Journal of New Theory
ISSN: 2149-1402

29 (2019) 71-78
Journal of New Theory
http://www.newtheory.org
Open Access

Bipolar Fuzzy *k*-Ideals in KU-Semigroups

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Article History
Received: 27.06.2019
Accepted: 16.10.2019
Published: 30.12.2019
Original Article

Abstract− We have studied some types of ideals in a KU-semigroup by using the concept of a bipolar fuzzy set. Bipolar fuzzy S-ideals and bipolar fuzzy *k*-ideals are introduced, and some properties are investigated. Also, some relations between a bipolar fuzzy *k*-ideal and *k*-ideal are discussed. Moreover, a bipolar fuzzy *k*-ideal under homomorphism and the product of two bipolar fuzzy *k*-ideals are studied.

Keywords− *KU-algebra, KU-semigroup, fuzzy S-ideal, bipolar fuzzy S-ideal, bipolar fuzzy k-ideal*

1. Introduction

In 1956, Zadeh [1] introduced the notion of fuzzy sets. This concept has been applied to many mathematical branches. In [2, 3], Mostafa et al. studied the fuzzy KU-ideals and investigated some basic properties. Intuitionistic fuzzy sets, interval-valued fuzzy sets and Bipolar-valued fuzzy sets are extension fuzzy sets theory. In 2000, Lee [4] introduced bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree from $[0,1]$ to $[-1,1]$. In bipolar-valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, while the membership degree $[-1,0)$ indicates that elements satisfy the implicit counter property. In [5-8], the authors introduced a bipolar-valued fuzzy set on different structures. In this work, we study the bipolar-valued fuzzy set theory to *k*-ideal of a KU-semigroup and discuss some relations between a bipolar fuzzy *k*-ideal and *k*-ideal. Also, a bipolar fuzzy *k*-ideal under homomorphism and the product of two bipolar fuzzy *k*-ideals are studied.

2. Preliminaries

In this part, we review some concepts related to KU-semigroup and a bipolar fuzzy logic.

Definition 2.1 [9] Algebra $(\aleph, *, 0)$ is a KU-algebra if, for all $\chi, \gamma, \tau \in \aleph$,

(ku_1) $(\chi * \gamma) * ((\gamma * \tau) * (\chi * \tau)) = 0$
(ku_2) $\chi * 0 = 0$
(ku_3) $0 * \chi = \chi$
(ku_4) $\chi * \gamma = 0$ and $\gamma * \chi = 0$ implies $\chi = \gamma$
(ku_5) $\chi * \chi = 0$

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On a KU-algebra $\aleph$, a relation $\leq$ is defined by $\leq \gamma \Leftrightarrow \gamma * \chi = 0$. Therefore $(\aleph, \leq)$ is a partially ordered set. It follows that 0 is the smallest element in $\aleph$.

Thus $(\aleph, * ,0)$ satisfies the following. For all $\chi, \gamma, \tau \in \aleph$,

(ku_1) $(\gamma * \tau) * (\chi * \tau) \leq (\chi * \gamma)$
(ku_2) $0 \leq \chi$
(ku_3) $\chi \leq \gamma, \gamma \leq \chi$ implies $\chi = \gamma$
(ku_4) $\gamma * \chi \leq \chi$

Theorem 2.2. [9] In a KU-algebra $\aleph$. The following axioms hold. For all $\chi, \gamma, \tau \in \aleph$,

i. $\chi \leq \gamma$ imply $\gamma * \tau \leq \chi * \tau$
ii. $\chi * (\gamma * \tau) = \gamma * (\chi * \tau)$
iii. $((\gamma * \chi) * \chi) \leq \gamma$

Definition 2.3. [10] A non-empty subset E of a KU-algebra $(\aleph, * ,0)$ is called KU-subalgebra of $\aleph$ if $\chi * \gamma \in E$ whenever $\chi, \gamma \in E$.

Definition 2.4. [10] A non-empty subset Ɽ of a KU-algebra $(\aleph, * ,0)$ is said to be an ideal of $\aleph$ if it satisfies, for any $\chi, \gamma \in \aleph$

i. $0 \in$ Ɽ and
ii. $* \gamma \in$ Ɽ, $\chi \in$ Ɽ imply that $\gamma \in$ Ɽ

Definition 2.5. [3] Let Ɽ be a nonempty subset of a KU-algebra $\aleph$. Then, Ɽ is said to be a KU-ideal of $\aleph$, if

(I_1) $0 \in$ Ɽ and

(I_2) $\forall \chi, \gamma, \tau \in \aleph$, $\chi * (\gamma * \tau) \in$ Ɽ and $\gamma \in$ Ɽ imply that $\chi * \tau \in$ Ɽ

Definition 2.6. [11] A KU-semigroup is a non-empty set $\aleph$ with two binary operations $*, \circ$ and constant 0 satisfying the following axioms

i. $(\aleph, * ,0)$ is a KU-algebra
ii. $(\aleph, \circ)$ is a semigroup
iii. The operation $\circ$ is distributive (on both sides) over the operation $*$, i.e.,

$$\chi \circ (\gamma * \tau) = (\chi \circ \gamma) * (\chi \circ \tau) \text{ and } (\chi * \gamma) \circ \tau = (\chi \circ \tau) * (\gamma \circ \tau), \forall \chi, \gamma, \tau \in \aleph$$

Example 2.7. [11] Let $\aleph = \{0,1,2,3\}$. Define $*$-operation and $\circ$-operation by the following tables

*	0	1	2	3
0	0	1	2	3
1	0	0	0	2
2	0	2	0	1
3	0	0	0	0

∘	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Then, $(\aleph, *, \circ, 0)$ is a KU-semigroup.

