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Some Properties of $\oplus-$Cofinitely $\delta-$Supplemented Modules

Figen ERYILMAZ*

Ondokuz Mayıs University, Faculty of Education, Department of Mathematics Education, Samsun, Turkey

fyuzbasi@omu.edu.tr, ORCID: 0000-0002-4178-971X

Abstract

In this paper, we study the properties of generalized $\oplus-$cofinitely $\delta-$supplemented modules or briefly $\oplus-gcof_{\delta}-$supplemented modules. We show that any direct sum of $\oplus-gcof_{\delta}-$supplemented modules is a $\oplus-gcof_{\delta}-$supplemented module. If M is a $\oplus-gcof_{\delta}-$supplemented module with SSP, then every direct summand of M is $\oplus-gcof_{\delta}-$ supplemented.

Keywords: $\delta-$small submodule, $\delta-$supplemented module, Cofinite submodule, $\oplus-$cofinitely supplemented module

$\oplus-$Dual Sonlu $\delta-$Tümlenmiş Modüllerin Bazı Özellikleri

Öz

Bu çalışmada, genelleştirilmiş $\oplus-$dual sonlu $\delta-$tümlenmiş modüllerin özellikleri çalışıldı. Bu modüller kısaca $\oplus-gcof_{\delta}$ ile gösterildi. Genelleştirilmiş $\oplus-$dual sonlu $\delta-$tümlenmiş modüllerin keyfi toplamının da genelleştirilmiş $\oplus-$dual sonlu $\delta-$tümlenmiş modül olduğu gösterildi. M modülünün direkt toplam terimlerinin toplama özelliğine sahip (DDT) genelleştirilmiş $\oplus-$dual sonlu $\delta-$tümlenmiş bir modül olması durumunda M modülünün her bir direkt toplam teriminin de genelleştirilmiş $\oplus-$dual sonlu $\delta-$tümlenmiş modül olduğu ispatlandı.

* Corresponding Author

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Anahtar Kelimeler: $\delta-$küçük alt modül, $\delta-$tümlenmiş modül, Dual sonlu alt modül, $\oplus-$ dual sonlu tümlenmiş modül

1. Introduction

In this study D is used to show a ring which is associative and has an identity. All mentioned modules will be unital left $D-$module. The notation $A \leq B$ means A is a submodule of B. Any submodule A of an $D-$module B is called *small* in B and showed by $A \ll B$ whenever $A+C \neq B$ for all proper submodule C of B. Dually, a submodule A of a $D-$module B is called to be *essential* in B which is showed by $A \trianglelefteq B$ where $A \cap K \neq 0$ for each nonzero submodule K of B. A module B is called *singular* when $B \cong \frac{A}{K}$ for any module A and a submodule K of A with $K \trianglelefteq A$.

Zhou firstly mentioned the definiton of "$\delta-$small submodule" as a generalization of small submodules in [1]. Remember that a submodule A of a module B is called as $\delta-$*small* in B and which is showed by $A \ll_{\delta} B$ if $B \neq A+X$ for any submodule X of B where $\frac{B}{X}$ singular. The symbol $\delta(B)$ will be used for the sum of all $\delta-$small submodules, that represents a preradical on the category of $D-$modules.

Let A and K be submodules of B. Then A is called a *supplement* of K in B when A is minimal with the property $B=A+K$; in other words $B=A+K$ and $A \cap K \ll A$. Definiton of *supplemented module* B is every submodule of B has a supplement in B. There are a lot of papers related with supplemented modules. One can examine the manuscripts [2,3].

Let A be a submodule of B. The submodule K is called a $\delta-$*supplement* of A in B if $B=A+K$ and $B \neq A+X$ for any proper submodule X of K where $\frac{K}{X}$ singular, in other words $B=A+K$ and $A \cap K \ll_{\delta} K$. Therefore B is called $\delta-$supplemented if all submodules of B have $\delta-$supplement in B [4,5]. Nevertheless, B is said to be $\oplus-\delta-$*supplemented* whether all submodules of B have $\oplus-\delta-$supplement that is a direct summand of B [6].

