BŞEÜ Fen Bilimleri Dergisi
7(2), 698-713, 2020

BSEU Journal of Science
DOI: 10.35193/bseufbd.668272

e-ISSN: 2458-7575 (http://dergipark.gov.tr/bseufbd)

Araştırma Makalesi - Research Article

Modifiye Edilmiş Coulomb Potansiyelli Conformable Sturm-Liouville Problemi

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Geliş / Received: 31/12/2019 ***Revize / Revised: 13/06/2020*** ***Kabul / Accepted: 25/06/2020***

ÖZ

Bu makalede, modifiye edilmiş Coloumb Potansiyele sahip Sturm Liouville probleminin uyumlu mertebeli versiyonu elde edilmiştir. Çalışılan sistem sınır koşullarıyla Sturm Liouville operatörünün uyumlu türevli daha genel bir formatı ispatlanmıştır. Ayrıca, gözününe alınan bu problem için özdeğerlerin reeliği ve özfonksiyonların α- ortoganalliğini ispatlamıştır. İlaveten modifiye edilmiş Coloumb Potansiyele sahip Sturm Liouville problemin çözümünün görüntüsü bulunmuştur. Sonuçlar grafiklerle karşılaştırmalı olarak gösterilmiştir.

Anahtar Kelimeler- Conformable (uyumlu) Türev, Özdeğer, Özfonksiyonlar, Spektral, Coulomb Potansiyeli

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BSEU Journal of Science
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Conformable Sturm-Liouville Problem with Modified Coulomb Potential

ABSTRACT

In this paper, Conformable derivative order version of the Sturm-Liouville problem having modified Coulomb potential is obtained. The studied system proves the shape of the conformable derivative general statement of the Sturm-Liouville operator with boundary conditions. Furthermore, real of eigenvalues and α-orthogonal of eigenfunctions have been proved for the problem considered. Additionally, the representation of the solution of the Sturm-Liouville problem having modified Coulomb potential is found. The results are shown comparatively by figures.

Keywords-Conformable Derivative, Eigenvalue, Eigenfunctions, Spectral, Coulomb Potential

BSEU Journal of Science
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I. INTRODUCTION

The idea of the fractional computation is as old as usual calculus. This concept was made when L'Hospital in 1695 asked specifically what does the means of $\frac{d^n f}{dx^n}$where $n = \frac{1}{2}$. From that time on, intensive research of fractional calculus in the last and current centuries has performed many types of research[1–4]. Some researchers like Fourier, Liouville, Weyl, Riemann, Abel, Leibniz, Grünwald, and Letnikov attempted to put a definition of the fractional derivative. Some of them used an integral form for the fractional derivative. From these outcomes, the most well-known notions of the fractional derivative are Riemann–Liouville definition also Caputo definition[5–8].

I. Riemann–Liouville definition. Assume that $0 < \alpha \leq 1$. Then the derivative of the function$f$ with order α is defined by

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \left[\int_a^t \frac{f(t)}{(t-x)^{\alpha-n+1}} dx \right]. \tag{1}$$

II. Caputo definition. The derivative of the function$f$ with order $\alpha$for$\alpha \in [n-1, n)$ is defined as follows

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \left[\int_a^t \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}} dx \right]. \tag{2}$$

Where D_a^{α} is an operator of the left fractional derivative and $\frac{d^n}{dt^n}$ is an operator of the derivative with integer order, the acquired fractional derivatives in this measurement appeared difficult also do not satisfy some of the main properties the same as normal derivatives for example chain rule, the product rule, and so on. The concept Riemann–Liouville derivative does not fulfill$D_a^{\alpha}(1) = 0$, $({}^c D_a^{\alpha}(1) = 0$ for the Caputo definition), where α does not belong to a natural number. However, some properties of these fragmentary administrators carry on well at times[9-10]. In recent times, Khalil et al. give a new-well definition of local derivative named conformable derivative which is applied a limit form same ordinary derivative form and satisfies all previous properties. Additionally, they define a conformable integral with order $\alpha \epsilon (0,1]$. They also really evaluated and showed conformable Rolle's theorem and mean value theorem through definition that founded by them [11]–[14].

II. BASIC DEFINITION AND THEOREM

Definition 2.1. [11]Assume that $f: [0, \infty) \to \mathbb{R}$ be a function, then the definition of the left conformable derivative of the function$f$ with order α, where $0 < \alpha \leq 1$ is defined as follows

$$(T_\alpha^a f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon(t-a)^{1-\alpha}) - f(t)}{\varepsilon}, \tag{3}$$

for every $t > a$. write T_α, where $a = 0$. If $(T_\alpha^a f)(t)$ exists on (a, b), then we have

$$(T_\alpha^a f)(a) = \lim_{t \to a^+} (T_\alpha^a f)(t). \tag{4}$$

In [15] let us f be a differentiable function on the interval(a, b) and $\alpha \in (0, 1]$, so that