Definition 2.8. [11] A nonempty subset R of $\aleph$ is called a sub-KU-semigroup of $\aleph$, if $\chi * \gamma, \chi \circ \gamma \in R$, for all $\chi, \gamma \in R$.

Definition 2.9. [11] A non-empty subset R of a KU-semigroup$\aleph$ is an S-ideal of $\aleph$, if

i. R is an ideal of $\aleph$
ii. For all $\chi \in \aleph$, and $a \in R$, we have $\chi \circ a \in R$ and $a \circ \chi \in R$

Definition 2.10. [11] A subset R of a KU-semigroup $\aleph$ is a k-ideal of $\aleph$, if

i. R is a KU-ideal of $\aleph$
ii. For all $\chi \in \aleph$, and $a \in R$, we have $\chi \circ a \in R$ and $a \circ \chi \in R$

Definition 2.11. [11] Let $\aleph$ and $\aleph'$ be two KU-semigroups. A mapping $f: \aleph \to \aleph'$ is called a KU-semigroup homomorphism if $f(\chi * \gamma) = f(\chi) * f(\gamma)$ and $f(\chi \circ \gamma) = f(\chi) \circ f(\gamma)$ for all $\chi, \gamma \in \aleph$. The set $\{\chi \in \aleph: f(\chi) = 0\}$ is called the kernel of f and denote by $ker\ f$ Moreover, the set $\{ f(\chi) \in \aleph' : \chi \in \aleph\}$ is called the image of f and denote by imf.

We review some concepts of fuzzy logic.

Let $\aleph$ be the collection of objects, then a fuzzy set $\mu(\chi)$ in $\aleph$ is defined as $\mu: \aleph \to [0,1]$, where $\mu(\chi)$ is called the membership value of χ in $\aleph$ and $0 \leq \mu(\chi) \leq 1$. The set $U(\mu, t) = \{\chi \in \aleph : \mu(\chi) \geq t\}$, where $0 \leq t \leq 1$ is said to be a level set of $\mu(\chi)$.

Definition 2.12. [12] Let $\mu(\chi)$ be a fuzzy set in $\aleph$, then $\mu(\chi)$ is called a fuzzy sub KU-semigroup of $\aleph$ if it satisfies the following condition : for all $\chi, \gamma \in \aleph$.

i. $\mu(\chi * \gamma) \geq \min\{\mu(\chi), \mu(\gamma)\}$
ii. $\mu(\chi \circ \gamma) \geq \min\{\mu(\chi), \mu(\gamma)\}$

Definition 2.13. [12] A fuzzy set $\mu(\chi)$ in $\aleph$ is called a fuzzy S-ideal of $\aleph$ if for all $\chi, \gamma \in \aleph$

i. $\mu(0) \geq \mu(\chi)$
ii. $\mu(\gamma) \geq \min\{\mu(\chi * \gamma), \mu(\chi)\}$
iii. $\mu(\chi \circ \gamma) \geq \min\{\mu(\chi), \mu(\gamma)\}$

Definition 2.14. [12] A fuzzy set $\mu(\chi)$ in $\aleph$ is called a fuzzy k-ideal, if it satisfies the following condition: for all $\chi, \gamma \in \aleph$

(k_1) $\mu(0) \geq \mu(\chi)$
(k_2) $\mu(\chi * \tau) \geq \min\{\mu(\chi * (\gamma * \tau)), \mu(\gamma)\}$
(k_3) $\mu(\chi \circ \gamma) \geq \min\{\mu(\chi), \mu(\gamma)\}$

Example 2.15. [12] Let $\aleph = \{0, a, b, c, d\}$ be a set. Define $*$-operation and $\circ$-operation by the following tables

*	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	c	d
b	0	a	0	c	d
c	0	a	0	0	d
d	0	0	0	0	0

∘	0	a	b	c	d
0	0	0	0	0	0
a	0	0	0	0	0
b	0	0	0	0	b
c	0	0	0	b	c
d	0	a	b	c	d

Then, $(\aleph, *, \circ, 0)$ is a KU-semigroup. Define a fuzzy set $\mu: \aleph \to [0,1]$ by $\mu(0) = \mu(a) = 0.4, \mu(b) = \mu(c) = 0.2$, $\mu(d) = 0.1$. Then, it is easy to see $\mu(\chi), \forall \chi \in \aleph$ is a fuzzy k-ideal.

We will refer to $\aleph$ is a KU-semigroup unless otherwise indicated.

3. Bipolar fuzzy *k*-ideals of a KU-semigroup

In this section, we give the definition and properties of bipolar fuzzy ideals of $\aleph$. Now, A bipolar valued fuzzy subset B in a nonempty set $\aleph$ is an object having the form $B = \{(\chi, \mu^-(\chi), \mu^+(\chi) | \chi \in \aleph\}$ where $\mu^-: \aleph \to [-1,0]$ and $\mu^+: \aleph \to [0,1]$ are two mappings. The membership degree $\mu^+(\chi)$ denotes the satisfaction degree of

χ to the property corresponding of B, and the membership degree $\mu^-(\chi)$ denotes the satisfaction degree of χ to some implicit counter-property of B. We shall use the symbol $B = (\chi, \mu^-, \mu^+)$, for $B = \{(\chi, \mu^-(\chi), \mu^+(\chi)) : \chi \in \aleph\}$, and use the concept of a bipolar fuzzy set instead of the concept of bipolar-valued fuzzy set.