A submodule A is named with *cofinite* in B as quotient module $\frac{B}{A}$ is finitely generated. Also the module B is named with "*cofinitely supplemented* if every cofinite submodule has a supplement in B" [7].

Following [8], if all submodules of B have a supplement which is a direct summand of B, then B is named with $\oplus-$*supplemented.* In [9], $\oplus-$cofinitely supplemented modules was examined and founded as a proper generalization of $\oplus-$supplemented modules,. Any module B is named $\oplus-$*cofinitely supplemented* if each cofinite submodule of B get a supplement which is a direct summand of B. As a result of this definition, finitely generated $\oplus-$cofinitely supplemented modules are already $\oplus-$supplemented. Basic properties of these modules we refer to [10,11]. Another generalization of these modules was studied in [12].

According to [13], a $D-$module B is named as $\oplus-cof_{\delta}-$*supplemented* if all cofinite submodules of B have $\delta-$supplement which is a direct sum term of B. Some properties of these modules we refer to [14, 15].

Talebi defined generalized $\delta-$supplemented modules. He called a submodule A of B is a *generalized* $\delta-$*supplement* submodule of B if one can find a submodule K of B where $B=A+K$ and $A\cap K\leq\delta(A)$. A module B is called a *generalized* $\delta-$*supplemented* or shortly $\delta-$GS if each submodule of B possessed of a generalized $\delta-$supplement in B [16].

B is called *generalized* $\oplus-$ *cofinitely* $\delta-$*supplemented* module provided that each cofinite submodule of B possessed of a generalized $\delta-$supplement with direct summand of B. In place of writing generalized $\oplus-$cofinitely $\delta-$supplemented module, we choose to use $\oplus-gcof_{\delta}-$*supplemented.* In the next section, some fundamental properties of $\oplus-gcof_{\delta}-$supplemented modules will be examined.

2. Main Results

Theorem 1. Any direct sum of $\oplus-gcof_{\delta}-$supplemented modules is a $\oplus-gcof_{\delta}-$supplemented module for any ring D.

Proof. Let $\{B_i\}_{i\in I}$ be a collection of generalized $\oplus-$cofinitely $\delta-$supplemented modules over an arbitrary ring D and let $B=\underset{i\in I}{\oplus}B_i$. Suppose that A is a cofinite submodule of B. Then $B=A+\left(\overset{n}{\underset{j=1}{\oplus}}B_{i_j}\right)$ can be written and it is easy to find that $\{0\}$ is a trivial generalized $\delta-$supplement of $B=B_{i_1}+\left(\left(\overset{n}{\underset{j=2}{\oplus}}B_{i_j}\right)+A\right)$. If we remember the following isomorphisms

$$\frac{B_{i_1}}{B_{i_1}\cap\left(A+\left(\overset{n}{\underset{j=2}{\oplus}}B_{i_j}\right)\right)}\cong\frac{B}{\left(\overset{n}{\underset{j=2}{\oplus}}B_{i_j}\right)+A}\cong\frac{(B/A)}{\left(\left(\overset{n}{\underset{j=2}{\oplus}}B_{i_j}\right)+A\right)/A},$$

then we have $B_{i_1}\cap\left(A+\left(\overset{n}{\underset{j=2}{\oplus}}B_{i_j}\right)\right)$ is a cofinite submodule of B_{i_1}. Onwards B_{i_1} is $\oplus-gcof_\delta-$supplemented, $B_{i_1}\cap\left(A+\left(\overset{n}{\underset{j=2}{\oplus}}B_{i_j}\right)\right)$ has a generalized $\delta-$supplement U_{i_1} in B_{i_1} where U_{i_1} is a direct summand term of B_{i_1}. With Lemma 2.4 in [17], U_{i_1} is a generalized $\delta-$supplement of $A+\left(\overset{n}{\underset{j=2}{\oplus}}B_{i_j}\right)$ in B. U_{i_1} is also a direct summand of B, forwhy B_{i_1} is direct summand of B. If one continues in this way, it will be obtained that A will have generalized $\delta-$supplement $U_{i_1}+U_{i_2}+\cdots+U_{i_j}$ in B such that every U_{i_j} is a direct summand of B_{i_j} for $1\le j\le A$. Since every B_{i_j} is a direct summand of B, one can get $\overset{n}{\underset{j=1}{\oplus}}U_{i_j}$ is a direct summand of B. Therefore, the module B is $\oplus-gcof_\delta-$ supplemented.

