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The Matrix Representation of A Rule of Cellular Automata and An Application to Coding Theory

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Abstract

In this paper we studied the behavior of a family of three dimensional cellular automata under periodic boundary condition by using matrix algebra. We obtained representation matrix of the this family with the help of polinomal algebra. We gave an application of obtained block matrices to coding theory over the ternary field.

Keywords: Three dimensional cellular automata, Rule matrix, Error correcting codes

Hücresel Dönüşümlerin Bir Kuralının Matris Temsili ve Kodlama Teorisinde Bir Uygulaması

Öz

Bu çalışmada, matris cebiri yardımıyla üç boyutlu bir hücresel dönüşüm ailesinin periyodik sınır şartı altında davranışını inceledik. Polinom cebiri yardımıyla bu ailenin temsil matrisini elde ettik. Elde edilen blok matrislerin üçlü cisimler üzerinde bir kodlama teorisi uygulamasını verdik.

Anahtar Kelimeler: Üç boyutlu hücresel dönüşüm, Kural matris, Hata düzelten kodlar

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1. Introduction and Basics

Three dimensional cellular automata (3D-CA) have been studied a lot recently for their applications in many areas. The state space of these works are mainly binary field with two elements $0, 1$ and so called as binary 3D-CA. One dimensional cellular automata (1D-CA) originally was introduced by Ulam and von Neumann in [1] and Wolfram investigated the complex behavior of 1D-CA rules (see [2]).

For a particular step of time,which we call t, each cell of cellular grid has a state value and synchronously updates its state at the next time step $t+1$ depending on its neighbors and local rule. If this dependence is formulated by a relation amongst the neighbors of the cell that is applied to all cells at each time step then these CA are called regular. Regular CA is model of different physical events or applications. Besides all these applications, the reversibility problem of CA is studied as a crucial research topic due to its important role in many applications.

The study of reversibility of CAs have received remarkable attention in the last few years due to its several applications in many disciplines (e.g., mathematics, physics, computer science, biology (see [3]), chemistry and so on) with different purposes (e.g., simulation of natural phenomena, pseudo-random number generation, image processing, analysis of universal model of computations, cryptography) (see [4]). For some of these applications, the inverse of CA are computed (see [5-10]). Most of these works done over one and two dimensional cellular automata (see [10-16]).

However, lately three dimensional cellular automata hasn't just much investigated, Hemmingson studied behavior of 3D-CA in [17]. Tsalides et al. studied the characterization of 3D-CA with the help of matrix algebra in [18]. They obtained matrix algebraic formulas concerning some exceptional rules of 3D-CA.Youbin et al. investigated 3D-CA model for HIV dynamics. in [19].

In this work, we define 3D-CA and then we obtain representation matrix for characterizing via matrix algebra. Finally we make an application about with coding theory over the ternary field.

2. Three Dimensional Cellular Automata

In this section ,we describe of 3D-CAs over the field $\mathbb{Z}_m$ with the aid of some local rules. Let $\mathbb{Z}_m$ be states set and $\mathbb{Z}_m^{\mathbb{Z}^3}$ is cells spaces. £ is local rule and F is global transition function

$$£: \mathbb{Z}_m^{\mathbb{Z}^3} \longrightarrow \mathbb{Z}_m,\ F: \mathbb{Z}_m^{\mathbb{Z}^3} \longrightarrow \mathbb{Z}_m^{\mathbb{Z}^3}.$$

For 3D-CA there is some classical type of neighborhoods.In this work, we only restrict ourselves to the adjacent neighbors which have found in more applications and they are very common cases. So, we define the $(t+1)^{th}$ state of the $(i,j,k)^{th}$ cell as the following.