Now, let $B = (\chi, \mu^-, \mu^+)$ be a bipolar fuzzy set and $(s, t) \in [-1,0] \times [0,1]$.

The set $B_s^- = \{\chi \in \aleph: \mu^-(\chi) \leq s\}$ and $B_t^+ = \{\chi \in \aleph: \mu^+(\chi) \geq t\}$ which are called the negative s-cut and the positive t-cut of $B = (\chi, \mu^-, \mu^+)$, respectively.

Definition 3.1. A fuzzy set μ in $\aleph$ is called a bipolar fuzzy sub-KU-semigroup of $\aleph$ if it satisfies the following condition : for all $\chi, \gamma \in \aleph$

i. $\mu^-(\chi * \gamma) \leq \max\{\mu^-(\chi), \mu^-(\gamma)\}$ and $\mu^+(\chi * \gamma) \geq \min\{\mu^+(\chi), \mu^+(\gamma)\}$
ii. $\mu^-(\chi \circ \gamma) \leq \max\{\mu^-(\chi), \mu^-(\gamma)\}$ and $\mu^+(\chi \circ \gamma) \geq \min\{\mu^+(\chi), \mu^+(\gamma)\}$

Proposition 3.2. Let $B = (\chi, \mu^-, \mu^+)$ be a bipolar fuzzy sub-KU-semigroup. Then, $\mu^-(0) \leq \mu^-(\chi)$ and $\mu^+(0) \geq \mu^+(\chi)$, for all $\chi \in \aleph$.

Proof. Clear.

Example 3.3. Let $\aleph = \{0, a, b, c\}$ be a set. Define $*$-operation and $\circ$-operation by the following tables

*	0	a	b	c
0	0	a	b	c
a	0	0	0	c
b	0	a	0	c
c	0	0	0	0

∘	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

Then, $(\aleph, *, \circ, 0)$ is a KU-semigroup. Define $B = (x, \mu^-, \mu^+)$ by $B = \{(0, -0.6, 0.7), (a, -0.5, 0.5), (b, -0.3, 0.4), (c, -0.2, 0.1)\}$. Then, we can prove that B is a bipolar fuzzy sub-KU-semigroup of $\aleph$.

Definition 3.4. A bipolar fuzzy set $B = (\chi, \mu^-, \mu^+)$in X is called a bipolar fuzzy S-ideal of $\aleph$ if it satisfies, for all $\chi, \gamma \in \aleph$

(Bf_1) $\mu^-(0) \leq \mu^-(\chi)$ and $\mu^+(0) \geq \mu^+(\chi)$
(Bf_2) $\mu^-(\gamma) \leq \max\{\mu^-(\chi * \gamma), \mu^-(\chi)\}$ and $\mu^+(\gamma) \geq \min\{\mu^+(\chi * \gamma), \mu^+(\chi)\}$
(Bf_3) $\mu^-(\chi \circ \gamma) \leq \max\{\mu^-(\chi), \mu^-(\gamma)\}, \mu^+(\chi \circ \gamma) \geq \min\{\mu^+(\chi), \mu^+(\gamma)\}$

Definition 3.5. A bipolar fuzzy set $B = (\chi, \mu^-, \mu^+)$ in $\aleph$ is called a bipolar fuzzy k-ideal of $\aleph$ if it satisfies: for all $\chi, \gamma, \tau \in \aleph$

(BF_1) $\mu^-(0) \leq \mu^-(\chi)$ and $\mu^+(0) \geq \mu^+(\chi)$
(BF_2) $\mu^-(\chi * \tau) \leq \max\{\mu^-(\chi * (\gamma * \tau)), \mu^-(\gamma)\}$ and $\mu^+(\chi * \tau) \geq \min\{\mu^+(\chi * (\gamma * \tau)), \mu^+(\gamma)\}$
(BF_3) $\mu^-(\chi \circ \gamma) \leq \max\{\mu^-(\chi), \mu^-(\gamma)\}, \mu^+(\chi \circ \gamma) \geq \min\{\mu^+(\chi), \mu^+(\gamma)\}$

Example 3.6. Let $\aleph = \{0, a, b, c\}$ with $*$ defined as in Example (3.3), and $B = (x, \mu^-, \mu^+)$ be a bipolar fuzzy set in $\aleph$ given by the following $B = \{(0, -0.7, 0.6), (a, -0.4, 0.2), (b, -0.4, 0.2), (c, -0.3, 0.1)\}$. Then, $B = (\chi, \mu^-, \mu^+)$ is a bipolar fuzzy k-ideal of $\aleph$.

Theorem 3.7. Let $\aleph$ be a KU-semigroup, a bipolar fuzzy set $B = (\chi, \mu^-, \mu^+)$ of $\aleph$ is a bipolar fuzzy k-ideal of $\aleph$ if and only if B is a bipolar fuzzy S-ideal of $\aleph$.

Proof.

