Araştırma Makalesi / Research Article

A Note on Finsler Version of Ambrose Theorem

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Abstract
Let $(\aleph, F)$ be a forward complete and connected Finsler manifold of dimensional $n \geq 2$. In this study, we extend Ambrose's compactness theorem in Riemannian manifolds to Finsler manifolds by using the weighted Ricci curvature. We use the Bochner Weitzenböck formula and suitable sequence choices for the proofs of the desired results.

Keywords: Finsler manifold, S-curvature, weighted Ricci curvature.

Ambrose Teoreminin Finsler Versiyonu Üzerine Bir Not

Öz
$(\aleph, F)$ manifoldu forward tam, bağlantılı ve $n \geq 2$ boyutlu bir Finsler manifold olsun. Bu çalışmada, Riemann manifoldlarında elde edilen Ambrose kompaktlık teoremi, ağırlıklı Ricci eğriliği kullanılarak Finsler manifoldlara genişletilmiştir. İstenilen sonuçların kanıtları için Bochner Weitzenböck formülü ve uygun dizi seçimleri kullanılmıştır.

Anahtar kelimeler: Finsler manifold, S-eğriliği, ağırlıklı Ricci eğriliği.

1. Introduction

Finsler geometry includes analogues for many of the natural objects in Riemannian geometry. The recent works have shown that some well-known results in Riemannian geometry have been extended to the Finsler setting. For examples in this scope, the reader is referred to [1-3] and references therein.
In the Riemannian case, Ambrose [4] proved a compactness theorem by a condition on the integral of Ricci tensor along geodesics. Later, this theorem was generalized to the Finsler manifolds by Anastasiei (see, [5]). Besides, Kim [6] has established the Finsler version of Galloway's compactness theorem [7] by using Ricci scalar.

In this work, we will obtain the corresponding Zhang's theorem (see Theorem 1.4 in [8]) and Cavalcante-Oliveira-Santos's theorem (see Theorem 2.1 in [9]), which are the generalizations of Ambrose compactness theorem, for the weighted Ricci curvature $\mathrm{R}ic_\infty$ and $\mathrm{R}ic_N$ on Finsler manifolds, respectively. In particular, we will use Zhang's approach to give proofs of the following results.

Theorem 1. *Let $\aleph$ be a forward complete and connected n-dimensional Finsler manifold. Assume every geodesic $\gamma(t)$ issuing from point $p \in \aleph$ satisfies*

$$\int_0^\infty \mathrm{R}ic_\infty(\gamma'(t))dt = \infty, \tag{1}$$

and the S-curvature $|S(\gamma'(t))| \leq H/r$, where $r(x) = d(x,p)$ is the distance function from $p \in \aleph$, then manifold is compact.

Next we obtain a compactness theorem for weighted Ricci tensor $\mathrm{R}ic_N$.

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Theorem 2. *Let $\aleph$ be a forward complete and connected n-dimensional Finsler manifold. Assume every geodesic $\gamma(t)$ issuing from point $p \in \aleph$ satisfies*

$$\int_0^\infty \mathrm{R}ic_N(\gamma'(t))dt = \infty, \tag{2}$$

then manifold is compact.

The above theorems are not require the weighted Ricci tensor $\mathrm{R}ic_\infty$ and $\mathrm{R}ic_N$ to be everywhere non-negative.

Now, we review below basic informations of the Finsler geometry to be used in the proofs of main theorems.

2. Finsler Geometry

Let $(\aleph, F)$ be a Finsler n-manifold equipped by Finsler metric $F: T\aleph \to [0, \infty)$. Let (x, y) be a point of $T\aleph$ such that $x \in \aleph$ and $y \in T_x\aleph$ and $\Omega: T\aleph \to \aleph$ be the natural projection. A *Finsler metric* is a $\mathcal{C}^\infty$-Finsler structure of $\aleph$ satisfying the following statements:

1. (Regularity) F is smooth on $T\aleph\backslash 0$,
2. (Positive homogeneity) $F(u, \mu v) = \mu F(u, v)$ for all $\mu > 0$,
3. (Strong convexity) The $n \times n$ matrix (fundamental quadratic form)

$$g_{ij} := \frac{1}{2}[F^2]_{v^i v^j} \tag{3}$$

is positively definite at each point of $T\aleph\backslash 0$.
The *Chern curvature* R^V for vectors fields $X, Y, Z \in T_x\aleph\backslash 0$ is defined by

$$R^V(U, V)Z := \nabla_U^V \nabla_V^V Z - \nabla_V^V \nabla_U^V Z - \nabla_{[U,V]}^V Z, \tag{4}$$

and for given linearly independent vectors $V, W \in T_x\aleph\backslash 0$, the *flag curvature* is defined as follows:

$$K(W, V) := \frac{g_V(R^V(W,V)V,W)}{g_V(W,W)g_V(V,V) - g_V(W,V)^2} \tag{5}$$

Then the *Ricci curvature* of V is defined as

$$\mathrm{R}ic(V) := \sum_{i=1}^{n-1} K(V, E_i), \tag{6}$$

where $\{e_1, e_2, \ldots, e_{n-1}, V/F(V)\}$ is an orthonormal basis of $T_x\aleph$ with respect to g_V.
Let $d\mu = \sigma_F(x)dx^1dx^2\ldots dx^n$ be the volume form on $\aleph$. A vector $W \in T_x\aleph\backslash 0$,

$$\tau(x, W) := \ln\frac{\sqrt{\det(g_{ij}(x,W))}}{\sigma_F(x)} \tag{7}$$

is a scalar function on $T_x\aleph\backslash 0$ and called the *distortion* of $(\aleph, F, d\mu)$.
Setting

$$S(x, W) := \frac{d}{dt}(\tau(\gamma(t), \dot\gamma(t)))|_{t=0}, \tag{8}$$

where γ is the geodesic with $\gamma(0) = x$, $\dot\gamma(0) = W$. $S(x, \lambda W) = \lambda S(x, W)$ for all $\lambda > 0$. S is a scalar function on $T_x\aleph\backslash 0$ and called the *S-curvature*. S-curvature measures the change rate of the distortion along geodesics in the direction $W \in T_x\aleph$.

For all $N \in (n, \infty)$, we define the *weighted Ricci curvature* of $(\aleph, F, d\mu)$ as follows (see [1]):

$$\begin{cases} \mathrm{R}ic_N(W) := \mathrm{R}ic(W) + \dot{S}(\mathrm{W}) - \dfrac{S(W)^2}{N-n}, \\ \mathrm{R}ic_\infty(W) := \mathrm{R}ic(\mathrm{W}) + \dot{S}(W), \\ \mathrm{R}ic_n(W) \ := \begin{cases} \mathrm{R}ic + \dot{S}(W), & if \quad S(W) = 0 \\ -\infty & otherwise. \end{cases} \end{cases}$$

Also $\mathrm{R}ic_N(kW) := k^2 \mathrm{R}ic_N(W)$ for $k > 0$.
A Finsler manifold $(\aleph, F)$ is *forward complete* if every geodesic $\sigma: [0, L] \to \aleph$, can be expanded to a geodesic on $[0, \infty)$. According to the Hopf-Rinow theorem, every pair of points $p, q \in \aleph$ can be joined by a minimal geodesic.

The *Legendre transformation* $\mathcal{L}: T\aleph \to T^*\aleph$ is defined as

$$\mathcal{L}(W) := \begin{cases} g_W(W, .), & W \neq 0, \\ 0 & W = 0. \end{cases}$$

Let $\mu: \aleph \to R$ be a smooth function, the *gradient* of μ at $y \in \aleph$ is defined as $\nabla\mu(y) := \mathcal{L}^{-1}(d\mu)$.
The *divergence* of a vector field $Z = Z^i \partial/\partial x^i$ on $\aleph$ is given as

$$\mathrm{d}ivZ := \sum_{i=1}^n \left(\frac{\partial Z^i}{\partial x^i} + Z^i \frac{\partial \varphi}{\partial x^i}\right). \tag{9}$$

for an arbitrary volume form $d\mu = e^\varphi dx^1 dx^2 \ldots dx^n$. Then we define the *Finsler-Laplacian* of μ by $\Delta\mu := \mathrm{d}iv(\nabla\mu) = \mathrm{d}iv(\mathcal{L}^{-1}(d\mu))$.
The following lemma will be very helpful in the proofs of main results (see [10]).