$$x_{(i,j,k)}^{t+1} = £\begin{pmatrix} x_{(i-1,j-1,k-1)}^{(t)}, x_{(i-1,j,k-1)}^{(t)}, x_{(i-1,j,k+1)}^{(t)}, x_{(i-1,j-1,k)}^{(t)}, \\ x_{(i-1,j-1,k+1)}^{(t)}, x_{(i-1,j,k)}^{(t)}, x_{(i-1,j+1,k)}^{(t)}, x_{(i-1,j+1,k-1)}^{(t)}, \\ x_{(i-1,j+1,k+1)}^{(t)}, x_{(i,j-1,k-1)}^{(t)}, x_{(i,j,k-1)}^{(t)}, x_{(i,j,k+1)}^{(t)}, \\ x_{(i,j-1,k)}^{(t)}, x_{(i,j-1,k+1)}^{(t)},, x_{(i,j,k)}^{(t)}, x_{(i,j+1,k)}^{(t)}, \\ x_{(i,,j+1,k-1)}^{(t)}, x_{(i,,j+1,k+1)}^{(t)}, x_{(i+1,j-1,k-1)}^{(t)}, x_{(i+1,j,k-1)}^{(t)}, \\ x_{(i+1,j,k+1)}^{(t)}, x_{(i+1,j-1,k)}^{(t)}, x_{(i+1,j-1,k+1)}^{(t)}, x_{(i+1,j,k)}^{(t)}, \\ x_{(i+1,j+1,k)}^{(t)}, x_{(i+1,j+1,k-1),}^{(t)} x_{(i+1,j+1,k+1)}^{(t)}) \end{pmatrix} \tag{1}$$

$$= a_0 x_{(i-1,j-1,k-1)}^{(t+1)} + a_1 x_{(i-1,j,k-1)}^{(t+1)} + \cdots + a_{26} x_{(i+1,j+1,k+1)}^{(t+1)} (modm).$$

The value of each cell for the next state may not depend upon all 27 neighbors.The linear combination of the neighboring cells on which each cell value determines the rule number of the 3D-CA.Regarding the neighborhood of the extreme cells, there exist some approaches (for details see [20]). we can use periodic boundary condition.Now we can define it as follows:

A Periodic Boundary CA is the one in which the extreme cells are adjacent to each other.

In this paper, in order to obtain representation matrix for characterizing 3D-CA, we can use the following local rule,which help of defining the rule matrix:

$$x_{(i,j,k)}^{t+1} = a.x_{(i,j,k+1)}^{(t+1)} + b.x_{(i,j+1,k)}^{(t+1)} + c.x_{(i,j-1,k)}^{(t+1)} \tag{2}$$

$$+d.x_{(i,j,k-1)}^{(t+1)} + e.x_{(i-1,j,k)}^{(t+1)} + f.x_{(i+1,j,k)}^{(t+1)}$$

where $a, b, c, d, e, f \in Z_m - \{0\}$.

In order to characterize 3-D PBCA with the local rules in Eq. (2), we get rule matrix for $m, n, s \geq 3$ $(m, n, s \in \mathbb{Z}^+)$ as follows:

$$(T_{RP})_{mns\times mns} = \begin{pmatrix} K_s & E_s & O_s & O_s & \dots & O_s & O_s & F_s \\ F_s & K_s & E_s & O_s & \dots & O_s & O_s & O_s \\ O_s & F_s & K_s & E_s & \dots & O_s & O_s & O_s \\ O_s & O_s & F_s & K_s & \dots & O_s & O_s & O_s \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ O_s & O_s & O_s & O_s & \dots & K_s & E_s & O_s \\ O_s & O_s & O_s & O_s & \dots & F_s & K_s & E_s \\ E_s & O_s & O_s & O_s & \dots & O_s & F_s & K_s \end{pmatrix},$$

K_s, E_s, O_s, F_s are $s\times s$ block matrices where $s = m\times n$.

Their sub matrices are as follows:

$$K_s = \begin{pmatrix} S_n(c,b) & d.I_n & O_n & \dots & O_n & a.I_n \\ a.I_n & S_n(c,b) & d.I_n & \dots & O_n & O_n \\ O_n & a.I_n & S_n(c,b) & \dots & O_n & O_n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ O_n & O_n & O_n & \dots & S_n(c,b) & d.I_n \\ d.I_n & O_n & O_n & \dots & a.I_n & S_n(c,b) \end{pmatrix}_{ns\times ns.},$$

$$E_s = \begin{pmatrix} e.I_n & O_n & O_n & \dots & O_n & O_n \\ O_n & e.I_n & O_n & \dots & O_n & O_n \\ O_n & O_s & e.I_n & \dots & O_n & O_n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ O_n & O_n & O_n & \dots & e.I_n & O_n \\ O_n & O_n & O_n & \dots & O_n & e.I_n \end{pmatrix}_{ns\times ns},$$