($\Rightarrow$) Let $B = (\chi, \mu^-, \mu^+)$ be a bipolar fuzzy k-ideal of $\aleph$. If we put $\chi = 0$ in (BF_2), then $\mu^-(\tau) \leq \max\{\mu^-(\gamma * \tau), \mu^-(\gamma)\}$ and

$\mu^+(\tau) \geq \min\{\mu^+(\gamma * \tau), \mu^+(\gamma)\}$. Also, since $B = (\chi, \mu^-, \mu^+)$ is a bipolar fuzzy k-ideal of KU-semigroup, then (BF_3) is true. Hence, $B = (\chi, \mu^-, \mu^+)$ is a bipolar fuzzy S-ideal of $\aleph$.

($\Leftarrow$) Let $B = (\chi, \mu^-, \mu^+)$ be a bipolar fuzzy S-ideal of $\aleph$, then $\mu^-(\chi * \tau) \leq \max\{\mu^-(\gamma * (\chi * \tau), \mu^-(\gamma)\}$ and $\mu^+(\chi * \tau) \geq \min\{\mu^+(\gamma * (\chi * \tau), \mu^+(\gamma)\}$. And by Theorem (2.2)(2), we get $\mu^-(\chi * \tau) \leq \max\{\mu^-(\chi * (\gamma * \tau), \mu^-(\gamma)\}$ and $\mu^+(\chi * \tau) \geq \min\{\mu^+(\chi * (\gamma * \tau), \mu^+(\gamma)\}$. Also, since $B = (\chi, \mu^-, \mu^+)$ is a bipolar fuzzy S-ideal of KU-semigroup, then (Bf_3) is true. Hence, $B = (\chi, \mu^-, \mu^+)$ is a bipolar fuzzy k-ideal of $\aleph$.

Proposition 3.8. Let $B = (\chi, \mu^-, \mu^+)$ be a bipolar fuzzy k-ideal of $\aleph$. If the inequality $\chi * \gamma \leq \tau$ holds in $\aleph$, then $\mu^-(\gamma) \leq \max\{\mu^-(\chi), \mu^-(\tau)\}$ and $\mu^+(\gamma) \geq \min\{\mu^+(\chi), \mu^+(\tau)\}$, for all $\chi, \gamma, \tau \in \aleph$.

Proof.

Assume that the inequality $\chi * \gamma \leq \tau$ holds in $\aleph$, then $\tau * (\chi * \gamma) = 0$ and by (BF_2)

$$\mu^-(\chi * \gamma) \leq \max\{\mu^-(\chi * (\tau * \gamma)), \mu^-(\tau)\}$$
$$= \max\{\mu^-(\tau * (\chi * \gamma)\}, \mu^-(\tau)\}$$
$$= \max\{\mu^-(0), \mu^-(\tau)\} = \mu^-(\tau) \ldots\ldots (1)$$

Now, $\mu^-(0 * \gamma) = \mu^-(\gamma) \leq max\{\mu^-(0 * (\chi * \gamma)), \mu^-(\chi)\} = \max\{\mu^-(\chi * \gamma), \mu^-(\chi)\} \leq \max\{\mu^-(\tau), \mu^-(\chi)\}$ (by using (1)) i.e. $\mu^-(\gamma) \leq \max\{\mu^-(\chi), \mu^-(\tau)\}$. Similarly,

$$\mu^+(\chi * \gamma) \geq \min\{\mu^+(\chi * (\tau * \gamma)), \mu^+(\tau)\} = \min\{\mu^+(\tau * (\chi * \gamma)\}, \mu^+(\tau)\} = \min\{\mu^+(0), \mu^+(\tau)\} = \mu^+(\tau) \ldots (2)$$

Now, $\mu^+(0 * \gamma) = \mu^+(\gamma) \geq min\{\mu^+(0 * (\chi * \gamma)), \mu^+(\chi)\} = \min\{\mu^+(\chi * \gamma), \mu^+(\chi)\} \geq \min\{\mu^+(\tau), \mu^+(\chi)\}$ (by using (2)) i.e. $\mu^+(\gamma) \geq \min\{\mu^+(\chi), \mu^+(\tau)\}$.

Theorem 3.9. Let $\aleph$ be a KU-semigroup, a bipolar fuzzy set $B = (\chi, \mu^-, \mu^+)$ of $\aleph$ is a bipolar fuzzy k-ideal of $\aleph$ if and only if B is a bipolar fuzzy sub-KU-semigroup of $\aleph$.

Proof. ($\Rightarrow$) Let $B = (\chi, \mu^-, \mu^+)$ be a bipolar fuzzy k-ideal of $\aleph$. By Theorem (3.7), B is a bipolar fuzzy S-ideal of $\aleph$. For any $\chi, \gamma \in \aleph$, from (ku_4) we have $\chi * \gamma \leq \gamma$, then by Proposition (3.2) $\mu^-(\chi * \gamma) \leq \mu^-(\gamma)$ and $\mu^+(\chi * \gamma) \geq \mu^+(\gamma)$. And by Proposition (3.8) $\mu^-(\gamma) \leq \max\{\mu^-(\chi), \mu^-(\tau)\}$ and $\mu^+(\gamma) \geq \min\{\mu^+(\chi), \mu^+(\tau)\}$, Hence, $\mu^-(\chi * \gamma) \leq \max\{\mu^-(\chi), \mu^-(\tau)\}$ and $\mu^+(\chi * \gamma) \geq \min\{\mu^+(\chi), \mu^+(\tau)\}$. Then, B is a bipolar fuzzy sub-KU-semigroup of $\aleph$.

