambda_s(x),\lambda_t(x)\right\rangle_{L_2(0,l)}=0$ and $\left\langle \Delta\lambda_s(x),\lambda_t(x)\right\rangle_{L_2(0,l)}=0$ as $k\neq l$.

The equality (25) can be written in the matrix form of

$$\begin{gathered}\frac{d^2C^N(t)}{dt^2}+D(t)C^N(t)=F(t)\\ C^N(0)=E,\frac{dC^N(0)}{dt}=G.\end{gathered} \tag{26}$$

Here $C^N(t)=\left[c_1^N(t),c_2^N(t),...,c_N^N(t)\right]^T$ is the matrix of searched functions. Also

$$F(t)=\int_0^l F(x,t)\lambda_t(x)dx, E=\int_0^l \varphi_1(x)\lambda_t(x)dx, G=\int_0^l \varphi_2(x)\lambda_t(x)dx \tag{27}$$

From system of second order ODE, we calculate the coefficients $c_s^N(t)$ and we achieve the approximate solution $\psi^N(x,t)$. Similarly, we have the approximate solution $\eta^N(x,t)$ of the adjoint problem (15)-(17).

3.2. Minimization process: Gradient method

In this section, we search the optimal solution of the optimization problem for which we use Gradient Method. In optimal control theory, this method is the most common optimization algorithm which utilizes from the first derivative. So, we will use the gradient of the functional given by

$$\left(\tilde{J}_\alpha^N\right)'\left((\varphi_1)_s\right)=-\eta^N(x,0)+2\alpha\left((\varphi_1)_s-\kappa\right). \tag{28}$$

The sequence $\{(\varphi_1)_s\}$ is known as the minimizing sequence. Selecting an initial element $(\varphi_1)_0\in V$ and using the following algorithm,

$$(\varphi_1)_{s+1}=(\varphi_1)_s-\varepsilon_s\left(\tilde{J}_\alpha\right)'\left((\varphi_1)_s\right), s=0,1,2,... \tag{29}$$

we find the optimal solution. Here the parameter $\varepsilon_s>0$ is the size of the step. Although the parameter ε_s is arbitrary, too small ε_s may cause slow convergence and too large ε_s could cause overshoot the minima and diverge. So, it is highly important to choose the parameter ε_s such that $\tilde{J}_\alpha\left((\varphi_1)_{s+1}\right)<\tilde{J}_\alpha\left((\varphi_1)_s\right)$. For small enough, we have

$$\tilde{J}_\alpha\left((\varphi_1)_{s+1}\right)-\tilde{J}_\alpha\left((\varphi_1)_s\right)=\varepsilon_s\left(-\left\|\tilde{J}_\alpha'\left((\varphi_1)_s\right)\right\|^2+\frac{o(\varepsilon_s)}{\varepsilon_s}\right)<0 \tag{30}$$

where $(\varphi_1)_s$ is a stationary point of the minimization problem. If $\tilde{J}_\alpha'\left((\varphi_1)_s\right)=0,$ we stop the iteration and the point $(\varphi_1)_s$ is a solution of the minimization problem.

The numerical algorithm to solve optimal control problem is outlined as follows. We constitute minimizing sequence $\{(\varphi_1)_s\}$. We will have the optimal solution by the following iteration.

1. Choose an control $(\varphi_1)_0\in V$.
2. From the problem (1)-(3) , get the function ψ_s .
3. Using the function ψ_s , get the function η_s .
4. Using ψ_s and η_s , calculate $\tilde{J}_\alpha'(\varphi_1)$ given by (18).
5. Choose an appropriate parameter ε_s such that $\tilde{J}_\alpha\left((\varphi_1)_{s+1}\right)<\tilde{J}_\alpha\left((\varphi_1)_s\right)$.

We will stop the iterations with the criteria of $\tilde{J}_\alpha\left((\varphi_1)_{s+1}\right)-\tilde{J}_\alpha\left((\varphi_1)_s\right)<\sigma$ for small enough σ The parameter ε_s will be shortened until the condition holds for each value of s .

