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Inverse Modeling Problems and Task Enrichment: Analysis of Two Experiences with Spanish Prospective Teachers

Victor Martinez-Luaces[1], José Antonio Fernández-Plaza and Luis Rico

Department of Didactics of Mathematics, University of Granada, Spain

ABSTRACT

Problem solving and problem posing are both important topics in mathematics education. Since in many branches of science and technology, typical problems are posed in an inverse form, we will focus on inverse problems that require modeling skills in order to be solved, i.e., the so-called inverse modeling problems. In this article, we will analyze them from the view point of their potential for task enrichment. For this purpose, a research project was carried out, by using inverse modeling problems to develop prospective teacher`s task enrichment skills. The results of this experience, that took place in 2017, showed that only few participants were very creative, whereas many others posed trivial problems or simply imitated examples previously analyzed. After that, a new research essay was implemented during the first months of 2019, with the aim of avoiding – or at least attenuating – those difficulties observed in the previous field work. The new results showed some few similarities and very interesting differences, when compared with the other experience. In this article, we comment our findings and some conclusions are reported.

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1. Introduction

Educational research should prepare prospective teachers in order to promote their competence and their favorable attitudes towards sciences and mathematics, improving the richness and creativity of the problems and tasks that link them. One of the challenges consists in developing prospective teachers' task enrichment skills (Martinez-Luaces, Rico, Ruiz-Hidalgo & Fernandez-Plaza, 2018) and for this purpose, inverse problems (Groestch, 1999, 2001) are relevant since in many branches of other sciences and technology, typical problems are posed in an inverse form. In previous works, when modeling skills were combined with inverse problems, we called them *inverse modeling problems* (Martinez-Luaces, 2013, 2016).

In this article we focus on their posing competence for task enrichment purposes. We describe the research carried out during the last four years in our work with prospective teachers at the University of Granada, Spain (UGR). Some of our most recent findings are reported and discussed in the following sections. For this purpose, we focus on a couple of creative proposals and their corresponding didactic analysis from the viewpoint of the participants.

2. Theoretical framework

In our research relatively simple problems – where only fundamental concepts of calculus, linear algebra and geometry are necessary to solve them – are proposed to prospective teachers. Our main purpose consists in analyzing easy problems, susceptible of being reformulated in the form of an inverse problem by the participants.

[1]Corresponding author's address: Facultad de Ciencias de la Educación. Campus Universitario de Cartuja C.P 18071 Granada 18071. Spain.
e-mail: victorml@correo.ugr.es

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Prospective teachers' competence in their future reformulations is expected to be richer than the original statements in accordance with Lester and Cai (2016), who observed "…teachers can develop worthwhile mathematical tasks by simply modifying problems from the textbooks" (p. 124).

The latter links our work with a traditional area of research in mathematics education, known as *problem posing*, which is the first subsection of our theoretical framework.

2.1. Problem Posing

There is a long tradition in the literature in English regarding problem-solving research. In this sense the work of Brown and Walter (2005, 2014) and Kilpatrick (1987), among others, deserve to be mentioned. In their works about problem posing, these authors analyze the competence for the formulation of new problems and the reformulation of problems previously proposed (Silver, 1994, 1997; English, 1997; Silver & Cai, 1996).

A particular case happens when students pose a new problem during the resolution of one of greater complexity (Silver, Mamona-Downs, Leung & Kenney, 1996). This is a situation that appears in the work of Polya (1957), that proposes as a possible strategy: the approach of the problem in a different way or the establishment of variants, discarding some of its conditions.

In other works, the formulation of problems does not have to be linked to the resolution of a specific problem, like in some cases where the invention of problems is proposed starting from a certain situation or experience (Silver, 1994, 1997).

Another interesting option consists in combining the two previous approaches and asks the students to give a solution after changing a condition or the final question of the problem. Obviously this procedure creates a new problem to be solved (Silver, 1994).

Other researchers such as Brown and Walter (2005, 2014), propose another strategy that they call "What if not?", which consists in changing conditions and/or restrictions of a given problem, obtaining a new one.

Stoyanova (1998) identifies three possible structured situations to apply strategies for the formulation and invention of new problems: free situations, semi-structured and structured situations. In the first one, there are no restrictions on the invention of problems. In the second one, the problem is proposed based on any experience or quantitative information. Lastly, in the third one, a certain given problem is reformulated or some of its conditions are changed.

In our research in the UGR the participants are given a direct modeling problem, which should be reformulated in the form of an inverse one and then, this can be considered as a structured situation, following the previous Stoyanova (1998) classification.