($\Leftarrow$) Let $B = (\chi, \mu^-, \mu^+)$ be a bipolar fuzzy sub-KU-semigroup. We have

(*i*) $\mu^-(0) \leq \mu^-(\chi)$ and $\mu^+(0) \geq \mu^+(\chi)$, for all $\chi \in \aleph$.

(*ii*) By Theorem (2.2) (2) and (3), we have $(\gamma * (\chi * \tau)) * (\chi * \tau) = (\chi * (\gamma * \tau)) * (\chi * \tau) \leq \gamma$, for all $\chi, \gamma, \tau \in \aleph$. It follows from Proposition (3.3.7) that $\mu^-(\chi * \tau) \leq \max\{\mu^-(\chi * (\gamma * \tau)), \mu^-(\gamma)\}$ and $\mu^+(\chi * \tau) \geq \min\{\mu^+(\chi * (\gamma * \tau)), \mu^+(\gamma)\}$, for all $\gamma, \tau \in \aleph$. Also, since $B = (\chi, \mu^-, \mu^+)$ is a bipolar fuzzy sub-KU-semigroup, then (BF_3) is true. Therefore, $B = (\chi, \mu^-, \mu^+)$ is a bipolar fuzzy k-ideal of $\aleph$.

Proposition 3.10. If $B = (\chi, \mu^-, \mu^+)$ is a bipolar fuzzy k-ideal of $\aleph$, then the sets $J = \{\chi \in \aleph : \mu^+(\chi) = \mu^+(0)\}$ and $K = \{\chi \in \aleph : \mu^-(\chi) = \mu^-(0)\}$ are k-ideals of $\aleph$.

Proof. Since $0 \in \aleph, \mu^+(0) = \mu^+(0)$ and $\mu^-(0) = \mu^-(0)$ implies $0 \in J$ and $0 \in K$, so $J \neq \emptyset, K \neq \emptyset$. Let $(\chi * (\gamma * \tau)) \in J$ and $\gamma \in J$ implies $\mu^+(\chi * (\gamma * \tau)) = \mu^+(0)$ and $\mu^+(\gamma) = \mu^+(0)$. Since $\mu^+(\chi * \tau) \geq \min\{\mu^+(\chi * (\gamma * \tau)), \mu^+(\gamma)\} = \mu^+(0) \Rightarrow \mu^+(\chi * \tau) \geq \mu^+(0)$ but $\mu^+(0) = \mu^+(\chi * \tau)$. It follows that $(\chi * \tau) \in J$, for all $\chi, \gamma, \tau \in \aleph$.

Also, let $\chi \in J$ and $\gamma \in J$ implies $\mu^+(\chi) = \mu^+(0)$ and $\mu^+(\gamma) = \mu^+(0)$. Since, $\mu^+(\chi \circ \gamma) \geq \min\{\mu^+(\chi), \mu^+(\gamma)\} = \mu^+(0)$, then $\mu^+(\chi \circ \gamma) = \mu^+(0)$. It follows that $\chi \circ \gamma \in J$, similarly $\gamma \circ \chi \in J$. Hence, J is k-ideal of $\aleph$. Similarly, we can prove K is k-ideal of $\aleph$.

Theorem 3.11. For a bipolar fuzzy set $B = (\chi, \mu^-, \mu^+)$ in $\aleph$, the following are equivalent:

(1) $B = (\chi, \mu^-, \mu^+)$ is a bipolar fuzzy k-ideal of $\aleph$.

(2) $B = (\chi, \mu^-, \mu^+)$ is satisfies the following:

i. $\forall s \in [-1,0], (B_s^- \neq \emptyset \Rightarrow B_s^-)$ is a k-ideal of $\aleph$.
ii. $\forall t \in [0,1], (B_t^+ \neq \emptyset \Rightarrow B_t^+)$ is a k-ideal of $\aleph$.

Proof. (1) $\Rightarrow$(2) (i) Let $s \in [-1,0]$ be such that $B_s^- \neq \emptyset$. Then, there exists $\gamma \in B_s^-$ and so $\mu^-(\gamma) \leq s$. It follows from (BF1) that $\mu^-(0) \leq \mu^-(\gamma) \leq s$, then $0 \in B_s^-$. Let,$\gamma, \tau \in B_s^-$, such that $(\chi * (\gamma * \tau)) \in B_s^-$ and $\gamma \in B_s^-$. Then, $\mu^-(\chi * (\gamma * \tau)) \leq s$ and $\mu^-(\gamma) \leq s$. By using (BF2), we have $\mu^-(\chi * \tau) \leq \max\{\mu^-(\chi * (\gamma * \tau)), \mu^-(\gamma)\} = \max\{s, s\} = s$, which implies that $(\chi * \tau) \in B_s^-$. By using (BF3), we have $\mu^-(\chi \circ \gamma) \leq \max\{\mu^-(\chi), \mu^-(\gamma)\} = \max\{s, s\} = s$, which implies that $(\chi \circ \gamma) \in B_s^-$ (res. $(\gamma \circ \chi) \in B_s^-$). Therefore, B_s^- is a k-ideal of $\aleph$.