Proposition 1. *If B is a $\oplus-gcof_\delta-$supplemented module, then each cofinite submodule of $\frac{B}{\delta(B)}$ is a direct summand.*

Proof. Assume that B is a $\oplus-gcof_{\delta}-$supplemented module. We know that every cofinite submodule of $\frac{B}{\delta(B)}$ has the form $\frac{U}{\delta(B)}$, such as U is a cofinite submodule of B and $\delta(B)\leq U$. By using hypothesis, we get $B=A+U$, $A\cap U\leq\delta(A)$ and $B=A\oplus K$ such that $A,K\leq B$. Since $\delta(A)\leq\delta(B)$, we have $A\cap U\leq\delta(B)$ and so

$$\frac{B}{\delta(B)}=\frac{A+U}{\delta(B)}=\left(\frac{A+\delta(B)}{\delta(B)}\right)\oplus\left(\frac{U}{\delta(B)}\right).$$

Consequently, $\frac{U}{\delta(B)}$ is a direct summand of $\frac{B}{\delta(B)}$.

A submodule A of a $D-$module B is named "*fully invariant* if one have $f(A)\subseteq A$ for all $f\in S$ where where $S=End_D(B)$" [3]. If $B=U\oplus V$ and A is a fully invariant submodule of B, then we obtain $A=(A\cap U)\oplus(A\cap V)$. $\delta(B)$ is a fully invariant submodule of B. A left $D-$module B is called a "*duo module* if any submodule of B is fully invariant" [18].

Proposition 2. *Suppose that B is a $\oplus-gcof_{\delta}-$supplemented module. If U is a fully invariant submodule of B. Then $\frac{B}{U}$ is a $\oplus-gcof_{\delta}-$supplemented module.*

Proof. Assume that $\frac{K}{U}$ is a cofinite submodule of $\frac{B}{U}$. Therefore K is a cofinite submodule of B. As B is a $\oplus-gcof_{\delta}-$supplemented module, one can find C,A submodules of B where $B=K+C$, $K\cap C\leq\delta(C)$ and $B=C\oplus A$. Using Proposition 2.9 in [16], we can see that $\frac{C+U}{U}$ is one of the generalized $\delta-$supplement of $\frac{K}{U}$ in $\frac{B}{U}$. If we remember that $U=(U\cap C)\oplus(U\cap A)$ can be written because of being fully invariant submodule of B, then we have $\frac{B}{U}=\left(\frac{C+U}{U}\right)\oplus\left(\frac{A+U}{U}\right)$. Therefore $\frac{C+U}{U}$ is a generalized $\delta-$supplement of $\frac{K}{U}$ such that $\frac{C+U}{U}$ is a direct summand of $\frac{B}{U}$.

Corollary 1. *Let B be a $\oplus-gcof_{\delta}-$supplemented and a duo module. Then every factor module of B is a $\oplus-gcof_{\delta}-$supplemented module.*

Proposition 3. *Assume that B is $\oplus-gcof_{\delta}-$supplemented, A is a fully invariant submodule of B and a cofinite direct summand of B. Then it's $\oplus-gcof_{\delta}-$ supplemented.*

Proof. Assume that A is a cofinite direct summand of B. Then, there exists a submodule $A^{'}$ of B with $B = A \oplus A^{'}$. If U is cofinite submodule of A, then $\frac{A}{U}$, $A^{'}$ are finitely generated and U is cofinite. As B is $\oplus-gcof_{\delta}-$supplemented, we have $B = U + K$, $U \cap K \le \delta(K)$ and $B = K \oplus K^{'}$ such that $K, K^{'} \le B$. By Lemma 2.1 in [18], we have $A = (A \cap K) \oplus (A \cap K^{'})$. Since $B = U + K$, we have $A = U + A \cap K$. Also $A \cap K$ is a direct summand of B. Hence, $A \cap K$ is a $\delta-$supplement submodule in B. By using Lemma 2.2 in [16], we can obtain $U \cap (A \cap K) \le \delta(A \cap K)$. Thus $A \cap K$ is a generalized $\delta-$supplement of U in A. This means A is $\oplus-gcof_{\delta}-$supplemented.