2.2. Inverse Problems

According to Groestch (1999, 2001), traditional mathematics courses are usually dominated by routine exercises that require memory skills and, at best, appear mixed with a few direct problems. However, real life problems are mostly inverse problems, indeed, the opposite of the exercises found in conventional courses.

Moreover, inverse problems are usually more challenging and interesting than the corresponding direct ones, largely due to either multiple solutions or no solution (Bunge, 2006). For instance, if n and m are prime numbers, where $n \neq 2$ and $m \neq 2$, it follows that $s = n + m \geq 6$ is an even number, whereas the inverse problem of the decomposition of an even number to a sum of a couple of prime numbers is extremely difficult. Furthermore, this problem is related to the even Goldbach conjecture, one of the oldest and best-known unsolved problems in number theory and in all mathematics (Oliveira e Silva, Herzog & Pardi, 2014).

In mathematics education the situation is completely different, as inverse problems are almost ignored. Moreover, though inverse problems require modeling skills in order to be solved –

mathematically enriching them – their presence in school mathematics courses and textbooks is rare and similarly, hardly seen in assessment tasks.

For all these reasons our research was oriented towards task enrichment by using inverse modeling problems and several of our recent papers (Martinez-Luaces, Fernández-Plaza, Rico & Ruiz-Hidalgo, 2019) and book chapters (Martinez-Luaces, Rico, Ruiz-Hidalgo & Fernández-Plaza, 2018; Martinez-Luaces, Fernández-Plaza & Rico, 2019) addressed this topic.

Groestch (1999, 2001) stated that direct problems are those that provide the required information in order to follow a well-defined stable procedure leading to a single correct solution. Based on that idea, the process of solving a direct problem can be schematized as in Figure 1.

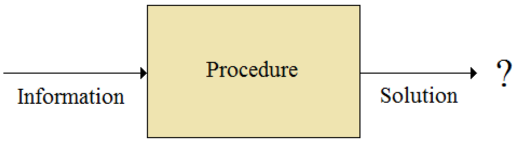

Figure 1. Scheme for direct problems

There are two types of inverse problems; firstly, the causation problem, where the procedure is well-known and the question is about the required data to obtain a given result. This situation is schematized in Figure 2.

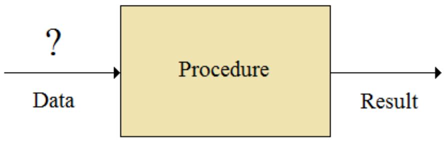

Figure 2. Scheme for inverse causation problems

The other possible inverse problem is the specification problem. In this new case data and result are given and the question is concerned with which procedure can let reach the desired result (output) with the chosen data (input). This process is schematized in Figure 3.

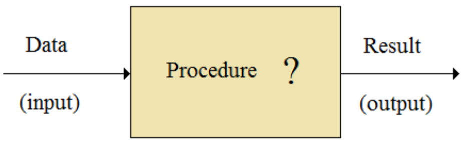

Figure 3. Scheme for inverse specification problems

Both kinds of inverse problems are common in experimental sciences and real life situations, as noted in previous research (Martinez-Luaces, 2013, 2016; Martinez-Luaces, Rico, Ruiz-Hidalgo & Fernández-Plaza, 2018; Martinez-Luaces, Fernández-Plaza & Rico, 2019; Martinez-Luaces, Fernández-Plaza, Rico & Ruiz-Hidalgo, 2019).

2.3. Mathematical Modeling

As it was observed in the preliminary discussion document to the ICMI (International Commission on Mathematical Instruction) Study 14 (Blum, 2002), the term “modeling” focuses on the competence to select data and abstract real-world structures with which to obtain a mathematical expression for them; on the contrary, the term "application" refers to the competence that develops mathematical structures in the opposite direction and interprets them in real-world sectors (Figure 4).

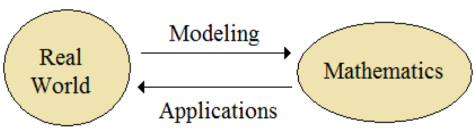

Figure 4. Comparison scheme between modeling and application problems

A more extended discussion about modeling and applications problems and competences can be found in our previous research (Martinez-Luaces, 2013, 2016; Martinez-Luaces, Rico, Ruiz-Hidalgo & Fernández-Plaza, 2018).

2.4. Didactic Analysis

The teacher professional knowledge helps to establish the meaning of the school concepts, in order to implement and to evaluate them, following a method that Rico and colleagues called Didactic Analysis (Rico, Lupiañez & Molina, 2013; Rico & Moreno, 2016).