(ii) Assume that $B_t^+ \neq \emptyset$, for $t \in [0,1]$ and let $a \in B_t^+$. Then, $\mu^+(a) \geq t$ and $\mu^+(0) \geq \mu^+(a) \geq t$ by (BF1), thus $0 \in B_t^+$. Let $\chi, \gamma, \tau \in \aleph$ be such that $(\chi * (\gamma * \tau)) \in B_t^+$ and $\gamma \in B_t^+$. Then, $\mu^+(\chi * (\gamma * \tau)) \geq t$ and $\mu^+(\gamma) \geq t$.

It follows from (BF2) that $\mu^+(\chi * \tau) \geq \min\{\mu^+(\chi * (\gamma * \tau)), \mu^+(\gamma)\} = \min\{t, t\} = t$, so that $(\chi * \tau) \in B_t^+$. Also, by (BF3), $\mu^+(\chi \circ \gamma) \geq \min\{\mu^+(\chi), \mu^+(\gamma)\} = \min\{t, t\} = t$, then $(\chi \circ \gamma) \in B_t^+$(res. $(\gamma \circ \chi) \in B_t^+$). Hence, B_t^+ is a k-ideal of $\aleph$.

(2) $\Rightarrow$(1) Assume that there exists $a \in \aleph$ such that $\mu^-(0) \geq \mu^-(a)$. Taking $s_0 = \frac{1}{2}(\mu^-(0) + \mu^-(a))$, for some $s_0 \in [-1,0]$ implies that $\mu^-(a) < s_0 < \mu^-(0)$. This is a contradiction, and thus $\mu^-(0) \leq \mu^-(\gamma)$, for all $\gamma \in \aleph$. Suppose that $\mu^-(\chi * \tau) \leq \max\{\mu^-(\chi * (\gamma * \tau)), \mu^-(\gamma)\}$, for some $\chi, \gamma, \tau \in \aleph$, and let $s_1 = \frac{1}{2}(\mu^-(\chi * \tau) + \max\{\mu^-(\chi * (\gamma * \tau)), \mu^-(\gamma)\})$. Then, $\max\{\mu^-(\chi * (\gamma * \tau)), \mu^-(\gamma)\} < s_1 < \mu^-(\chi * \tau)$, which is a contradiction. Therefore, $\mu^-(\chi * \tau) \leq \max\{\mu^-(\chi * (\gamma * \tau)), \mu^-(\gamma)\}$, for all $\chi, \gamma, \tau \in \aleph$. Suppose that $\mu^-(\chi \circ \gamma) \leq \max\{\mu^-(\chi), \mu^-(\gamma)\}$, for some $\chi, \gamma \in \aleph$, and let $s_2 = \frac{1}{2}(\mu^-(\chi \circ \gamma) + \max\{\mu^-(\chi), \mu^-(\gamma)\}$. Then, $\max\{\mu^-(\chi), \mu^-(\gamma)\} < s_2 < \mu^-(\chi \circ \gamma)$, which is a contradiction. Therefore, $\mu^-(\chi \circ \gamma) \leq \max\{\mu^-(\chi), \mu^-(\gamma)\}$, for all $\chi, \gamma \in \aleph$.

Now, if $\mu^+(0) < \mu^+(\gamma)$, for some $\gamma \in \aleph$, then $\mu^+(0) < t_0 < \mu^+(\gamma)$, for some $t_0 \in (0,1]$. This is a contradiction. Thus $\mu^+(0) \geq \mu^+(\gamma)$, for all $\gamma \in \aleph$.

If $\mu^+(\chi * \tau) \leq \min\{\mu^+(\chi * (\gamma * \tau)), \mu^+(\gamma)\}$, for some $\chi, \gamma, \tau \in \aleph$. Then, there exists $t_1 \in (0,1]$, such that $\mu^+(\chi * \tau) < t_1 \leq \min\{\mu^+(\chi * (\gamma * \tau)), \mu^+(\gamma)\}$. We get $\chi * (\gamma * \tau) \in B_{t_1}^+$ and $\gamma \in B_{t_1}^+$ but $\chi * \tau \notin B_{t_1}^+$. This is a contradiction. Consequently, $\mu^+(\chi * \tau) \geq \min\{\mu^+(\chi * (\gamma * \tau)), \mu^+(\gamma)\}$, for all $\chi, \gamma, \tau \in \aleph$. And if $\mu^+(\chi \circ \gamma) \leq \min\{\mu^+(\chi), \mu^+(\gamma)\}$, for some, $\gamma \in \aleph$.

Then, there exists $t_2 \in (0,1]$ such that $\mu^+(\chi \circ \gamma) < t_2 \leq \min\{\mu^+(\chi), \mu^+(\gamma)\}$. It follows that $\chi \in B_{t_2}^+$ and $\gamma \in B_{t_2}^+$ but $(\chi \circ \gamma) \notin B_{t_2}^+$, which is a contradiction. Hence, $\mu^+(\chi \circ \gamma) \geq \min\{\mu^+(\chi), \mu^+(\gamma)\}$, for all $\chi, \gamma \in \aleph$.

Therefore $B = (\chi, \mu^-, \mu^+)$ is a bipolar fuzzy k-ideal of $\aleph$.

4. Bipolar fuzzy *k*-ideals under homomorphism

Definition 4.1. For any $\chi \in \aleph$. We define a new bipolar fuzzy set $B_f = (\chi, \mu_f^-, \mu_f^+)$ in $\aleph$ by $\mu_f^-(\chi) = \mu^-(f(\chi))$ and $\mu_f^+(\chi) = \mu^+(f(\chi))$, where $f: \aleph \to \aleph'$ is a KU-semigroup homomorphism.