$$(T_\alpha^a f)(t) = (t-a)^{1-\alpha} f'(x). \tag{5}$$

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Definition 2.2. [11]The right conformable derivative of $f:[0,\infty)\to\mathbb{R}$ with order α,$\alpha \in (0,1]$ is defined as follows

$$\left({}_{\alpha}^{b}Tf\right)(t) = \lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon(b-t)^{1-\alpha}\right)-f(t)}{\varepsilon}, \tag{6}$$

for every $t<b$. We write ${}_{\alpha}T$, where$b=0$. If $\left({}_{\alpha}^{b}Tf\right)(t)$ exists on (a,b), then we have

$$\left({}_{\alpha}^{b}Tf\right)(b) = \lim_{t\to b^{-}}\left({}_{\alpha}^{b}Tf\right)(t). \tag{7}$$

In [15] if f is differentiable function on the interval(a,b), then we can define the right conformable derivative of $0<\alpha\le 1$ as

$$\left({}_{\alpha}^{b}Tf\right)(t) = -(b-t)^{1-\alpha}f'(t). \tag{8}$$

If the conformable derivative of the function$f$exists, then the given function$f$ is said to be α-differentiable [16].

Definition 2.3.[16]Suppose that $\alpha\,\epsilon\,(n,n+1]$ for $n\,\epsilon\,\mathbb{N}$, and f be n-differentiable function at a point$t$ for $t>0$, the conformable derivative with order α of f is defined by

$$(T_{\alpha}f)(t) = \lim_{\varepsilon\to 0}\frac{f^{(\lceil\alpha\rceil-1)}\left(t+\varepsilon t^{\lceil\alpha\rceil-\alpha}\right)-f^{(\lceil\alpha\rceil-1)}(t)}{\varepsilon}. \tag{8}$$

Where $\lceil\alpha\rceil$ be the smallest integer number that greater or equal to α.Besides, for $\alpha\,\epsilon\,(n,n+1]$, n is a positive integer we can say that

$$(T_{\alpha}f)(t) = t^{(\lceil\alpha\rceil-\alpha)}f^{\lceil\alpha\rceil}(t). \tag{9}$$

From the above definition, we can say f is $(n+1)$-differentiable function at non-negative number t.

Definition 2.4.[15]Assume that $f:(0,\infty)\to\mathbb{R}$ be a given function and $\alpha\in(0,1]$. so that, the definition of left conformable integral of f with order $\alpha$is given by

$$\begin{aligned}(I_{\alpha}^{a}f)(t) &= \int_{a}^{t} f(x)d_{\alpha}(x,a)\\ &= \int_{a}^{t}(x-a)^{\alpha-1}f(x)dx.\end{aligned} \tag{10}$$

We are writing I_{α} and $d_{\alpha}x$ where $a=0$.

Definition 2.5.[15] The right conformable integral with order α of the function f where $f:(0,\infty)\to\mathbb{R}$ and $\alpha\in[0,1]$ is defined by

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$$\begin{aligned}({}_{\alpha}^{b}If)(t) &= \int_t^b f(x)d_\alpha(b,x) \\ &= \int_t^b (b-x)^{\alpha-1} f(x)dx.\end{aligned} \tag{11}$$

If $b=0$, then we write ${}_\alpha I$ and$d_\alpha x$.

Definition 2.6. [15] Let f be a given function and $\beta = \alpha - n$ where $0 < \alpha \le 1$, then the left conformable integral definition that starting from $a > 0$ of f is

$$(I_\alpha^a f)(t) = \boldsymbol{I}_{n+1}^a\left((t-a)^{\beta-1} f\right) = \frac{1}{n!}\left[\int_a^t (t-x)^n (x-a)^{\beta-1} f(x)dx\right]. \tag{12}$$

Perceive that if $\alpha = n+1$, then we obtain $\beta = 1$. Therefore, by way of Cauchy formula, the iterative integral of f over $n+1$ times on the interval$(a,t]$ is provided

$$(I_\alpha^a f)(t) = (\boldsymbol{I}_{n+1}^a f)(t) = \frac{1}{n!}\left[\int_a^t (t-x)^n f(x)dx\right]. \tag{13}$$

Theorem 2.7.[15] Assume that $\alpha \in (0,1]$ and$f: (0,\infty) \to \mathbb{R}$ be given continuous function, for every $t > a$

$$T_\alpha^a I_\alpha^a f(t) = f(t). \tag{14}$$

And

$${}_\alpha^b T\, {}_\alpha^b I f(t) = f(t). \tag{15}$$

If $f^n(t)$ is continuous, then for $\alpha \in (n, n+1]$ both equations (15) and (16) hold.