Lemma 3. *Let $(\aleph, F, d\mu)$ be a Finsler n-manifold, and $h: \aleph \to R$ a smooth function on $\aleph$. Then on $\mathcal{U} = \{x \in \aleph : \nabla h|_x \neq 0\}$ we have*

$$\Delta h = \sum_i H(h)(E_i, E_i) - S(\nabla h) := \mathrm{tr}_{\nabla h} H(h) - S(\nabla h), \tag{10}$$

where $E_1, E_2, \ldots, E_n$ is a local $g_{\nabla h}$-orthonormal frame on $\mathcal{U}$.
Finally, the *reversibility* λ of $\aleph$ is defined as

$$\lambda_F := \sup_{x \in M, y \in TM \setminus 0} \frac{F(x, -y)}{F(x, y)}. \tag{11}$$

Clearly, $\lambda_F \in [1, \infty]$, and $(\aleph, F)$ is reversible if $\lambda_F = 1$.

3. Proofs of the Theorems

Given $(\aleph, F, d\mu)$ Finsler manifold of dimensional n and $r(x) = d(x, q)$ distance function with respect to a fixed point $q \in \aleph$. The distance function r is only smooth on $\aleph - (C_q \cup \{q\})$ where C_q is the cut locus of the point $q \in \aleph$. Let γ be a minimal unit speed geodesic segment. We have $\nabla r = \gamma'(t)$ and $F(\nabla r) = 1$ (see [11]). Besides, we obtain a weighted Riemannian metric $g_{\nabla r}$ by using the Finsler metric. Thus we can apply Riemannian calculation for $g_{\nabla r}$ (on $\aleph - (C_q \cup \{q\})$).
Firstly we shall achieve the proof of Theorem 1.

Proof of Theorem 1. Assume by contradiction that $\aleph$ is non-compact Finsler manifold and let $\gamma(t)$ be a unit speed ray starting from $p \in \aleph$. For every $t > 0$, denote by $\eta(t) := (\Delta r)(\gamma(t))$ the Finsler-Laplacian of distance function. In the Finsler case, by Lemma 3, the Bochner Weitzenböck formula [12] says

$$0 \geq \mathrm{R}ic_{\infty}(\gamma'(t)) + \eta'(t) + \frac{1}{n-1}(\eta(t) + S(\gamma'(t)))^2. \tag{12}$$

By the inequality $(x+y)^2 \geq \frac{1}{\alpha+1}x^2 - \frac{1}{\alpha}y^2$, $\alpha > 0$, we have

$$0 \geq \mathrm{R}ic_{\infty}\big(\gamma'(t)\big) + \eta'(t) + \frac{\eta(t)^2}{(n-1)(\alpha+1)} - \frac{S(\gamma'(t))^2}{(n-1)\alpha}. \tag{13}$$

Let us the modified Finsler-Laplacian denote by $\tilde{\eta}(t) := \eta(t) + h(t)$, where $h(t)$ is a smooth function. Then from (13) and the assumption given in Theorem 1, we get

$$\begin{aligned} 0 &\geq \mathrm{R}ic_{\infty}(\gamma'(t)) + (\tilde{\eta}(t) - h(t))' + \frac{(\tilde{\eta}(t) - h(t))^2}{(n-1)(\alpha+1)} - \frac{S(\gamma'(t))^2}{(n-1)\alpha} \\ &\geq \mathrm{R}ic_{\infty}(\gamma'(t)) + \tilde{\eta}'(t) + \frac{\tilde{\eta}(t)^2}{(n-1)(\alpha+1)(\beta+1)} \\ &\quad -h'(t) - \frac{h(t)^2}{(n-1)(\alpha+1)\beta} - \frac{H^2}{(n-1)\alpha r^2} \end{aligned} \tag{14}$$

for every $\beta > 0$. Taking $(n-1)(\alpha+1)\beta = k > 0$ and $(n-1)\alpha = l > 0$, we have

$$0 \geq \mathrm{R}ic_{\infty}(\gamma'(t)) + \tilde{\eta}'(t) + \frac{\tilde{\eta}(t)^2}{l+k+n-1} - h'(t) - \frac{h(t)^2}{k} - \frac{H^2}{lr^2} \tag{15}$$