Theorem 4.2. Let $f: \aleph \to \aleph'$ be a KU-semigroup homomorphism and onto mapping. Then, $B = (\chi', \mu^-, \mu^+)$ is a bipolar fuzzy k-ideal of $\aleph'$ if and only if $B_f = (\chi, \mu_f^-, \mu_f^+)$ is a bipolar fuzzy k-ideal of $\aleph$.

Proof: For any $\chi' \in \aleph'$ there exists $\chi \in \aleph$ such that $f(\chi) = \chi'$, we have

$$\mu_f^+(0) = \mu^+(f(0)) = \mu^+(0') \geq \mu^+(\chi') = \mu^+(f(\chi)) = \mu_f^+(\chi)$$

and

$$\mu_f^-(0) = \mu^-(f(0)) = \mu^-(0') \leq \mu^-(\chi') = \mu^-(f(\chi)) = \mu_f^-(\chi).$$

Let $\chi, \tau \in \aleph$, $\gamma' \in \aleph'$ then there exists $\gamma \in \aleph$ such that $f(\gamma) = \gamma'$. We have

$$\begin{aligned}\mu_f^+(\chi * \tau) = \mu^+(f(\chi * \tau)) = \mu^+(f(\chi) * f(\tau)) &\geq \min\{\mu^+(f(\chi) * (\gamma' * f(\tau)), \mu^+(\gamma')\} \\ &= \min\{\mu^+(f(\chi) * (f(\gamma) * f(\tau))\}, \mu^+(f(\gamma)\} = \min\{\mu_f^+(\chi * (\gamma * \tau)), \mu_f^+(\gamma)\}\end{aligned}$$

and

$$\begin{aligned}\mu_f^-(\chi * \tau) = \mu^-(f(\chi * \tau)) = \mu^-(f(\chi) * f(\tau)) &\leq \max\{\mu^-(f(\chi) * (\gamma' * f(\tau)), \mu^-(\gamma')\} \\ &= \max\{\mu^-(f(\chi) * (f(\gamma) * f(\tau))\}, \mu^-(f(\gamma)\} = \max\{\mu_f^-(\chi * (\gamma * \tau)), \mu_f^-(\gamma)\}\end{aligned}$$

Hence, $B_f = (\chi, \mu_f^-, \mu_f^+)$ is a bipolar fuzzy k-ideal of $\aleph$.

Conversely, since $f: \aleph \to \aleph'$ is an onto mapping, then for any $\chi, \gamma, \tau \in \aleph'$.

It follows that there exists $a, b, c \in \aleph$ such that $f(a) = \chi, f(b) = \gamma$ and $f(c) = \tau$. We have

$$\begin{aligned}\mu^+(\chi * \tau) = \mu^+(f(a) * f(c))) = \mu^+(f(a * c)) = \mu_f^+(a * c) &\geq \min\{\mu_f^+(a * (b * c)), \mu_f^+(b)\} \\ &= \min\{\mu^+(f(a) * (f(b) * f(c))\}, \mu^+(f(b)\} = \min\{\mu^+(\chi * (\gamma * \tau)), \mu^+(\gamma)\}.\end{aligned}$$

and

$$\begin{aligned}\mu^-(\chi * \tau) = \mu^-(f(a) * f(c))) = \mu^-(f(a * c)) = \mu_f^-(a * c) &\leq \max\{\mu_f^-(a * (b * c)), \mu_f^-(b)\} \\ &= \max\{\mu^-(f(a) * (f(b) * f(c))\}, \mu^-(f(b)\} = \max\{\mu^-(\chi * (\gamma * \tau)), \mu^-(\gamma)\}\end{aligned}$$

Therefore, $B = (\chi, \mu^-, \mu^+)$ is a bipolar fuzzy k-ideal of $\aleph'$.

Now, we introduce the product of bipolar fuzzy k-ideals in a KU-semigroup, and we study some results.

Definition 4.3. Let $B_1 = (\chi, \mu_1^-, \mu_1^+)$ and $B_2 = (\gamma, \mu_2^-, \mu_2^+)$ be two bipolar fuzzy sets of $\aleph$. The product $B_1 \times B_2 = ((\chi, \gamma), \mu_1^- \times \mu_2^-, \mu_1^+ \times \mu_2^+)$ is defined by the following: $(\mu_1^- \times \mu_2^-)(\chi, \gamma) = \max\{\mu_1^-(\chi), \mu_2^-(\gamma)\}$ and $(\mu_1^+ \times \mu_2^+)(\chi, \gamma) = \min\{\mu_1^+(\chi), \mu_2^+(\gamma)\}$, where $\mu_1^- \times \mu_2^-: \aleph \times \aleph \to [-1,0]$ and $\mu_1^+ \times \mu_2^+: \aleph \times \aleph \to [0,1]$, for all $\chi, \gamma \in \aleph$.

Theorem 4.4. Let $B_1 = (\chi, \mu_1^-, \mu_1^+)$ and $B_2 = (\gamma, \mu_2^-, \mu_2^+)$ be two bipolar fuzzy k-ideals of KU-semigroup $\aleph$, then $B_1 \times B_2$ is a bipolar fuzzy k-ideal of $\aleph \times \aleph$.