Theorem 2. *Assume that B is direct sum of submodules B_1 and B_2. B_2 is $\oplus-gcof_{\delta}-$supplemented $\Leftrightarrow$ there is a submodule K of B_2 with K is one of the direct summand of B and $B = U + K$, $U \cap K \le \delta(K)$ for each cofinite submodule $\frac{U}{B_1}$ of $\frac{B}{B_1}$.*

Proof. Let $\frac{U}{B_1}$ be any cofinite submodule of $\frac{B}{B_1}$. If we remember $\frac{(B/B_1)}{(U/B_1)} \cong \frac{B}{U}$ and $\frac{B}{U} \cong \frac{B_2}{(U \cap B_2)}$, then it follows that $U \cap B_2$ is a cofinite submodule of B_2. By the assumption, there are $K, K^{'}$ submodules of B_2 with $B_2 = (U \cap B_2) + K$, $(U \cap B_2) \cap K \le \delta(K)$ and $B_2 = K \oplus K^{'}$. Therefore the equalities $B = B_1 + B_2 = B_1 + (U \cap B_2) + K = U + K$ can be obtained. Also we get $(U \cap B_2) \cap K = U \cap K \le \delta(K)$ and so K is direct summand of B.

Conversely, let's take A as any cofinite submodule of B_2. Note that

$$\frac{\left(\frac{B}{B_1}\right)}{\left(\frac{(A+B_1)}{B_1}\right)} \cong \frac{B}{(A+B_1)} = \frac{(A+B_1+B_2)}{(A+B_1)} \cong \frac{B_2}{(B_2 \cap (A+B_1))} = \frac{B_2}{A}.$$

Since the last module $\frac{B_2}{A}$ is finitely generated , $\frac{(A+B_1)}{B_1}$ is a cofinite submodule of $\frac{B}{B_1}$. By the assumption , we have a submodule K in B_2 where K is a direct summand of B with $B = K + A + B_1$ and $(A+B_1) \cap K \leq \delta(K)$. Then it follows that $B_2 = A + K$ and $A \cap K \leq \delta(K)$ and so B_2 is $\oplus - gcof_\delta -$supplemented.

Proposition 4. *Let B be $\oplus - gcof_\delta - supplemented$ with $\delta(B) =_\delta B$. Then B is a $\oplus - cof_\delta - supplemented$ module.*

Proof. Let A be any cofinite submodule of B. As B is $\oplus - gcof_\delta -$supplemented, there are submodules K and K' of B where $B = A + K$ and $A \cap K \leq \delta(K)$, $B = K \oplus K'$. Remember that $A \cap K \leq \delta(K) \leq \delta(B) =_\delta B$. Therefore $A \cap K =_\delta K$ by Lemma 1.1 in [14]. As a result, B is $\oplus - cof_\delta -$supplemented.

Theorem 3. *Let B be $\oplus - gcof_\delta - supplemented$ and $U \leq B$. If $\frac{(U+W)}{U}$ is a direct summand of $\frac{B}{U}$ for all direct summand of W of B, then $\frac{B}{U}$ is $\oplus - gcof_\delta -$ supplemented.*