3.3. Numerical simulation

The following numerical tests are performed for two examples.

Example 1 Consider the minimization problem

$$J_\alpha(\varphi_1) = \iint_\Omega \left[\psi(x,t) - \cos 3\pi t \sin \pi x\right]^2 dxdt + \int_0^l \left[\psi_t(x,1) - 0\right]^2 dx + \alpha \left\|\varphi_1 - \kappa\right\|^2_{H^1(0,l)}$$

governed by the following hyperbolic problem defined on the domain $\Omega = (0,1)\times(0,1)$ *;*

$$\psi_{tt} - \psi_{xx} + 5\psi = -\cos 3\pi t \sin \pi x\left(8\pi^2 - 5\right), \quad (x,t)\in\Omega$$

$$\psi(x,0) = \varphi_1(x), \quad \psi_t(x,0) = 0, \quad x\in(0,l)$$
$$\psi(0,t) = 0, \quad \psi(l,t) = 0, \quad t\in(0,T)$$

For this problem, if we take the initial element as $(\varphi_1)_0 = 10$ and $\kappa = 0.1$ *;* we obtain the following optimal controls. Here the stopping criteria is $\tilde{J}_\alpha\left((\varphi_1)_{s+1}\right) - \tilde{J}_\alpha\left((\varphi_1)_s\right) < 0.001$ and the functional $\tilde{J}_\alpha(\varphi_1)$ values are calculated via the control functions.

Table 1. Optimal controls, cost functional values and the norm $\left\|\varphi_1^* - \kappa\right\|^2_{L_2(0,l)}$ for different α values.

α	φ_1^*	$\left\|\varphi_1^* - \kappa\right\|^2_{L_2(0,l)}$	$\tilde{J}_\alpha(\varphi_1)$
0.1	$0.28370637 - 0.30634135\times10^{-8}\sin\pi x$ $+0.19637542\times10^{-19} I \sin\pi x$ $+0.16952105\times10^{-18}\sin 2\pi x$ $+0.28708092\times10^{-28} I \sin 2\pi x$	0.083706	0.021681
0.15	$0.246522803 - 0.17044938\times10^{-8}\sin\pi x$ $-0.33314723\times10^{-18} I \sin\pi x$ $+0.87458425\times10^{-19}\sin 2\pi x$ $+0.47461819\times10^{-29} I \sin 2\pi x$	0.046528	0.030801
0.3	$0.20903746 - 0.65367567\times10^{-9}\sin\pi x$ $-0.51037812\times10^{-22} I \sin\pi x$ $+0.2665707\times10^{-19}\sin 2\pi x$ $-0.20450360\times10^{-28} I \sin 2\pi x$	0.009037	0.060061
0.45	$0.20242657 - 0.38101656\times10^{-9}\sin\pi x$ $-0.33177510\times10^{-22} I \sin\pi x$ $+0.15212996\times10^{-19}\sin 2\pi x$ $+0.77164621\times10^{-29} I \sin 2\pi x$	0.002426	0.090006
0.7	$0.20302737 - 0.13356239\times10^{-9}\sin\pi x$ $+0.30247963\times10^{-22} I \sin\pi x$ $+0.45312092\times10^{-20}\sin 2\pi x$ $+0.55025682\times10^{-29} I \sin 2\pi x$	0.003027	0.140016