The didactic analysis is structured according to four kinds of analysis, focused on different purposes as study objects, regarding the dimensions of the mathematics curriculum. Firstly, it begins with the analysis of the meaning for the mathematical content, where the teacher identifies, selects and organizes concepts and procedures suitable for instruction. Our notion of meaning is based on Frege's semiotic triangle, as considered by Rico and Moreno (2016) – conceptual structure, representational systems and modes of use –. Secondly, it performs the cognitive analysis of the content, which determines the intentionality, commitments and conditions for the achievement of the corresponding learning. For this purpose, expectations are expressed, limitations are analyzed and the selection of the tasks to be performed is organized. Next, the instructional analysis considers the tasks, the organization and resources necessary for the teaching of the mathematical content. Thus, the teacher designs and proposes task sequences, selects materials and delimits classroom management. In the evaluation analysis, the lessons learned are assessed, the information is collected and finally, the teacher makes educational decisions.

The previous description gives rise to a cyclical structure, where the information obtained in a given analysis will be essential for a new implementation of the didactic analysis.

The descriptors utilized in the didactic analysis include: previous knowledge, mathematical content activated by the task, challenge, task completion, event, question, purpose, language, data, goal, formulation, materials and resources, grouping, learning situation, timing, mathematical content, situation and complexity.

The didactic analysis descriptors are listed and briefly explained in Table 1.

Table 1. Descriptors of the didactic analysis and explanation

Descriptors	Brief explanation of the descriptor
Prior knowledge	It refers to the content that the students already know and the task is based on these concepts and procedures that the students already possess.
Mathematical content activated by the task	Concepts and procedures that the teacher wants to develop through the task work.
Challenge	This item asks if the task can be considered a challenge for the students and if they are interested in it or not.
Task completion: recognition / justification	Students should be able to recognize if the task has been done successfully and also if their response is accurate, providing explanations to decide if the given response completes the task or not.
Event	The task refers to an event that happened before, or if it has a real chance of happening.
Question	The question of the task can be considered as consistent with the expected question in the real life.
Purpose	The purpose of the task is consistent with the one that could be proposed in a real life situation.
Language	The language in which the task is expressed is appropriate.
Data	The given data are realistic.
Goal	This is about the learning expectation developed by the task.
Formulation	This item considers the way in which the task is presented (written text, oral, video, or other formats)
Materials and resources	This is about the materials and resources needed to complete the task.
Grouping	About the ways of organizing the students when the working on the given task.
Learning situation	The place or the physical situation where the task is carried out.
Timing	The timing for the work to be done in order to complete the task.
Mathematical content	Quantity, space and form, uncertainty and data, change and relationships.
Situation	Personal, educational / labor, public, scientific.
Complexity	Reproduction, connection and reflection.

3. Our previous experiences at UGR

In the University of Granada our research was designed to work with the students of two groups of the course named "Learning and Teaching Mathematics in Secondary School", which is part of the Master's Degree in Teaching Secondary Education (Martinez-Luaces, Rico, Ruiz-Hidalgo & Fernández-Plaza, 2018).

In the 2016-2017 academic year at the UGR, the first group consisted of 33 students and 41 students formed the second group, with regular class attendance. In our research, we had the collaboration of two of the master courses university teacher trainers.

In a first class about task enrichment and problem posing, the prospective teachers of both groups worked on a problem about the filling of a swimming pool. In the first session this direct problem was proposed and future professors were asked to reformulate it, as a task enrichment proposal to be used with secondary school students.

The productions of the prospective teachers underwent a first analysis, and among them, three reformulations were highlighted and selected since they had been posed spontaneously in an inverse form.

Then, in a second work session, showing these reformulations, they were given by trainers a brief explanation about direct and inverse problems. After that, prospective teachers were proposed a new direct problem (the sheep problem) and they were asked to reformulate it in an inverse manner for task enrichment purposes.

The prospective teachers' most creative productions were analyzed in a previous book chapter (Martinez-Luaces, Rico, Ruiz-Hidalgo & Fernández-Plaza, 2018). It was observed that several participants were particularly creative in their proposals; nevertheless, the vast majority opted for a standard approach and in some cases, for a trivialization of the problem.

Despite our research is not focused on prospective teachers, since our main objective is to analyze the productions from the task enrichment point of view, it can be observed that most of the subjects in our field work had difficulties to propose inverse modeling reformulations. It can be mentioned that more than 50 % of the productions in 2017 were ill posed problems, or trivialized proposals, or they simply inverted the given function in a classic way. Moreover, only 20 % of the productions were considered as creative proposals. These results confirmed that teacher training courses really need to pay attention to these topics (modeling and inverse problems) as particular ways for developing their task enrichment skills.