Proof. Assume that $\frac{A}{U}$ is a cofinite submodule in $\frac{B}{U}$ where A cofinite submodule of B and $U \leq A$. Since B is a $\oplus - gcof_\delta -$supplemented module, one can find a direct summand V' of B such that $B = A + W$, $A \cap W \leq \delta(W)$ and $B = W \oplus V'$ where V' is any submodule of B. Now, we have $\frac{B}{U} = \frac{A}{U} + \left(\frac{U+W}{U}\right)$. Also, by hypothesis, $\frac{U+W}{U}$ is a direct summand of $\frac{B}{U}$. Let $f: B \to \frac{B}{U}$ be canonical epimorphism. Since $A \cap W \le \delta(W)$ and

$$\left(\frac{A}{U}\right) \cap \left(\frac{U+W}{U}\right) = \frac{A \cap (U+W)}{U} = \frac{U + (A \cap W)}{U}$$
$$= f(A \cap W) \le f(\delta(W)) \le \delta\left(\frac{U+W}{U}\right)$$

by Lemma 1.5 in [1], it follows that $\frac{U+W}{U}$ is generalized $\delta-$supplement of $\frac{A}{U}$ in $\frac{B}{U}$ which is a direct summand.

A $D-$module B has *SSP* *"*(*Summand Sum Property*) if the sum of two direct summand of B is again a direct summand of B *"* [3].

Theorem 4. *If* B *is* $\oplus - gcof_{\delta} -$*supplemented with SS property, then each direct summand of* B *is* $\oplus - gcof_{\delta} -$*supplemented.*

Proof. Assume that U_1 is a direct summand of B. Therefore we get $B = U_1 \oplus U'$ for $U' \le B$. Let A be a direct summand of B. Having SS property of B, we can write that $B = (U' + A) \oplus K$ such that $K \le B$. Thus, the equality $\frac{B}{U'} = \frac{(U' + A)}{U'} \oplus \frac{(K + U')}{U'}$ implies that $\frac{B}{U'}$ is a $\oplus - gcof_{\delta} -$supplemented module by Theorem 3.

We already know from [19] that, a $D-$module B is named *weakly distributive* if each submodule of B is a weak distributive submodule of B, H is called a *weak distributive submodule* of B if $H = (H \cap M) + (H \cap N)$ for all submodules of B where $B = M + N$.

Theorem 5. *If* B *is* $\oplus - gcof_{\delta} -$*supplemented and weakly distributive, then* $\frac{B}{E}$ *is* $\oplus - gcof_{\delta} -$*supplemented for each submodule* E *of* B.

Proof. Suppose that M is a direct summand of B. Then $B = M \oplus N$ for some submodule N of B and we can write $\frac{B}{E} = \left(\frac{E+M}{E}\right) + \left(\frac{E+N}{E}\right)$. By distributive property of B, we have $E = E + (M \cap N) = (E+M) \cap (E+N)$. This equality says that $\frac{B}{E} = \left[\frac{E+M}{E}\right] \oplus \left[\frac{E+N}{E}\right]$ and therefore $\frac{B}{E}$ is a $\oplus - gcof_{\delta}-$supplemented module by Theorem 3.

A $D-$module B is said to have property (D_3) "if B_1 and B_2 are direct summands of B with $B = B_1 + B_2$, then $B_1 \cap B_2$ is also a direct summand of B" [2].

Theorem 6. *If B is a $\oplus - gcof_{\delta}-$module with the property (D_3), then all cofinite direct summand terms of B is a $\oplus - gcof_{\delta}-$module.*

Proof. Assume that U is any cofinite direct summand of B. Then we can find a submodule U' of B where $B = U \oplus U'$ and U' is finitely generated. If A is any cofinite submodule of U, then A is also cofinite submodule of B by the fact that $\frac{B}{A} \cong \frac{U}{A} \oplus U'$ is finitely generated. Since B is $\oplus - gcof_{\delta}$, we have a direct summand W of B whereas $B = A + W$, $A \cap W \leq \delta(W)$. Note that $B = U + W$ and $U = A + (U \cap W)$. Having (D_3) property of B implies that $U \cap W$ is a direct summand of B. Thus $A \cap (W \cap U) = A \cap W \leq \delta(W \cap U)$ by using Lemma 2.2 of [16]. Furthermore $U \cap W$ is a direct summand of U forwhy so does $W \cap U$ of B. Hence U is $\oplus - gcof_{\delta}-$supplemented.

Lastly, there is a module example which is $\oplus - gcof_{\delta}-$supplemented but not $\oplus - \delta -$supplemented.