Here, if $h(t)$ and l are chosen to be $h(t) = k/2r$ and $l = 4H^2/k$, then the term $-h'(t) - \frac{h(t)^2}{k} - \frac{H^2}{lr^2}$ in (15) equals to zero. Therefore we have

$$\mathrm{R}ic_{\infty}\big(\gamma'(t)\big) \leq -\tilde{\eta}'(t) - \frac{k\tilde{\eta}(t)^2}{4H^2+k^2+(n-1)k} \tag{16}$$

Integrating both sides of the inequality (16) from 1 to t, we obtain

$$\int_1^t \mathrm{R}ic_{\infty}(\gamma'(s))ds \leq -\tilde{\eta}(t) + \tilde{\eta}(1) - \int_1^t \frac{k\tilde{\eta}(s)^2}{4H^2+k^2+(n-1)k}ds. \tag{17}$$

On the other hand, under the assumption

$$\int_0^{\infty} \mathrm{R}ic_{\infty}(\gamma'(t))dt = \infty, \tag{18}$$

given in Theorem 1, we have

$$\lim_{t\to\infty} -\tilde{\eta}(t) - \int_1^t \frac{k\tilde{\eta}(s)^2}{4H^2+k^2+(n-1)k}ds = \infty. \tag{19}$$

Here, multiplying by $\frac{k}{4H^2+k^2+(n-1)k}$ on both sides then yields

$$\lim_{t\to\infty} - \frac{k\tilde{\eta}(t)}{4H^2+k^2+(n-1)k} - \int_1^t \left(\frac{k\tilde{\eta}(s)}{4H^2+k^2+(n-1)k}\right)^2 ds = \infty. \tag{20}$$

Therefore, given $\vartheta > 1$ there exists $t_1 > 1$ such that

$$-\frac{k\tilde{\eta}(t)}{4H^2+k^2+(n-1)k} - \int_1^t \left(\frac{k\tilde{\eta}(s)}{4H^2+k^2+(n-1)k}\right)^2 ds \geq \vartheta \tag{21}$$

for all $t \geq t_1$.
Let us set

$$t_{\ell+1} = t_\ell + \vartheta^{1-\ell} \quad for \ \ell \geq 1. \tag{22}$$

The one that seems $\{t_\ell\}$ is an increasing sequence and it converges to $\Upsilon := t_1 + \frac{\vartheta}{\vartheta - 1}$ as $\ell \to \infty$.
We claim the fact that

$$-\tilde{\eta}(t) \geq \frac{4H^2+k^2+(n-1)k}{k}\vartheta^\ell \quad for \ all \ t \geq t_\ell. \tag{23}$$

To prove the claim, we use induction argument. It is trivial from inequality (21) for $\ell = 1$. By induction, we get the claim for ℓ. Then we must prove that $-\tilde{\eta}(t) \geq \frac{4H^2+k^2+(n-1)k}{k}\vartheta^{\ell+1}$ for all $t \geq t_{\ell+1}$. By means of the inequality (21), we obtain

$$\begin{aligned}
-\tilde{\eta}(t) &\geq \frac{4H^2 + k^2 + (n-1)k}{k}\vartheta + \frac{k}{4H^2 + k^2 + (n-1)k}\int_1^t \tilde{\eta}(s)^2 ds \\
&\geq \frac{k}{4H^2 + k^2 + (n-1)k}\int_1^{t_\ell} \tilde{\eta}(s)^2 ds + \frac{k}{4H^2 + k^2 + (n-1)k}\int_{t_\ell}^t \tilde{\eta}(s)^2 ds \\
&\geq \frac{k}{4H^2 + k^2 + (n-1)k}\int_{t_\ell}^t \tilde{\eta}(s)^2 ds \\
&\geq \frac{4H^2 + k^2 + (n-1)k}{k}\vartheta^{2\ell}(t - t_\ell) \\
&\geq \frac{4H^2+k^2+(n-1)k}{k}\vartheta^{2\ell}(t_{\ell+1} - t_\ell) = \frac{4H^2+k^2+(n-1)k}{k}\vartheta^{\ell+1}.
\end{aligned} \tag{24}$$