For these reasons, a new research essay was proposed, with the aim of avoiding – or at least attenuating – to the prospective teachers those difficulties observed in the previous field work. In particular, both classroom experiences had three main differences:

- Prospective teachers were asked to solve the original direct problem, before proposing their inverse reformulations.
- Before proposing to them this new task, several examples about inverse problems were discussed. However, none of them were about the sheep problem. The main reason for this decision was to avoid simple imitation or adaptation of a given model.
- Prospective teachers were asked to solve their own reformulated problem – or at least write a sketch of the solution – with the aim of reducing the number of non-well-posed problems

The new results showed interesting some important differences and few similarities which are analyzed in next sections of this article.

4. Results of the new fieldwork

In this section we first consider the original version of the sheep problem, posed in a direct form. After that, we present two of the most creative reformulations proposed by the prospective teachers as inverse modeling problems.

4.1. The Sheep Problem

In this direct problem, a sheep is grazing in a square field with side length L and it is tied at the point $(L/2\,,\,0)$ with a rope that has a length R as shown in Figure 5.

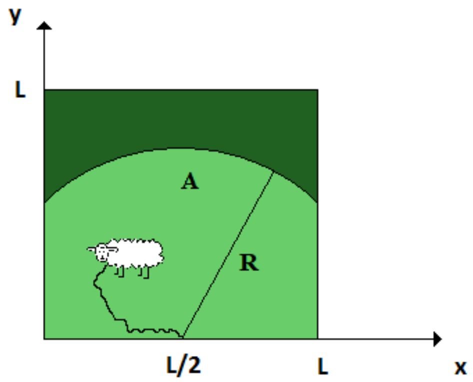

Figure 5. Part of the field accessible for the sheep

As it can be observed, A represents the area of the sector where the sheep may graze. Two dimensionless numbers can be defined: $r = \frac{R}{L}$, the ratio of the rope length to field side length and $f = \frac{A}{L^2}$, the fraction of the total area accessible for the sheep.

Obviously, f is a function of the ratio r, that can be obtained by integration techniques. Then, the typical direct problem consists in supplying students with this figure and asking them to obtain f corresponding to one or more values of r.

4.2. An Unusual Specification Problem

4.2.1. The prospective teacher No. 1's proposal

One prospective teacher observed that the sheep problem can be modeled by using a circumference, which equation can be written as: $\left(\frac{L}{2}-x\right)^2 + y^2 = R^2$ and then, the area accessible for the sheep is the following integral: $A = \int_0^L \sqrt{R^2 - \left(\frac{L}{2}-x\right)^2}\,dx$. After that, by using the change of variables $x = \frac{L}{2} - R\sin t$, he writes the integral as: $A = -R^2 \int_{\arcsin(L/2R)}^{\arcsin(-L/2R)} \sqrt{1-\sin^2(t)}\cdot\cos(t)dt$ (Taneja, 2010). The integrand is $\cos^2(t) = \frac{1+\cos(2t)}{2}$, and so: $A = -\frac{R^2}{2}\left(t + \frac{1}{2}\sin(2t)\right)\Big|_{\arcsin(L/2R)}^{\arcsin(-L/2R)}$. Then he arrives to the following formula as the solution for the direct problem:

$$A = -\frac{R^2}{2}\left[\arcsin\left(\frac{-L}{2R}\right) - \arcsin\left(\frac{L}{2R}\right) + \frac{1}{2}\left(\sin\left[2\arcsin\left(\frac{-L}{2R}\right)\right] - \sin\left[2\arcsin\left(\frac{L}{2R}\right)\right]\right)\right]$$

Obviously, this formula can be simplified, but the prospective teacher leaves it in the long version, as showed above.

After this classical solution, the prospective teacher obtains particular results for $R=\frac{L}{2}$ and $R=\frac{L}{\sqrt{2}}$, i.e. $r=\frac{1}{2}$ and $r=\frac{1}{\sqrt{2}}$, getting $f=\frac{\pi}{8}$ and $f=\frac{\pi}{8}+\frac{1}{4}$, respectively.

In his inverse reformulation, he proposes to get the solution in a geometrical way and compare the final result with the one previously obtained by integration.

This proposal can be considered a specification problem, since data and final result are known and he asks for another procedure in order to get the desired result.

When the prospective teacher solves his own reformulation, he decides to divide the area accessible for the sheep into three parts: a circular sector and two triangles, as it can be observed in Figure 6.