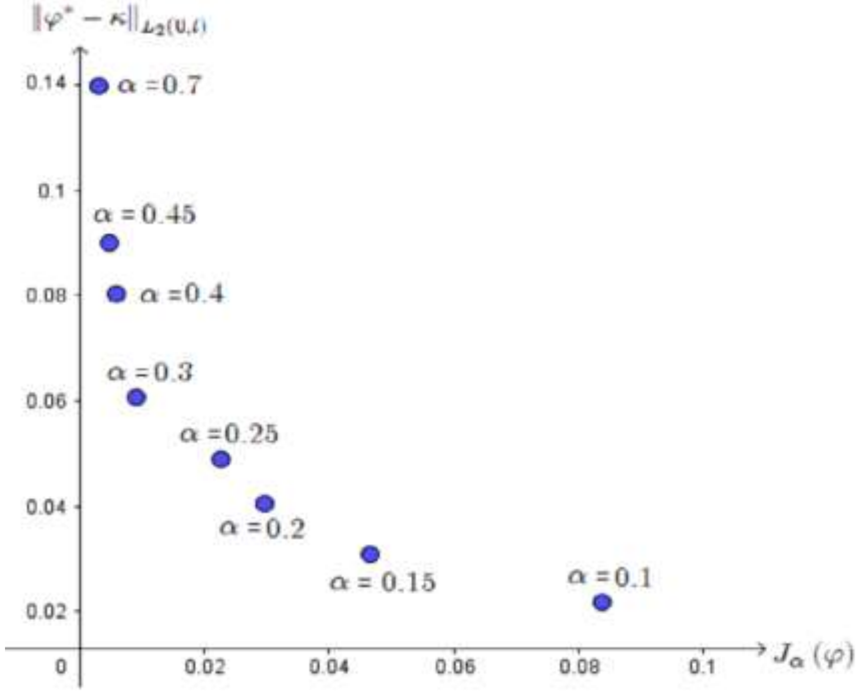

Figure 1. Numerical simulation of the functional $\tilde{J}_\alpha(\varphi_1)$ and the norm $\|\varphi_1^* - \kappa\|^2_{L_2(0,l)}$.

In addition to, we have mentioned from importance of the parameter α above. In this example, if we take $\alpha = 0$, then the norm $\|\varphi_1^* - \kappa\|^2_{L_2(0,l)}$ is calculated as 9.99997. Namely, the parameter α enables to avoid too large controls.

Example 2 On the domain $\Omega = (0,1)\times(0,2)$, let us consider an example where the optimal control problem is governed by

$$\begin{aligned}
&\psi_{tt} - \psi_{xx} + 10\psi = e^{-t}\left(11x^2 - 11x - 2\right), \quad (x,t)\in\Omega \\
&\psi(x,0) = \varphi_1(x), \quad \psi_t(x,0) = -x(x-1), \quad x\in(0,l) \\
&\psi(0,t) = 0, \quad \psi(l,t) = 0, \quad t\in(0,T)
\end{aligned}$$

by the cost functional

$$J_\alpha(\varphi_1) = \iint_\Omega \left[\psi(x,t) - e^{-t}x(x-1)\right]^2 dxdt + \int_0^l \left[\psi_t(x,1) - e^{-1}x(1-x)\right]^2 dx + \alpha\|\varphi_1 - \kappa\|^2_{H^1(0,l)}$$

For this problem, if we take the initial element as $(\varphi_1)_0 = x + 10$ and $\kappa = x - 0.1$;we obtain the following optimal controls by the stopping criteria $\tilde{J}_\alpha\left((\varphi_1)_{s+1}\right) - \tilde{J}_\alpha\left((\varphi_1)_s\right) < 0.001$ and the functional $\tilde{J}_\alpha(\varphi_1)$ values are calculated via the control functions.

Table 1. Optimal controls, cost functional values and the norm $\left\|\varphi_1^* - \kappa\right\|_{L_2(0,l)}^2$ for different α values.