Lemma 2.8. [15]Assume $0 < \alpha \le 1$ and$f: [a,b) \to \mathbb{R}$ be a differentiable function, then for all $t > 0$

$$I_\alpha^a T_\alpha^a f(t) = f(t) - f(a). \tag{16}$$

And

$${}_\alpha^b I\, {}_\alpha^b T(f)(t) = f(t) - f(b). \tag{17}$$

Theorem 2.9. [15](Conformable integration by parts).Assume that f and g be two given differentiable functions such that $f, g: [a,b] \to \mathbb{R}$, then

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$$\int_a^b f(x)\, T_\alpha^a(g)(x) d_\alpha(x,a) = fg|_a^b - \int_a^b g(x)\, T_\alpha^a(f)(x) d_\alpha(x,a). \tag{18}$$

Theorem 2.10.[17]Let$p$ be a point in real number, then for $p \geq 1$ the set $L_\alpha^p([a,b],\mathbb{R})$, $(a \geq 0)$ be a Banach space along with the norm such that defined for $f \in L_\alpha^p([a,b],\mathbb{R})$ as

$$\|f\|_{L_\alpha^p([a,b],\mathbb{R})} = \left(\int_a^b |f(t)|^p\, d_\alpha t\right)^{1/p}. \tag{19}$$

Furthermore, space$L_\alpha^2([a,b],\mathbb{R})$ is a Hilbert space along with inner product provided for all$(f,g) \in L_\alpha^p([a,b],\mathbb{R}) \times L_\alpha^p([a,b],\mathbb{R})$

$$\langle f,g\rangle_{L_\alpha^2([a,b],\mathbb{R})} = \int_a^b f(t)g(t)\, d_\alpha t. \tag{20}$$

III. MAIN RESULTS

For $a < x < b$, we are looking at the conformable extension thorough account of the Sturm-Liouville eigenvalue problem having modified Coulomb potential

$$-T_\alpha^a T_\alpha^a y + \alpha^2\left(\frac{A}{x^\alpha} + q(x)\right)y = \alpha^2 \lambda y. \tag{21}$$

where$\frac{y(x)}{x^\alpha} \in C^{2\alpha}[0,\pi]$. We analyze (22) with boundary conditions

$$\begin{aligned} c_1 y(0) + c_2 y'(0) = 0, &\quad {c_1}^2 + {c_2}^2 > 0, \\ r_1 y(\pi) + r_2 y'(\pi) = 0, &\quad {r_1}^2 + {r_2}^2 > 0. \end{aligned} \tag{22}$$

If $T_\alpha^a T_\alpha^a y$ is continuous on $[a.b]$ then we conclude that y is 2α-continuously differentiable. Also, $y \in C^{2\alpha}[a,b]$ where$y \in C^1[a,b]$ and be 2α- continuously differentiable on $[a,b]$.

Suppose that L be a linear operator described on certain elements such that

$$L(y,\alpha) = -T_\alpha^a T_\alpha^a y + \alpha^2\left(\frac{A}{x^\alpha} + q(x)\right)y, \tag{23}$$

then we can write (22) as

$$L(y,\alpha) = \alpha^2 \lambda y. \tag{24}$$

In this process, we generalize the easy outcome of a well-known Lagrange identity.

Theorem 3.1. Let y_1, y_2 be 2α-continuously differentiable functions on the interval $[0,\lambda]$, so that the following equation maintain true

BİLECİK ŞEYH EDEBALİ ÜNİVERSİTESİ
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$$\int_0^\pi (y_2\, L(y_1,\alpha) - y_1 L(y_2,\alpha))d_\alpha(x) = (y_2 T_\alpha^a y_1 - y_1 T_\alpha^a y_2)|_0^\pi. \tag{25}$$

This theorem is said to be the conformable Lagrange identity.

Proof. From (22), we see that

$$\begin{aligned} y_2 L(y_1,\alpha) - y_1 L(y_2,\alpha) &= -y_2 T_\alpha^a T_\alpha^a y_1 + \alpha^2\left(\frac{A}{x^\alpha} + q(x)\right) y_1 y_2 \\ &+ y_1 T_\alpha^a T_\alpha^a y_2 - \alpha^2 \left(\frac{A}{x^\alpha} + q(x)\right) y_1 y_2 \\ &= y_1 T_\alpha^a T_\alpha^a y_2 - y_2 T_\alpha^a T_\alpha^a y_1 \end{aligned} \tag{26}$$

by using conformable integration with order α of (27) and using conformable integration by parts from Theorem 3.1, we obtain

$$\begin{aligned} &\int_0^\pi (y_2\, L(y_1,\alpha) - y_1 L(y_2,\alpha))d_\alpha(x) \\ &= \int_0^\pi y_1 T_\alpha^a T_\alpha^a y_2\, d_\alpha(x) - \int_0^\pi y_2 T_\alpha^a T_\alpha^a y_1\, d_\alpha(x) \\ &= y_1 T_\alpha^a y_2|_0^\pi - \int_0^\pi T_\alpha^a y_1 T_\alpha^a y_2\, d_\alpha(x) \\ &- y_2 T_\alpha^a y_1|_0^\pi + \int_0^\pi T_\alpha^a y_1 T_\alpha^a y_2\, d_\alpha(x) \\ &= (y_1 T_\alpha^a y_2 - y_2 T_\alpha^a y_1)|_0^\pi. \end{aligned} \tag{27}$$