$$F_s = \begin{pmatrix} f.I_n & O_n & O_n & \dots & O_n & O_n \\ O_n & f.I_n & O_s & \dots & O_n & O_n \\ O_n & O_n & f.I_n & \dots & O_n & O_n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ O_n & O_n & O_n & \dots & f.I_n & O_n \\ O_n & O_n & O_n & \dots & O_n & f.I_n \end{pmatrix}_{ns\times ns},$$

$$O_s = \begin{pmatrix} O_n & O_n & O_n & \dots & O_n & O_n \\ O_n & O_n & O_n & \dots & O_n & O_n \\ O_n & O_n & O_n & \dots & O_n & O_n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ O_n & O_n & O_n & \dots & O_n & O_n \\ O_n & O_n & O_n & \dots & O_n & O_n \end{pmatrix}_{ns\times ns}.$$

I_n is $n\times n$ identity matrix. O_n is $n\times n$ zero matrix and then $S_n(c,b)$ is as follow:

$$S_n(c,b) = \begin{pmatrix} 0 & b & 0 & 0 & \dots & 0 & 0 & c \\ c & 0 & b & 0 & \dots & 0 & 0 & 0 \\ 0 & c & 0 & b & \dots & 0 & 0 & 0 \\ 0 & 0 & c & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & b & 0 \\ 0 & 0 & 0 & 0 & \dots & c & 0 & b \\ b & 0 & 0 & 0 & \dots & 0 & c & 0 \end{pmatrix}_{n\times n}.$$

Example 1. If we take $m = 3, n = 3,\ s = 3,$ then we get the rule matrix T_{RP} of order 27×27. In this situation we have 5 configurations and then we consider a configuration of size 3×3×3 with periodic boundary condition.

$$\begin{matrix} x_{131} & x_{111} & x_{121} & x_{131} & x_{111} \\ x_{133} & x_{113} & x_{123} & x_{133} & x_{113} \\ x_{132} & x_{112} & x_{122} & x_{132} & x_{112}, \\ x_{131} & x_{111} & x_{121} & x_{131} & x_{111} \\ x_{133} & x_{113} & x_{123} & x_{133} & x_{113} \end{matrix}$$

$$\begin{matrix} x_{331} & x_{311} & x_{321} & x_{331} & x_{311} \\ x_{333} & x_{313} & x_{323} & x_{333} & x_{313} \\ x_{332} & x_{312} & x_{322} & x_{332} & x_{312}, \\ x_{331} & x_{311} & x_{321} & x_{331} & x_{311} \\ x_{333} & x_{313} & x_{323} & x_{333} & x_{313} \end{matrix}$$

$$\begin{matrix} x_{231} & x_{211} & x_{221} & x_{231} & x_{211} \\ x_{233} & x_{213} & x_{223} & x_{233} & x_{213} \\ x_{232} & x_{212} & x_{222} & x_{232} & x_{212}, \\ x_{231} & x_{211} & x_{221} & x_{231} & x_{211} \\ x_{233} & x_{213} & x_{223} & x_{233} & x_{213} \end{matrix}$$

$$
\begin{matrix}
x_{131} & x_{111} & x_{121} & x_{131} & x_{111} \\
x_{133} & x_{113} & x_{123} & x_{133} & x_{113} \\
x_{132} & x_{112} & x_{122} & x_{132} & x_{112}, \\
x_{131} & x_{111} & x_{121} & x_{131} & x_{111} \\
x_{133} & x_{113} & x_{123} & x_{133} & x_{113}
\end{matrix}
$$

$$
\begin{matrix}
x_{331} & x_{311} & x_{321} & x_{331} & x_{311} \\
x_{333} & x_{313} & x_{323} & x_{333} & x_{313} \\
x_{332} & x_{312} & x_{322} & x_{332} & x_{312}, \\
x_{331} & x_{311} & x_{321} & x_{331} & x_{311} \\
x_{333} & x_{313} & x_{323} & x_{333} & x_{313}
\end{matrix}
$$

we apply local rule all the cells and than we obtain new configurations is as follow:

$$b.x_{323} + d.x_{312} + c.x_{333} + a.x_{311} + e.x_{213} + f.x_{113} = y_{313}$$

$$b.x_{333} + d.x_{322} + c.x_{313} + a.x_{321} + e.x_{223} + f.x_{123} = y_{323}$$

$$b.x_{313} + d.x_{332} + c.x_{323} + a.x_{331} + e.x_{233} + f.x_{113} = y_{333}$$