Proof. For any $(\chi, \gamma) \in \aleph \times \aleph$, we have

$$(\mu_1^+ \times \mu_2^+)(0, 0) = \min\{\mu_1^+(0), \mu_2^+(0)\} \geq \min\{\mu_1^+(\chi), \mu_2^+(\gamma)\} = (\mu_1^+ \times \mu_2^+)(\chi, \gamma)$$

and

$$(\mu_1^- \times \mu_2^-)(0, 0) = \max\{\mu_1^-(0), \mu_2^-(0)\} \leq \max\{\mu_1^-(\chi), \mu_2^-(\gamma)\} = (\mu_1^- \times \mu_2^-)(\chi, \gamma)$$

Let $(\chi_1, \chi_2), (\gamma_1, \gamma_2)$ and $(\tau_1, \tau_2) \in \aleph \times \aleph$, then

$$\begin{aligned}(\mu_1^+ \times \mu_2^+)(\chi_1 * \tau_1, \chi_2 * \tau_2) &= \min\{\mu_1^+(\chi_1 * \tau_1), \mu_2^+(\chi_2 * \tau_2)\} \\ &\geq \min\{\min\{\mu_1^+(\chi_1 * (\gamma_1 * \tau_1)), \mu_1^+(\gamma_1)\}, \min\{\mu_2^+(\chi_2 * (\gamma_2 * \tau_2)), \mu_2^+(\gamma_2)\}\} \\ &= \min\{\min\{\mu_1^+(\chi_1 * (\gamma_1 * \tau_1)), \mu_2^+(\chi_2 * (\gamma_2 * \tau_2)\}, \min\{\mu_1^+(\gamma_1), \mu_2^+(\gamma_2)\} \\ &= \min\left\{\min(\mu_1^+ \times \mu_2^+)\left\{(\chi_1 * (\gamma_1 * \tau_1)), (\chi_2 * (\gamma_2 * \tau_2))\right\}, \{(\mu_1^+ \times \mu_2^+)(\gamma_1, \gamma_2)\}\right\}\end{aligned}$$

and

$$\begin{aligned}(\mu_1^- \times \mu_2^-)(\chi_1 * \tau_1, \chi_2 * \tau_2) &= \max\{\mu_1^-(\chi_1 * \tau_1), \mu_2^-(\chi_2 * \tau_2)\} \\ &\leq \max\{\max\{\mu_1^-(\chi_1 * (\gamma_1 * \tau_1)), \mu_1^-(\gamma_1)\}, \max\{\mu_2^-(\chi_2 * (\gamma_2 * \tau_2), \mu_2^-(\gamma_2)\} \\ &= \max\{\max\{\mu_1^-(\chi_1 * (\gamma_1 * \tau_1)), \mu_2^-(\chi_2 * (\gamma_2 * \tau_2)\}, \max\{\mu_1^-(\gamma_1), \mu_2^-(\gamma_2)\} \\ &= \max\left\{(\mu_1^- \times \mu_2^-)\left\{(\chi_1 * (\gamma_1 * \tau_1)), (\chi_2 * (\gamma_2 * \tau_2))\right\}, \{(\mu_1^- \times \mu_2^-)(\gamma_1, \gamma_2)\}\right\}\end{aligned}$$

and

$$\begin{aligned}(\mu_1^+ \times \mu_2^+)(\chi_1 \circ \gamma_1, \chi_2 \circ \gamma_2) &= \min\{\mu_1^+(\chi_1 \circ \gamma_1), \mu_2^+(\chi_2 \circ \gamma_2)\} \\ &\geq \min\{\min\{\mu_1^+(\chi_1), \mu_1^+(\gamma_1)\}, \min\{\mu_2^+(\chi_2), \mu_2^+(\gamma_2)\} \\ &= \min\{\min\{\mu_1^+(\chi_1), \mu_2^+(\chi_2)\}, \min\{\mu_1^+(\gamma_1), \mu_2^+(\gamma_2)\} \\ &= \min\{\{(\mu_1^+ \times \mu_2^+)(\chi_1, \chi_2)\}, \{(\mu_1^+ \times \mu_2^+)(\gamma_1, \gamma_2)\}\}\end{aligned}$$

and

$$\begin{aligned}(\mu_1^- \times \mu_2^-)(\chi_1 \circ \gamma_1, \chi_2 \circ \gamma_2) &= \max\{\mu_1^-(\chi_1 \circ \gamma_1), \mu_2^-(\chi_2 \circ \gamma_2)\} \\ &\leq \max\{\max\{\mu_1^-(\chi_1), \mu_1^-(\gamma_1)\}, \max\{\mu_2^-(\chi_2), \mu_2^-(\gamma_2)\} \\ &= \max\{\max\{\mu_1^-(\chi_1), \mu_2^-(\chi_2)\}, \max\{\mu_1^-(\gamma_1), \mu_2^-(\gamma_2)\} \\ &= \max\{\{(\mu_1^- \times \mu_2^-)(\chi_1, \chi_2)\}, \{(\mu_1^- \times \mu_2^-)(\gamma_1, \gamma_2)\}\}\end{aligned}$$

Therefore $B_1 \times B_2$ is a bipolar fuzzy *k*-ideal of $\aleph \times \aleph$.

Acknowledgement

The authors have greatly appreciated the referees for their valuable comments and suggestions for improving the paper.

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