Proposition 3.2. Assume that y_1 and y_2 be two given function in $C^1[0,\lambda]$ such that satisfy the boundary conditions (23), then the following holds true

$$(y_2 T_\alpha^a y_1 - y_1 T_\alpha^a y_2)|_0^\pi = 0. \tag{28}$$

Proof: In (29) we have

$$\begin{aligned} (y_2 T_\alpha^a y_1 - y_1 T_\alpha^a y_2)|_0^\pi &= y_2(\pi)(T_\alpha^a y_1)(\pi) - y_1(\pi)(T_\alpha^a y_2)(\pi) \\ &-(y_2(0)(T_\alpha^a y_1)(0) + y_1(0)(T_\alpha^a y_2)(0)). \end{aligned} \tag{29}$$

Since ${c_1}^2 + {c_2}^2 > 0$, and ${r_1}^2 + {r_2}^2 > 0$, in the beginning, we assume that $c_1 \neq 0$ and$r_1 \neq 0$, without loss of simplification and the proof of other cases going to be gotten similarly. Now in (23), we get

BİLECİK ŞEYH EDEBALİ ÜNİVERSİTESİ
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$$y(0) = -\frac{c_2}{c_1} y'(0),$$
$$y(\pi) = -\frac{r_2}{r_1} y'(\pi). \tag{30}$$

By the above definitions $(T_\alpha^a y_1)(x) = (x-a)^{1-\alpha} {y'}_1(x)$ and $(T_\alpha^a y_2)(x) = (x-a)^{1-\alpha} {y'}_2(x)$ because $y_1, y_2 \in C^1[a,b]$. Now by applying that and (29) in equation (28), we get

$$\begin{aligned}(y_2(\pi)(T_\alpha^a y_1)(\pi) - y_1(\pi)(T_\alpha^a y_2)(\pi)) \\ = -\frac{r_2}{r_1} {y'}_2(\pi)(T_\alpha^a y_1)(\pi) + \frac{r_2}{r_1} {y'}_1(\pi)(T_\alpha^a y_2)(\pi) \\ = -\frac{r_2}{r_1}\left({y'}_2(\pi)(\pi-a)^{1-\alpha}{y'}_1(\pi) - {y'}_1(\pi)(\pi-a)^{1-\alpha}{y'}_2(\pi)\right) \\ = 0.\end{aligned} \tag{31}$$

Similarly

$$y_2(0)(T_\alpha^a y_1)(0) + y_1(0)(T_\alpha^a y_2)(0) = 0. \tag{32}$$

Therefore the proof has been demonstrated.

Definition 3.3. [8] Let f and g be two given function then we say f and g are α-orthogonal in relation to the weight function $\mathcal{U}(t) \geq 0$, where

$$\int_0^\pi \mathcal{U}(x) f(x) g(x) d_\alpha(x) = 0. \tag{33}$$

Theorem 3.4. Both eigenfunctions y_1 and y_2 of the Sturm-Liouville eigenvalue problem $(22) - (23)$ corresponding to different eigenvalues λ_1 and λ_2 respectively are α-orthogonal where $(y_1, y_2) \in L_\alpha^2([0,\pi],\mathbb{R}) \times L_\alpha^2([0,\pi],\mathbb{R})$.

Proof. in (25) we see that

$$L(y_1, \alpha) = \alpha^2 \lambda_1 y_1 \tag{34}$$

$$L(y_2, \alpha) = \alpha^2 \lambda_2 y_2 \tag{35}$$

multiplying (35) by y_2 also, multiplying (36) by y_1 and subtracting both equations, then we receive that

$$(\lambda_1 - \lambda_2)\alpha^2 y_1 y_2 = y_2 L(y_1, \alpha) - y_1 L(y_2, \alpha). \tag{36}$$

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By using the conformable integration with order α and the conformable Lagrange identity theorem, we see that

$$\begin{aligned}(\lambda_1-\lambda_2)\alpha^2\int_0^\pi y_1y_2\, d_\alpha(x) &= \int_0^\pi (y_2\, L(y_1,\alpha)-y_1L(y_2,\alpha))d_\alpha(x)\\ &= [y_2T_\alpha^a y_1-y_1T_\alpha^a y_2]|_0^\pi \\ &= 0,\end{aligned} \tag{37}$$

it is according to Proposition 3.2. In (38) we get $\int_0^\pi y_1y_2\, d_\alpha(x)=0$ because the eigenvalues λ_1 and λ_2 are distinct and $\alpha^2\neq 0$, this fulfills the proof.