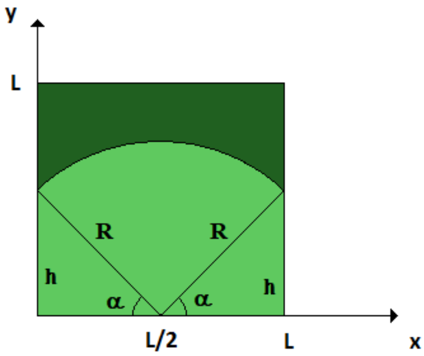

Figure 6. The accessible area divided into three parts

Finally, the accessible area can be written as: $A=\frac{L}{2}\sqrt{R^2-\frac{L^2}{4}}+\pi R^2\frac{\pi-2\arctan\left(\frac{2}{L}\sqrt{R^2-\frac{L^2}{4}}\right)}{2\pi}$.

The prospective teacher shows that both formulas give the same results for the particular cases previously considered: $R=\frac{L}{2}$ and $R=\frac{L}{\sqrt{2}}$. The participant ends his work observing that "as it was expected, both methods gave the same results".

It is important to mention that in the previous experience, carried out in 2017, all the reformulations were proposed as causation inverse problems. None of prospective teachers proposed an inverse specification problem like this creative production.

4.2.1. The prospective teacher No. 1's didactic analysis

This prospective teacher considers that both tasks are significant since they help students to express their ideas and improve their mathematical knowledge and skills, that is to say their competence. Regarding prior knowledge, he thinks that there exists a big difference between both proposals: in the first problem the student is asked to apply the integral calculus, while in the second the main purpose is to use geometric concepts to calculate the area, and establish a relationship between both methodologies. Therefore, the second task, also including geometric procedures, puts in practice more mathematical contents relations among them. The future professor considers that both tasks are challenging, but in a different way. In the first one this is due to the type of integral needed and its relationship with a realistic statement. In the second one, the problem has the same components as the direct one and it also includes a challenging geometric procedure which result is equivalent. In his opinion, checking that the result is the same makes it more interesting for the students. The future teacher says that in both tasks students can find a justification for the obtained result, which must be bounded between certain values with a certain geometric meaning. As a consequence, since the second task includes both procedures and obtains twice the same result, its justification is much better.

The prospective teacher says that both tasks can be considered authentic because they can be reproduced in reality and he adds "I have not found a relatively simple way to bring more authenticity to the problem". He founds that the event is realistic because it is an animal tied, that can access a certain area. About the questions, he says that they are realistic but do not happen in everyday life, however, he recognizes that in some scientific contexts about calculation of areas, similar questions may be presented, preparing students that can relate these tasks with the problems considered here. The purpose is similar in both problems, but he remarks that there is an important change: while the first problem is about an application of the integral calculus, the second one explores the relation between the integral and geometry, comparing both procedures and observing the same results. He mentions that language and data are the same in both cases, presenting both a verbal statement and a scheme which is required to understand the problem. He also remarks that in order to solve the second problem in a geometric way, there is a slight change in the representation. In fact, in the inverse problem the square can be considered as an isolated figure without including the coordinate axes, while in the case of the integral the axes must be considered in order to calculate the integral.

The prospective teacher notes that there are several changes between the first and the second task in what concern with the elements that form part of the tasks. He repeats his observation about the tasks goals: integral calculation in the first one and analyzing the relationship between two methods in the second. As a consequence, the student is expected to observe that there are at least two procedures to arrive at the same solution, depending on how the problem is modeled. Regarding the formulation, he observes once more that both texts are presented with a verbal statement and graphic representation, being this graphic representation essential to understand the task. He comments that in the formulation of the inverse problem only the statement is modified, by asking for a new procedure and a comparison of results, maintaining the scheme. He observes that materials and resources are the same (pen and paper in both cases), and he does not consider that at this level the use of other resources may be important. In his opinion, this is an individual task in both cases, to be done in the classroom. However, he recommends an important change in the timing, since the second task demands more work than the first one. In the first case, the requested integral is not easy and he recommends giving a time of about 20 or 25 minutes to complete its calculation. In the second, a geometric method and the comparison between both procedures are also requested and then, the time to work should be extended about 40 or 45 minutes.

The prospective teacher reflects that several changes occur in relation to the task variables, when comparing the tasks corresponding to the direct and the inverse problem. Regarding the mathematical content, both tasks use concepts of integral calculus and geometry, although in the second one the emphasis is placed on the relationship between them. Moreover, the student is asked to use both methods and compare their results. In the opinion of the prospective teacher both tasks correspond to an educational situation. Regarding the complexity of the task, the participant says that it can be classified as a connection task, since the students need to connect their knowledge of integral calculus with the application to calculation of areas. For the second task he considers that it corresponds to a level of reflection, since when solving the same problem by using two different approaches (the integral calculation and the geometric solution), there should be a reflection on whether and why the same results should be obtained or not.