$$b.x_{322} + d.x_{311} + c.x_{332} + a.x_{313} + e.x_{212} + f.x_{112} = y_{312}$$

$$b.x_{332} + d.x_{321} + c.x_{312} + a.x_{323} + e.x_{222} + f.x_{122} = y_{322}$$

$$b.x_{312} + d.x_{331} + c.x_{322} + a.x_{333} + e.x_{232} + f.x_{132} = y_{332}$$

$$b.x_{321} + d.x_{313} + c.x_{331} + a.x_{312} + e.x_{211} + f.x_{111} = y_{311}$$

$$b.x_{331} + d.x_{323} + c.x_{311} + a.x_{322} + e.x_{221} + f.x_{121} = y_{321}$$

$$b.x_{311} + d.x_{333} + c.x_{321} + a.x_{332} + e.x_{231} + f.x_{131} = y_{331}$$

$$b.x_{223} + d.x_{212} + c.x_{233} + a.x_{211} + e.x_{113} + f.x_{313} = y_{213}$$

$$b.x_{233} + d.x_{222} + c.x_{213} + a.x_{221} + e.x_{123} + f.x_{323} = y_{223}$$

$$b.x_{213} + d.x_{232} + c.x_{223} + a.x_{231} + e.x_{133} + f.x_{333} = y_{233}$$

$$b.x_{222} + d.x_{211} + c.x_{232} + a.x_{213} + e.x_{112} + f.x_{312} = y_{212}$$

$$b.x_{232} + d.x_{221} + c.x_{212} + a.x_{223} + e.x_{122} + f.x_{322} = y_{222}$$

$$b.x_{212} + d.x_{231} + c.x_{222} + a.x_{233} + e.x_{132} + f.x_{332} = y_{232}$$

$$b.x_{221} + d.x_{213} + c.x_{231} + a.x_{212} + e.x_{111} + f.x_{311} = y_{211}$$

$$b.x_{231} + d.x_{223} + c.x_{211} + a.x_{222} + e.x_{121} + f.x_{321} = y_{221}$$

$$b.x_{211} + d.x_{233} + c.x_{221} + a.x_{232} + e.x_{131} + f.x_{331} = y_{231}$$

$$b.x_{123} + d.x_{112} + c.x_{133} + a.x_{111} + e.x_{313} + f.x_{213} = y_{113}$$

$$b.x_{133} + d.x_{122} + c.x_{113} + a.x_{121} + e.x_{323} + f.x_{223} = y_{123}$$

$$b.x_{113} + d.x_{132} + c.x_{123} + a.x_{131} + e.x_{333} + f.x_{233} = y_{133}$$

$$b.x_{122} + d.x_{111} + c.x_{132} + a.x_{113} + e.x_{312} + f.x_{212} = y_{112}$$

$$b.x_{132} + d.x_{121} + c.x_{112} + a.x_{123} + e.x_{322} + f.x_{222} = y_{122}$$

$$b.x_{112} + d.x_{131} + c.x_{122} + a.x_{133} + e.x_{332} + f.x_{232} = y_{132}$$

$$b.x_{121} + d.x_{113} + c.x_{131} + a.x_{112} + e.x_{311} + f.x_{211} = y_{111}$$

$$b.x_{131} + d.x_{123} + c.x_{111} + a.x_{122} + e.x_{321} + f.x_{221} = y_{121}$$

$$b.x_{111} + d.x_{133} + c.x_{121} + a.x_{132} + e.x_{331} + f.x_{231} = y_{131}.$$

In order to obtain represantation matrix T_{RP} corresponding to the local rule applied over al the cells , we evaluate the basis vector as follows:

$$T_{RP}(E_1) = T_{RP}(100000000000000000000000000)^T$$

$$= (0\ c\ b\ a\ 0\ 0\ d\ 0\ 0\ f\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ e\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)^T,$$

$$T_{RP}(E_2) = T_{RP}(010000000000000000000000000)^T$$

$$= (b\ 0\ c\ 0\ a\ 0\ 0\ d\ 0\ 0\ f\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ e\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)^T.$$