Theorem 3.5. Eigenvalues of the conformable Sturm-Liouville eigenvalue problem $(21)-(23)$ are real.

Proof. Assume that y be a solution of the conformable Sturm-Liouville eigenvalue problem $(21)-(23)$. By applying the complex conjugate of $(21)-(23)$, then we obtain that

$$\begin{aligned}L(\bar{y},\alpha) &= -T_\alpha^a T_\alpha^a\bar{y}+\alpha^2\left(\frac{A}{x^\alpha}+q(x)\right)\bar{y}\\ &= \alpha^2\lambda\bar{y}.\end{aligned} \tag{38}$$

$$\begin{aligned}&c_1\bar{y}(0)+c_2\overline{y'}(0)=0,\quad {c_1}^2+{c_2}^2>0.\\ &r_1\bar{y}(\pi)+r_2\overline{y'}(\pi)=0,\quad {r_1}^2+{r_2}^2>0.\end{aligned} \tag{39}$$

By using comparable measures to the proof of Theorem 3.4 for $y_1=y$,$y_2=\bar{y}$, $\lambda_1=\lambda$ and $\lambda_2=\bar{\lambda}$, we get

$$\begin{aligned}\alpha^2(\lambda-\bar{\lambda})\int_0^\pi |y(x)|^2\, d_\alpha(x) &= \int_0^\pi (\bar{y}\, L(y,\alpha)-yL(\bar{y},\alpha))d_\alpha(x)\\ &= [\bar{y}T_\alpha^a y-y\overline{T_\alpha^a y_2}]|_0^\pi\\ &= 0\end{aligned} \tag{40}$$

So, $\int_0^\pi |y(x)|^2\, d_\alpha(x)\neq 0$because $|y(x)|^2$ is positive, since $\alpha^2>0$, Then $\lambda-\bar{\lambda}=0\Longrightarrow\lambda=\bar{\lambda}$. Hence, the eigenvalues must be real.

Definition 3.6.[7] We assume that $y_1(t),y_2(t),\cdots,y_n(t)$ are $(n-1)$ times α-differentiable functions and $0<\alpha\leq 1$. So that, we are denoting the conformable Wronskian of that functions by $W_\alpha(y_1,y_2,\cdots,y_n)$ and defined as

$$W_\alpha(y_1,y_2,\ldots,y_n)=\begin{vmatrix} y_1 & y_2 & \cdots & y_n\\ T_\alpha^a y_1 & T_\alpha^a y_2 & \cdots & T_\alpha^a y_n\\ \vdots & \vdots & \cdots & \vdots\\ {}^{(n-1)}T_\alpha^a y_1 & {}^{(n-1)}T_\alpha^a y_2 & \cdots & {}^{(n-1)}T_\alpha^a y_n\end{vmatrix}. \tag{41}$$

Theorem 3.7. Suppose that $y \in C^{2\alpha}[0,\pi]$, then the representation of the solution of the Sturm-Liouville eigenvalue problem having modified Coulomb potential

$$-T_\alpha T_\alpha y + \alpha^2\left(\frac{A}{x^\alpha} + q(x)\right)y = \alpha^2 \lambda y, \tag{42}$$

With both initial conditions $y(0,\lambda) = 1$ and $T_\alpha^0 y(0,\lambda) = h$ is

$$y(x) = \cos(\sqrt{\lambda}x^\alpha) + \frac{h}{\alpha\sqrt{\lambda}}\sin(\sqrt{\lambda}x^\alpha) + \frac{\alpha}{\sqrt{\lambda}}\int_0^x \left(\frac{A}{t^\alpha} + q(t)\right)y(t)\sin\left(\sqrt{\lambda}(x^\alpha - t^\alpha)\right)d_\alpha t. \tag{43}$$

Proof. To get a solution of (43) we need to find y_h and y_p.

$$T_\alpha T_\alpha y + \alpha^2 \lambda y = 0 \tag{44}$$

we look for $y = e^{r\frac{x^\alpha}{\alpha}}$, then $T_{\alpha\alpha}^{\,0} y = re^{r\frac{x^\alpha}{\alpha}}$ and $T_\alpha T_\alpha y = r^2 e^{r\frac{x^\alpha}{\alpha}}$. So, in (45) we get $(r^2 + \alpha^2\lambda)e^{r\frac{x^\alpha}{\alpha}} = 0 \Rightarrow$. Hence, we obtain that $r_1 = \alpha\sqrt{\lambda}i$ and $r_2 = -\alpha\sqrt{\lambda}i$. So,

$$y_h = c_1\cos(\sqrt{\lambda}x^\alpha) + c_2\sin(\sqrt{\lambda}x^\alpha). \tag{45}$$