This proves the above claim.
From hence, we have

$$\lim_{\ell\to\infty} -\tilde{\eta}(t_\ell) = -\tilde{\eta}(\Upsilon) \geq \lim_{\ell\to\infty} \frac{4H^2+k^2+(n-1)k}{k}\vartheta^\ell. \tag{25}$$

However, this result contradicts with the smoothness of $\tilde{\eta}(t)$. Namely, $\lim_{t\to\Upsilon^-} -\tilde{\eta}(t) = \infty$. Thus the proof of Theorem 1 is satisfied.
Now we hold with the proof of Theorem 2.

Proof of Theorem 2. In a similar way of that is made to prove Theorem 1, via the Bochner Weitzenböck formula for $\mathrm{R}ic_N$ weighted Ricci curvature [12], we have that

$$\mathrm{R}ic_N(\gamma'(t)) \leq -\eta'(t) + \frac{\eta(t)^2}{N}, \tag{26}$$

where $\eta(t) := (\Delta r)(\gamma(t))$. Integrating both sides of (26) and taking the limit as $t \to \infty$, we get

$$\lim_{t\to\infty} \int_1^t \mathrm{R}ic_N(\gamma'(s))ds \leq \lim_{t\to\infty} (-\eta(t) + \eta(1) - \int_1^t \frac{\eta(s)^2}{N} ds). \tag{27}$$

Under the assumption (2) given in Theorem 2, we have

$$\lim_{t\to\infty} (-\eta(t) - \int_1^t \frac{\eta(s)^2}{N} ds) = \infty. \tag{28}$$

Multiplying $1/N$ by (28), yields

$$\lim_{t\to\infty}\left(-\frac{\eta(t)}{N}-\int_1^t\left(\frac{\eta(s)}{N}\right)^2 ds\right)=\infty. \tag{29}$$

From the above equation, given $C>1$ there exists $t_1>1$ such that

$$-\frac{\eta(t)}{N}-\int_1^t\left(\frac{\eta(s)}{N}\right)^2 ds\geq C \tag{30}$$

for all $t\geq t_1$.
We consider an increasing sequence $\{t_\ell\}$ defined by

$$t_{\ell+1}=t_\ell+C^{1-\ell} \quad for \;\; \ell\geq 1, \tag{31}$$

such that $\{t_\ell\}$ converges to $T:=t_1+\frac{C}{C-1}$ as $\ell\to\infty$.
The claim that $-\eta(t)\geq NC^\ell$ for all $t\geq t_\ell$. By induction, similar computations as in Theorem 1 yields $-\eta(t)\geq NC^{\ell+1}$ for all $t\geq t_{\ell+1}$. Indeed, we have that

$$\begin{aligned}
-\eta(t) &\geq NC+\frac{1}{N}\int_1^t \eta(s)^2 ds\\
&\geq \frac{1}{N}\int_1^{t_\ell}\eta(s)^2 ds+\frac{1}{N}\int_{t_\ell}^t \eta(s)^2 ds\\
&\geq \frac{1}{N}\int_{t_\ell}^t \eta(s)^2 ds\\
&\geq NC^{2\ell}(t-t_\ell)\\
&\geq NC^{2\ell}(t_{\ell+1}-t_\ell)=NC^{\ell+1}.
\end{aligned} \tag{32}$$

Consequently, because of the same reasons in Theorem 1, we contradict with the smoothness of $\eta(t)$. Thus the proof of Theorem 2 is satisfied.

Authors' Contributions

All contributions (the original theoretical results and applicational calculations, analysis, literature search and writing manuscript) belongs to the author.

Statement of Conflicts of Interest

There is no conflict of interest between the authors.

Statement of Research and Publication Ethics

The author declares that this study complies with Research and Publication Ethics.

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