α	φ_1^*	$\left\|\varphi_1^* - \kappa\right\|_{L_2(0,l)}^2$	$\tilde{J}_\alpha(\varphi_1)$
0.1	$x + 0.11817326 - 0.20912775\times10^{-10}\sin\pi x$ $+0.29741453\times10^{-21} I \sin\pi x$ $+0.20367523\times10^{-11}\sin 2\pi x$ $-0.13122581\times10^{-22} I \sin 2\pi x$	0.218173	0.070091
0.2	$x - 0.00490614 - 0.16330348\times10^{-10}\sin\pi x$ $+0.11072268\times10^{-23} I \sin\pi x$ $+0.74076935\times10^{-12}\sin 2\pi x$ $-0.13849588\times10^{-22} I \sin 2\pi x$	0.095093	0.134542
0.5	$x - 0.083452 - 0.861038\times10^{-11}\sin\pi x$ $+0.12492278\times10^{-22} I \sin\pi x$ $+0.19513422\times10^{-12}\sin 2\pi x$ $+0.11463954\times10^{-22} I \sin 2\pi x$	0.016547	0.333043
0.7	$x - 0.093380 - 0.643049\times10^{-11}\sin\pi x$ $-0.235643\times10^{-24} I \sin\pi x$ $+0.130700\times10^{-12}\sin 2\pi x$ $+0.58650264\times10^{-23} I \sin 2\pi x$	0.006619	0.466130
0.9	$x - 0.099586 - 0.54190903\times10^{-11}\sin\pi x$ $-0.848498\times10^{-21} I \sin\pi x$ $+0.75648327\times10^{-13}\sin 2\pi x$ $+0.36167663\times10^{-22} I \sin 2\pi x$	0.000413	0.599274

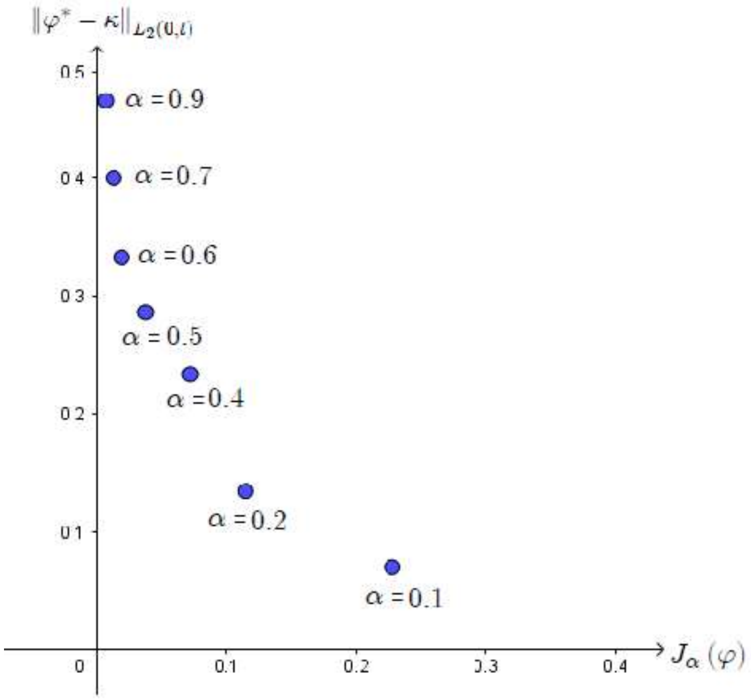

Figure 2. Numerical simulation of the functional $\tilde{J}_\alpha(\varphi_1)$ and the norm $\left\|\varphi_1^* - \kappa\right\|_{L_2(0,l)}^2$.

4. Conclusion

This study presents numerical results about optimal control problem including hyperbolic equation. For this problem, the initial status has been chosen as control function. We obtain an adjoint problem which is helpful for obtaining the derivative of the cost functional. It is obtained the gradient of the functional which is highly important for obtaining an optimal solution on the space $H^1(0,l)$. To find the desired optimal control, we construct an iterative algorithm. Numerical results are tested with two examples. It is easily seen that the method is efficient and helpful.

Authors' Contributions

Seda İğret Araz has performed all theoretical results and numerical calculations with final version of the study as the only author of the paper.

Statement of Conflicts of Interest

There is no conflict of interest between the authors.

Statement of Research and Publication Ethics

The author declares that this study complies with Research and Publication Ethics.

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