Transpose of $T_{RP}(E_1)$ and $T_{RP}(E_2)$ compose first and second columns of represantation matrix T_{RP}.we can similarly obtain the rest of the columns and we get represantation matrix $(T_{RP})_{27\times27}$ as follow:

$$
\begin{pmatrix}
0 & b & c & d & 0 & 0 & a & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c & 0 & b & 0 & d & 0 & 0 & a & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b & c & 0 & 0 & 0 & d & 0 & 0 & a & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 & b & c & d & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 & 0 & 0 \\
0 & a & 0 & c & 0 & b & 0 & d & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 & 0 \\
0 & 0 & a & b & c & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 \\
d & 0 & 0 & a & 0 & 0 & 0 & b & c & 0 & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 \\
0 & d & 0 & 0 & a & 0 & c & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 \\
0 & 0 & d & 0 & 0 & a & b & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f \\
f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & c & d & 0 & 0 & a & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & b & 0 & d & 0 & 0 & a & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & f & 0 & 0 & 0 & 0 & 0 & 0 & b & c & 0 & 0 & 0 & d & 0 & 0 & a & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & f & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & b & c & d & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & f & 0 & 0 & 0 & 0 & 0 & a & 0 & c & 0 & b & 0 & d & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 & 0 & 0 & a & b & c & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & d & 0 & 0 & a & 0 & 0 & 0 & b & c & 0 & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & d & 0 & 0 & a & 0 & c & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & d & 0 & 0 & a & b & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e \\
e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & c & d & 0 & 0 & a & 0 & 0 \\
0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & b & 0 & d & 0 & 0 & a & 0 \\
0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 & 0 & 0 & 0 & b & c & 0 & 0 & 0 & d & 0 & 0 & a \\
0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & b & c & d & 0 & 0 \\
0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 & 0 & 0 & a & 0 & c & 0 & b & 0 & d & 0 \\
0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 & 0 & 0 & a & b & c & 0 & 0 & 0 & d \\
0 & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & d & 0 & 0 & a & 0 & 0 & 0 & b & c \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & d & 0 & 0 & a & 0 & c & 0 & b \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & d & 0 & 0 & a & b & c & 0
\end{pmatrix}_{27\times 27}
$$

$$
= \begin{pmatrix} K_3 & E_3 & F_3 \\ F_3 & K_3 & E_3 \\ E_3 & F_3 & K_3 \end{pmatrix}_{27\times 27} .
$$

K_3, E_3 , F_3, O_3 are block matrices of $(T_{RP})_{27\times 27}$ which are given as follows:

$$
K_3 = \begin{pmatrix} S_3(c,b) & d.I_3 & O_3 \\ a.I_3 & S_3(c,b) & d.I_3 \\ O_3 & a.I_3 & S_3(c,b) \end{pmatrix}_{9\times 9} , \quad E_3 = \begin{pmatrix} e.I_3 & O_3 & O_3 \\ O_3 & e.I_3 & O_3 \\ O_3 & O_3 & e.I_3 \end{pmatrix}_{9\times 9} ,
$$

$$
F_3 = \begin{pmatrix} f.I_3 & O_3 & O_3 \\ O_3 & f.I_3 & O_3 \\ O_3 & O_3 & f.I_3 \end{pmatrix}_{9\times 9} , \qquad O_3 = \begin{pmatrix} O_3 & O_3 & O_3 \\ O_3 & O_3 & O_3 \\ O_3 & O_3 & O_3 \end{pmatrix}_{9\times 9} .
$$

3. Application of Error Correcting Code Based 3D-CA with PBC

1D-CA based bit error correcting binary codes (CA-ECC) were first proposed by Chowdhury et al. in [21]. This method recently has been generalized to error correcting codes over non binary fields by Koroglu et al. in [5]. It is also known that CA based error correcting codes have some advantages compared to the classical ones [5, 21, 22]. In this section, we present an application of CA based bit error correcting codes by applying reversible CA which fall into a 3D-CA family with periodic boundary condition. First we present the encoding and decoding process that is given in [5]:

Let T be a $n \times n$ nonsingular transition matrix. Assume that there exists $1 \leq k \leq n$, $k \in \mathbb{Z}^+$ such that $G = [I_n | T^k]$ (I_n, $n \times n$ identity matrix) generates a linear code that corrects up to t errors.

Encoding:

Let $I = (i_1, i_2, \cdots, i_n) \in \mathbb{Z}_3^n$ be an information vector, where n is the rank of the nonsingular transition matrix.Then, the encoded codeword is as follow:

$$CW = (I, T^k[I]) = (i_1, i_2, \ldots, i_n, c_{n+1}, c_{n+2}, \ldots, c_{2n}),$$

i.e.,

$$C = T^k[I] = (c_{n+1}, c_{n+2}, \ldots, c_{2n})$$

is the check vector.