Example 1. [13, Example 1]. *Let $D = \mathbb{Z}$ and $B = \bigoplus_{i \in I} B_i$, with each $B_i = \mathbb{Z}_{p^{\infty}}$, where p is a prime number. Then B_i are $\delta-$supplemented. By Theorem 1, B is $\oplus - gcof_{\delta}-$supplemented. However B is not $\oplus - \delta -$supplemented.*

References

[1] Zhou, Y., *Generalizations of perfect, semiperfect, and semiregular rings*, Algebra Colloqium, 7(3), 305-318, 2000.

[2] Mohamed, S.H., Müller, B.J., Continuous and Discrete Modules, London Mathematical Society, Cambridge University Press, Cambridge, 1990.

[3] Wisbauer, R., Foundations of Module and Ring Theory, Gordon and Breach, Philadelphia, 1991.

[4] Koşan, M.T., $\delta-$*lifting and* $\delta-$*supplemented modules,* Algebra Colloqium, 14(1), 53-60, 2007.

[5] Wang, Y., $\delta-$*small submodules and* $\delta-$*supplemented Modules*, International Journal of Mathematics and Mathematical Sciences, Article ID 58132, 8 pages, 2007.

[6] Ungor, B., Halıcıoğlu, S., Harmancı, A., *On a class of* $\delta-$*supplemented Modules*, Bulletin of the Malaysian Mathematical Sciences Society (2), 37(3), 703-717, 2014.

[7] Alizade, R., Bilhan, G., Smith, P.F., *Modules whose maximal submodules have supplements*, Communications in Algebra, 29, 2389-2405, 2001.

[8] Harmancı, A., Keskin, D., Smith, P.F., *On* $\oplus-$*supplemented modules,* Acta Mathematica Hungarica, 83, 161-169, 1999.

[9] Çalışıcı, H., Pancar, A., $\oplus-$*cofinitely supplemented modules*, Czechoslovak Mathematical Journal, 54 (129), 1083-1088, 2004.

[10] Wang, Y., Sun, Q., *A note on* $\oplus-$*cofinitely supplemented modules*, International Journal of Mathematics and Mathematical Sciences, Article ID 10836, 5 pages, 2007.

[11] Zeng, Q., $\oplus-$*cofinitely supplemented modules*, Southeast Asian Bulletin of Mathematics, 33, 805-819, 2009.

[12] Nişancı, B., Pancar, A., *On generalization of* $\oplus-$*cofinitely supplemented modules*, Ukrainian Mathematical Journal, 62(2), 203-209, 2010.

[13] Thuyet, L.V., Koşan, M.T., Quynh, T.C., *On cofinitely* $\delta-$*semiperfect modules*, Acta Mathematica Vietnamica, 33(2), 197-207, 2008.

[14] Alattass, A.Ö., *Cofinitely* δ_M-*supplemented and cofinitely* δ_M-*semiperfect modules*, International Journal of Algebra, 5(32), 1575-1588, 2011.

[15] Al-Takhman, K., *Cofinitely* $\delta-$*supplemented and cofinitely* $\delta-$*semiperfect modules*, International Journal of Algebra, 1(12), 601-613, 2007.

[16] Talebi, Y., Talaee, B., *On generalized* $\delta-$*supplemented modules*, Vietnam Journal of Mathematics, 37(4), 515-525, 2009.

[17] Yüzbaşı, F., Eren, Ş., *Generalized cofinitely* $\delta-$*semiperfect modules*, Analele Ştiinţ. University of Alexandru Ioan Cuza Iaşi Secţ. I. Mathematics, 59(2), 269-280, 2013.

[18] Özcan, A.Ç., Harmancı, A., Smith, P.F., *Duo modules*, Glasgow Mathematical Journal, Trust, 48, 533-545, 2006.

[19] Büyükaşık, E., Demirci, Y.M., *Weakly distributive modules, applications to supplement submodules*, Proceedings of the Indian Academy of Sciences: Mathematical Sciences, 120 (5), 525-534, 2010.