Suppose that $y_1 = \cos(\sqrt{\lambda}x^\alpha)$ and $y_2 = \sin(\sqrt{\lambda}x^\alpha)$, then we have

$$y_p = u_1 y_1 + u_2 y_2 \tag{46}$$

$$T_\alpha y_p = T_\alpha u_1 y_1 + u_1 T_\alpha y_1 + T_\alpha u_2 y_2 + u_2 T_\alpha y_2 \tag{47}$$

$$T_\alpha u_1 y_1 + T_\alpha u_2 y_2 = 0 \tag{48}$$

$$T_\alpha T_\alpha y_p = T_\alpha u_1 T_\alpha y_1 + u_1 T_\alpha T_\alpha y_1 + T_\alpha u_2 T_\alpha y_2 + u_2 T_\alpha T_\alpha y_{2.} \tag{49}$$

By taking (47) and (50) in (43), we get

$$-T_\alpha u_1 T_\alpha y_1 - u_1 T_\alpha T_\alpha y_1 - T_\alpha u_2 T_\alpha y_2 - u_2 T_\alpha T_\alpha y_2 + \alpha^2\left(\frac{A}{x^\alpha} + q(x)\right)y = \alpha^2\lambda u_1 y_1 + \alpha^2 \lambda u_2 y_2$$

$$T_\alpha u_1 T_\alpha y_1 + T_\alpha u_2 T_\alpha y_2 + u_1(T_\alpha T_\alpha y_1 + \alpha^2\lambda y_1) + u_2(T_\alpha T_\alpha y_2 + \alpha^2\lambda y_2) = \alpha^2\left(\frac{A}{x^\alpha} + q(x)\right)y$$

$$T_\alpha u_1 T_\alpha y_1 + T_\alpha u_2 T_\alpha y_2 = \alpha^2\left(\frac{A}{x^\alpha} + q(x)\right)y \tag{50}$$

So, by (49) and (51) we receive that

$$T_\alpha u_1 \cos(\sqrt{\lambda}x^\alpha) + T_\alpha u_2 \sin(\sqrt{\lambda}x^\alpha) = 0 \tag{51}$$

$$-\alpha\sqrt{\lambda}T_\alpha u_1 \sin(\sqrt{\lambda}x^\alpha) + \alpha\sqrt{\lambda}T_\alpha u_1 \cos(\sqrt{\lambda}x^\alpha) = \alpha^2\left(\frac{A}{x^\alpha} + q(x)\right)y \tag{52}$$

By applying Definition 3.6 to find α-Wronskian of y_1 and y_2 as

$$\begin{aligned} W_\alpha(y_1, y_2) &= y_1 T_\alpha y_2 - y_2 T_\alpha y_1 \\ &= \alpha\sqrt{\lambda}\,(\cos(\sqrt{\lambda}x^\alpha))^2 + \alpha\sqrt{\lambda}(\sin(\sqrt{\lambda}x^\alpha))^2 \\ &= \alpha\sqrt{\lambda} \end{aligned} \tag{53}$$

$$T_\alpha u_1 = \frac{\begin{vmatrix} 0 & \sin(\sqrt{\lambda}x^\alpha) \\ \alpha^2\left(\frac{A}{x^\alpha} + q(x)\right)y & \cos(\sqrt{\lambda}x^\alpha) \end{vmatrix}}{\alpha\sqrt{\lambda}} = -\frac{\alpha}{\sqrt{\lambda}}\left(\frac{A}{x^\alpha} + q(x)\right)y \sin(\sqrt{\lambda}x^\alpha). \tag{54}$$

$$T_\alpha u_2 = \frac{\begin{vmatrix} \cos(\sqrt{\lambda}x^\alpha) & 0 \\ \sin(\sqrt{\lambda}x^\alpha) & \alpha^2\left(\frac{A}{x^\alpha} + q(x)\right)y \end{vmatrix}}{\alpha\sqrt{\lambda}} = \frac{\alpha}{\sqrt{\lambda}}\left(\frac{A}{x^\alpha} + q(x)\right)y \cos(\sqrt{\lambda}x^\alpha). \tag{55}$$

$$u_1 = -\frac{\alpha}{\sqrt{\lambda}}\int_0^x \left(\frac{A}{t^\alpha} + q(t)\right)y(t)\sin(\sqrt{\lambda}t^\alpha)\, d_\alpha t. \tag{56}$$

$$u_2 = \frac{\alpha}{\sqrt{\lambda}}\int_0^x \left(\frac{A}{t^\alpha} + q(t)\right)y(t)\cos(\sqrt{\lambda}t^\alpha)\, d_\alpha t. \tag{57}$$