Now, we present a decoding scheme for ternary CA based error correcting codes.

Decoding:

Now suppose that the codeword $CW = (I, T^k[I])$ is sent and $CW' = (I', T^k[I]) = (i'_1, i'_2, \ldots, i'_n, c'_{n+1}, c'_{n+2}, \ldots, c'_{2n}) = (I \oplus I_e, T^k[I] \oplus C_e)$ (where the operator $\oplus$ represent modulo 3 addition) is the received word. Here, I_e and C_e represent the errors that have occurred in information and check bits respectively. We assume that the sum of the Hamming weight of I_e and C_e are less or equal to t i.e. if $w_H(I_e) \leq i$ and $w_H(C_e) \leq t - i$ ($i = 1,2, \ldots, t$), then $w_H(I_e) + w_H(C_e) \leq t$. The syndrome vector is defined by:

$$S = 2T^k[I'] \oplus C' = 2T^k[I_e] \oplus C_e. \tag{3}$$

The syndrome of both the information and check vectors is defined by

$$S_n = 2T^k[I'] \oplus C' \tag{4}$$

and

$$S_c = T^k[I'] \oplus 2C', \tag{5}$$

respectively.

The example given in the following realizes the encoding-decoding schemes above by using a 27×27 invertible rule matrix $T = T_{RP}$ of 3D cellular automata.

Example 2. Let $\mathrm{b = c = e = 1,\ a = f = 0,\ d = 2}$ be elements in ternary field $\mathrm{F_3}$.Then we have a 27×27 rule matrix $\mathrm{T = T_{RP}}$ with $\det(\mathrm{T}) = 2$ in $\mathrm{F_3}$. Thus the matrix T is non singular and for $\mathrm{k = 2}$ the matrix $\mathrm{G = [I_{27}|T^2]}$ generates a $[54,27,5]_3$ linear code with $\mathrm{d(C) = 5}$. It is known that, this code can correct all one and two errors.

Let $\mathrm{I = 111111111111111111111111111}$ be information part of a codeword. Then, the check part is $\mathrm{C = T^2[I] = 111111111111111111111111111}$ and so $\mathrm{CW = (I, T^2[I])}$ is a codeword of length 54.

Case 1. Suppose that one error occurs in the information part. For instance, suppose that the received word is

$$\begin{aligned} CW' &= \hat{2}11111111111111111111111111111111111111111111111111111 \\ &= (I'|C'). \end{aligned}$$

Now, we compute the syndrome as

$$S = 2T^2[I'] \oplus C' = 122200022200000000011000200.$$

The syndrome of the check part is

$$S_c = T^k[I'] \oplus 2C' = 000000000000000000000000000,$$

as we should expect since the errors are located in the information part as supposed.

$$S_{27} = S \oplus S_c = 122200022200000000011000200.$$

Therefore,

$$I_e = T^{-2}[S_{27}] = 200000000000000000000000000.$$

$$\begin{aligned} I = I' \oplus I_e &= 211111111111111111111111111 \\ &\oplus 200000000000000000000000000 \end{aligned}$$

and $C = C'$. Hence, the error vector is

$$E = 200000000000000000000000000000000000000000000000000000.$$

Case 2. Suppose that one error occurs in the check part. Let the received word be

$$CW' = 11111111111111111111111111111111111111111111111111111\hat{0}$$
$$= (I'|C').$$

The syndrome of the check part can be computed as

$$S = T^2[I'] \oplus 2C' = 000000000000000000000000001.$$

The syndromes of the information and the check parts are

$$S_{27} = 000000000000000000000000000$$

and

$$S_c = 000000000000000000000000001,$$

respectively. Next,

$$I_e = T^{-2}[S_{27}] = 000000000000000000000000000$$

and

$$C_e = S_c = 000000000000000000000000001.$$

Hence,

$$C = C' \oplus C_e = 111111111111111111111111111.$$

So, the error vector is

$$E = 200000000000000000000000000000000000000000000000000001.$$

4. Conclusion

In this paper, the author studied a family of three dimensional cellular automata. The algebraic representation of such 3D-CA is established. The author obtained representation matrice via matrice algebra and then author gave an important application about coding theory over the ternary field and we conclude by presenting an application to error correcting codes where reversibility of cellular automata is crucial.

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