4.3. An Arc Length Inverse Problem

4.3.1. The prospective teacher No. 2's proposal

Another prospective teacher solved the direct problem by using integrals, putting the area accessible for the sheep as: $A=\int_0^L\sqrt{R^2-\left(\frac{L}{2}-x\right)^2}\,dx$. In this case he uses a slightly different change of variables $x=\frac{L}{2}+R\sin t$ and he applies this change of variables to obtain the primitive function, not the definite integral as in the previous example. He arrives to the indefinite integral $R^2\int \cos^2(t)dt$ and then $A = R^2\left(\frac{\cos(t)\sin(t)+t}{2}\right)\Big|_0^L$, finally obtaining that $A = R^2\left(\frac{L}{2R}\sqrt{1-\left(\frac{L}{2R}\right)^2}+\arcsin\left(\frac{L}{2R}\right)\right)$ (Taneja, 2010).

The prospective teacher completes this result including the case $0 \le R \le \frac{L}{2}$, arriving to the following piecewise formula: $A = \begin{cases} \frac{1}{2}\pi R^2 & \text{if} \quad 0 \le R \le \frac{L}{2} \\ R^2\left(\frac{L}{2R}\sqrt{1-\left(\frac{L}{2R}\right)^2}+\arcsin\left(\frac{L}{2R}\right)\right) & \text{if} \quad \frac{L}{2} \le R \le L \end{cases}$.

So, this prospective teacher considers two different situations, depending on the comparison between R and $\frac{L}{2}$. As it can be observed, his solution of the direct problem does not consider other radius greater than L. It can be mentioned that this difficulty was observed in almost all the productions, since only a few prospective teachers considered all the cases that take place in the piecewise area function.

In his inverse reformulation he gives this piecewise function as part of the data and he informs that the shepherd decides to eliminate the rope and instead of it, he wants to build a circular fence like in Figure 5, i.e., the same as in the original problem. The cost of the fence is given (15 €/m) and question is about the final cost as a function of variable R.

It is interesting to note that the problem could be solved in a direct way, by using the arc length formula, then calculating the corresponding integral and lastly multiplying by the fence cost per meter. However, this solution does not use the given area function, which is the input of the inverse problem, so it cannot be considered as the solution required.

The participant solves his own problem by differentiating the given function, since he claims that $L = \frac{dA}{dR}$ – without any demonstration or justification – and finally, after obtaining $L(R)$ by differentiation, the price is easily obtained multiplying by the cost per meter.

It should be noted that he statement that makes possible this solution, i.e., $L = \frac{dA}{dR}$, is not true for every region in $\mathbf{R}^2$. Nevertheless, the result is correct in this case, since the region is composed of a circular sector and two triangles. This is just a particular case of the general conditions obtained by Dorff and Hall (2003), that gives some criteria in order to know when the derivative of the area gives the perimeter, and when the derivative of the volume of a solid gives the surface area.

As it was mentioned before, the prospective teacher does not give any explanation about this result. It would be easy in this particular case to proof the correctness of the procedure, which is obvious in the first case $\left(0 \le R \le \frac{L}{2}\right)$, since: $L(R) = \frac{dA}{dR} = \frac{d}{dR}\left(\frac{1}{2}\pi R^2\right) = \pi R$. In the second case $\left(\frac{L}{2} \le R \le L\right)$ the proof it is not so easy and it is better to do it directly from the integral formula: $L = \frac{dA}{dR} = \frac{d}{dR}\int_0^L \sqrt{R^2-\left(\frac{L}{2}-x\right)^2}\,dx$, since differentiation under the integral (Taneja, 2010) gives:

$$L = \int_0^L \frac{d}{dR}\sqrt{R^2-\left(\frac{L}{2}-x\right)^2}\,dx = \int_0^L R\Big/\sqrt{R^2-\left(\frac{L}{2}-x\right)^2}\,dx \ .$$

On the other hand, the circumference equation can be written as: $\left(x-\frac{L}{2}\right)^2+y^2=R^2$ and then, the arc length is given by the following integral: $L=\int_0^L\sqrt{1+(f'(x))^2}dx$ (Taneja, 2010), where $f(x)=\sqrt{R^2-\left(x-\frac{L}{2}\right)^2}$, being its derivative $f'(x)=\left(\frac{L}{2}-x\right)\Big/\sqrt{R^2-\left(x-\frac{L}{2}\right)^2}$, so it follows that $1+(f'(x))^2=(R^2)\Big/R^2-\left(x-\frac{L}{2}\right)^2$. Finally the arc length integral is: $L=\int_0^L\sqrt{1+(f'(x))^2}dx$, which can be written as: $L=\int_0^L R\Big/\sqrt{R^2-\left(\frac{L}{2}-x\right)^2}\,dx$, i.e., giving the same result previously obtained and then, proving the correctness of the procedure in this case.