By taking the value of u_1 and u_2 in (47) we obtain

$$\begin{aligned} y_p &= -\frac{\alpha\cos(\sqrt{\lambda}x^\alpha)}{\sqrt{\lambda}}\int_0^x \left(\frac{A}{t^\alpha} + q(t)\right)y(t)\sin(\sqrt{\lambda}t^\alpha)\, d_\alpha t \\ &\quad + \frac{\alpha\sin(\sqrt{\lambda}x^\alpha)}{\sqrt{\lambda}}\int_0^x \left(\frac{A}{t^\alpha} + q(t)\right)y(t)\cos(\sqrt{\lambda}t^\alpha)\, d_\alpha t \\ &= \frac{\alpha}{\sqrt{\lambda}}\int_0^x \left(\frac{A}{t^\alpha} + q(t)\right)y(t)[\sin(\sqrt{\lambda}x^\alpha)\cos(\sqrt{\lambda}t^\alpha) - \cos(\sqrt{\lambda}x^\alpha)\sin(\sqrt{\lambda}t^\alpha)]d_\alpha t \\ &= \frac{\alpha}{\sqrt{\lambda}}\int_0^x \left(\frac{A}{t^\alpha} + q(t)\right)y(t)\sin\left(\sqrt{\lambda}(x^\alpha - t^\alpha)\right) d_\alpha t. \end{aligned} \tag{58}$$

BSEU Journal of Science
DOI: 10.35193/bseufbd.668272

e-ISSN: 2458-7575 (http://dergipark.gov.tr/bseufbd)

Thus, from (46) and (59) we have

$$y(x) = c_1 \cos(\sqrt{\lambda}x^\alpha) + c_2 \sin(\sqrt{\lambda}x^\alpha) + \frac{\alpha}{\sqrt{\lambda}}\int_0^x \left(\frac{A}{t^\alpha} + q(t)\right) y(t) \sin\left(\sqrt{\lambda}(x^\alpha - t^\alpha)\right) d_\alpha t. \quad (59)$$

By applying the boundary conditions, we get that $1 = c_1 cos0 + c_2 sin0 \Rightarrow c_1 = 1$ and $h = -\alpha\sqrt{\lambda}c_1 sin0 + \alpha\sqrt{\lambda}c_2 cos0 \Rightarrow c_2 = \frac{h}{\alpha\sqrt{\lambda}}$, then in (60), we get

$$y(x) = \cos(\sqrt{\lambda}x^\alpha) + \frac{h}{\alpha\sqrt{\lambda}}\sin(\sqrt{\lambda}x^\alpha) + \frac{\alpha}{\sqrt{\lambda}}\int_0^x \left(\frac{A}{t^\alpha} + q(t)\right) y(t) \sin\left(\sqrt{\lambda}(x^\alpha - t^\alpha)\right) d_\alpha t. \quad (60)$$

Application 3.8. In [18] for $0 < \alpha \leq 1$ by using the Frobenius method we obtained the solution of modified power series within a regular singular point $x = 0$ of

$$-T_\alpha T_\alpha y + \alpha^2 \frac{1}{x^\alpha} y = 0, \quad (61)$$

as

$$y(x) = \sum_{n=0}^{\infty} \frac{c_0}{n!\,(n+1)!} x^{(n+1)\alpha}. \quad (62)$$

Equation (61) is the first solution of conformable Sturm-Liouville eigenvalue problem having modified Coulomb potential in the homogeneous case for A

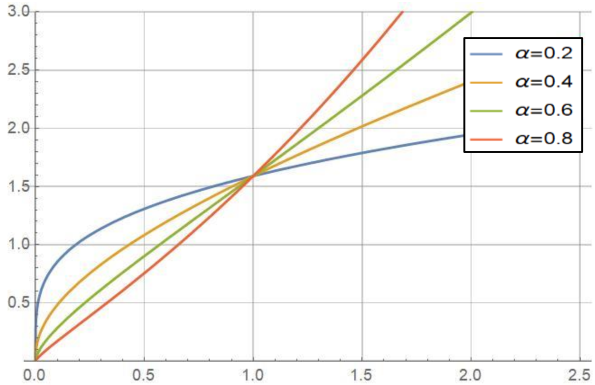

Figure. 1. The solution of the equation (62) when in(63)$c_0 = 1$.

BSEU Journal of Science
DOI: 10.35193/bseufbd.668272

e-ISSN: 2458-7575 (http://dergipark.gov.tr/bseufbd)

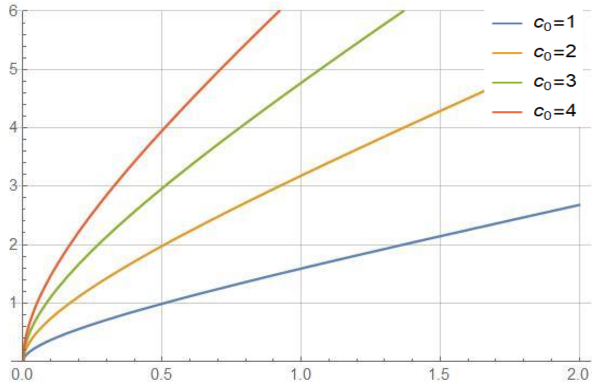

Figure 2. The solution of the equation (62) when in (63)$\alpha = 0.5$.