As it was mentioned before, the prospective teacher does not include this proof or any other justification of the procedure he followed. Nevertheless, the result is correct, but this kind of solution cannot be considered as a general one. Although, it works for this particular situation and the proposed procedure is simple, short and elegant.

In our previous experience, in 2017, only one prospective teacher proposed a reformulation involving arc length – probably due to the difficulties in the integral calculations – and it appeared in a problem where the accessible area was not an input. Then, this proposal should be considered as a different kind of problem. In fact, the arc length is the unique weak connection between both reformulations of the original direct problem.

4.3.2. The prospective teacher No. 2's didactic analysis

This other prospective teacher says that both statements are proposed for students of the last year of high school since knowledge about derivation and integration and their relationship with areas, and the arc length of circumference obtained by derivation are needed. Then, both statements are equally significant: in the first one the purpose consists in obtaining an area function by integration, while in the second the idea is to find the arc length by derivation. In his opinion, both of them suppose a great challenge for the students.

In what concerns to authenticity, the future teacher thinks that both the direct and the inverse proposals need some improvement. In his opinion, although the events described can occur, both the question and the purpose of the problem do not justify this context. In his opinion, a scientific context such as a bacteria culture that grows in a circular form starting from an inoculation point in a cell culture flask could be more suitable for proposing more authentic problems.

About the elements that form part of the task and the task variables, he thinks that they are the same for both the direct and the inverse problem.

5. Conclusions

A first conclusion of the fieldwork is that the results of both experiences – carried out in 2017 and 2019 – are absolutely different. Even within the same experience (2019), the prospective teachers show very different opinions in the two cases studied in the previous section. Indeed, one of them does not see significant changes between the two proposals, whereas the other sees in the inverse problem a more challenging task, with greater complexity, involving other mathematical knowledge and requiring much more time to be completed.

In 2017, the prospective teachers imitated previous examples provided for the first problem (the filling of a swimming pool) and then followed the same ideas for the sheep problem. In fact, most of the proposals – like in the swimming pool problem – inverted the function, changed the geometry, or included obstacles, among other ideas.

In 2019 the introductory examples were very simple and concerned other mathematics topics like proportions, arithmetic and geometric sequences, etc., and the prospective teachers were asked to solve the direct problem before proposing their own reformulation. These two facts led the future teachers' proposals in many different ways. For instance, one of them gave a formula as an input for the reformulated problem, and asked for another way to get the same result without using integrals. Other reformulations also gave a formula and asked for an interpretation of the parameters or a sketch of the corresponding region.

Another important difference was about the use of external variables (physical, chemical, economical or biological). Those kinds of variables were widely used in 2017; however, in 2019 they only appear in a few cases, like in the arc length example, where the cost of the fence per meter is given.

Although our research is focused on the proposals, not in the prospective teachers, it should be mentioned that in 2019, the productions that only inverted the function, plus the trivial and the ill-posed ones were less than 50 %, improving the results of year 2017. Moreover, the creative proposals in 2019 increased to more than 34 %, which represents 70 % more than those of 2017. These results suggest that the new experimental design (see Section 3) attenuated several difficulties observed two years before, as it was expected.

It can be observed that the proposals corresponding to 2019 are usually more challenging from a pure mathematical viewpoint and they ask for more conceptual issues. In fact, as it was mentioned, some proposals ask for an identification of a region, or give a meaning to one or more given variables in a certain formula, among other options that did not appear in 2017. Besides, the proposals in year 2017 were more practical, i.e., hands-on problems more involved with other disciplines and more connected with the reality and its mathematical modeling.

As a general conclusion it seems that the prospective teachers tend to propose the reformulations based on their own recent experiences. If they work with some previous examples, then they try to imitate them. If their experience consists mainly in solving the direct problem, they tend to use the solution as the main input for the problem posing.

Taking into account the fieldwork results, it is impossible to say that one of these experiences yielded better results than the other in terms of the participants posing competence for task enrichment purposes, which was the main purpose of the study. Indeed, in the first one, certain characteristics predominated, whereas in the second one, other different characteristics were observed. As a final comment, the resulting proposals in both experiences, more than antagonistic can be regarded as truly complementary.

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References

Blum, W. (2002). ICMI Study 14: applications and modelling in mathematics education – discussion document. *Educational Studies in Mathematics, 51 (1-2)*, 149-171.

Brown, S. I. & Walter, M. I. (2005). *The art of problem posing. 3rd ed*. Hove, UK: Psychology Press.

Brown S. I. & Walter, M. I. (2014). *Problem posing. Reflections and Applications*. Hove, UK: Psychology Press.

Bunge, M. (2006). *Problemas directos e inversos*. Available from: http://grupobunge.wordpress.com/2006/07/20/119.

Dorff, M. & Hall, L. (2003). Solids in $\mathbf{R}^n$ whose Area is the Derivative of the Volume, *The College Mathematics Journal*, *34 (5)*, 350-358.

English, L. (1997). The development of fifth-grade children's problem-posing abilities. *Educational Studies in Mathematics*, *34*, 183-217.

Groetsch, C. W. (1999). *Inverse problems: activities for undergraduates*, Washington, DC: Mathematical Association of America.

Groetsch, C. W. (2001). Teaching-Inverse problems: The other two-thirds of the story. *Quaestiones Mathematicae*, *24 (1)* Supplement, 89-94.

Kilpatrick, J. (1987). Problem formulating: Where do good problems come from? In: A. Shoenfeld (Ed.) *Cognitive science and mathematics education* (pp. 123-148), New Jersey: Lawrence Erlbaum Associates.

Lester, F. & Cai, J. (2016). Can mathematical problem solving be taught? Preliminary answers from 30 years of research. In: P. Felmer, E. Pehkonen & J. Kilpatrick (Eds.), *Posing and solving mathematical problems* (pp. 117-135), Cham: Springer International Publishing.

Martinez-Luaces, V. (2013). Inverse modelling problems in linear algebra undergraduate courses. *International Journal of Mathematical Education in Science and Technology*, *44 (7)*, 1056-1064.

Martinez-Luaces, V. (2016) Inverse Modeling Problems and their Potential in Mathematics Education. In: M. Vargas (Ed.), *Teaching and Learning: Principles, Approaches and Impact Assessment* (pp. 151-185). New York: Nova Science Publishers.

Martinez-Luaces, V., Fernández-Plaza, J.A. & Rico, L. (2019). Inverse Modeling Problems in Task Enrichment for STEM Courses [Online First], London, UK: IntechOpen. Available from: https://www.intechopen.com/online-first/inverse-modeling-problems-in-task-enrichment-for-stem-courses

Martinez-Luaces, V., Fernández-Plaza, J.A., Rico, L. & Ruiz-Hidalgo, J.F. (2019). Inverse reformulations of a modelling problem proposed by prospective teachers in Spain, *International Journal of Mathematical Education in Science and Technology*, DOI: 10.1080/0020739X.2019.1683773. Available from: https://www.tandfonline.com/doi/full/10.1080/0020739X.2019.1683773

Martinez-Luaces, V., Rico, L., Ruiz-Hidalgo, J. F. & Fernández-Plaza, J. A. (2018). Inverse Modeling Problems and Task Enrichment in Teacher Training Courses. In: R.V. Nata (Ed.), *Progress in Education*, *53*. Pp. 185-214. New York: Nova Science Publishers.

Oliveira e Silva, T., Herzog, S. & Pardi, S. (2014). Empirical verification of the even Goldbach conjecture and computation of prime gaps up to $4 \cdot 10^{18}$. *Mathematics of Computation, 83.288*, 2033-2060.

Polya, G. (1957). *How to Solve it? A new Aspect of Mathematical Method*. *2nd edition*. Princeton, NJ: Princeton Science Library.

Rico, L., Lupiañez, J. L., & Molina, M. (Eds.) (2013). *Análisis Didáctico en Educación Matemática: metodología de investigación, formación de profesores e innovación curricular*. Granada: Editorial Comares.

Rico, L. & Moreno, A. (Eds.) (2016*). Elementos de didáctica de la matemática para el profesor de secundaria*. Madrid: Editorial Pirámide.

Silver, E. A. (1994). On Mathematical Problem Posing. *For the Learning of Mathematics*, *14(1)*, 19-28.

Silver, E. A. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. *ZDM - Zentrallblatt fur Didaktik der Mathematik*, *29(3)*, 75-80.

Silver, E. & Cai, J. (1996). An analysis of arithmetic problem posing by middle school students. *Journal for Research in Mathematics Education*, *27(5)*, 521-539.

Silver, E., Mamona-Downs, J., Leung, S. & Kenney, P. (1996) Posing mathematical problem: An exploratory study. *Journal for research in mathematics education*, *27(3)*, 293-309.

Stoyanova, E. (1998). Problem posing in mathematics classrooms. In A. McIntosh & N. Ellerton (Eds.), *Research in Mathematics Education: a contemporary perspective* (pp. 164-185), Perth, WA: MASTEC, Mathematics, Science and Technology Education Centre, Edith Cowan University.

Taneja, H. C. (2010). *Advanced engineering mathematics, vol. 2.* New Delhi: I.K. International Publishing House.