Furthermore, the first solution to

$$T_\alpha T_\alpha y - \alpha^2 \frac{1}{x^\alpha} y = \alpha^2 \lambda y, \tag{63}$$

where $0 < \alpha \leq 1$ is

$$y(x) = c_0 \left(x^\alpha + \frac{1}{2} x^{2\alpha} + \frac{(2\lambda + 1)}{2.2.3} x^{3\alpha} + \frac{(8\lambda + 1)}{2.2.3.3.4} x^{4\alpha} + \frac{24\lambda^2 + (20\lambda + 1)}{2.2.3.3.4.4.5} x^{5\alpha} + \cdots \right). \tag{64}$$

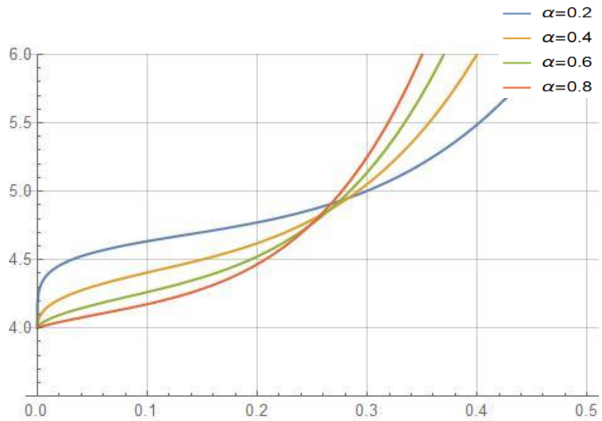

Figure 3. The solution of the equation (64) when in(65)$\lambda = 1, c_0 = 1$.

BŞEÜ Fen Bilimleri Dergisi
7(2), 698-713, 2020

BSEU Journal of Science
DOI: 10.35193/bseufbd.668272

e-ISSN: 2458-7575 (http://dergipark.gov.tr/bseufbd)

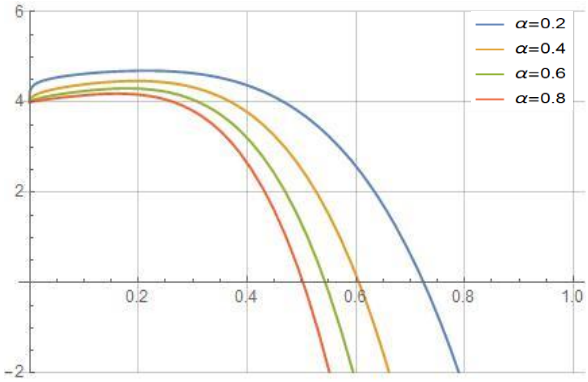

Figure 4. The solution of the equation (64) when in $(65) c_0 = 1, \lambda = -1$.

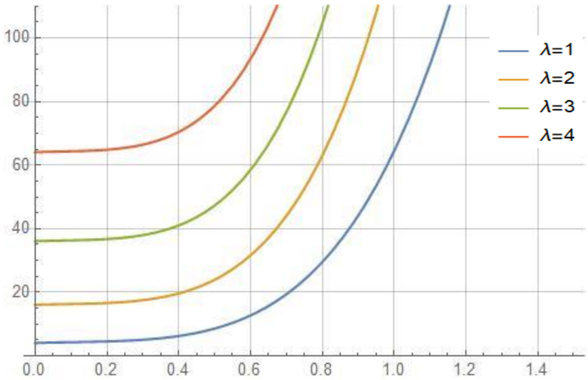

Figure 5. The solution of the equation (64) when in $(65) c_0 = 1, \alpha = 0.5$.

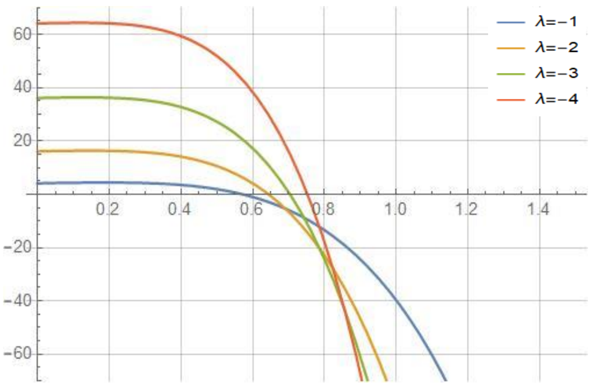

Figure 6. The solution of the equation (64) when in $(65) \alpha = 0.5, c_0 = 1$.

IV. CONCLUSION

The conformable extension through the study of the Sturm-Liouville eigenvalue problem having modified Coulomb potential is recognized. We proved the conformable Lagrange identity theorem by using conformable integration by parts. Defined orthogonality of two functions and using this definition can show that the eigenfunctions of the Sturm-Liouville eigenvalue problem corresponding to different eigenvalues are α-orthogonal. By applying the complex conjugate of the problem $(21) - (23)$ we showed that the eigenvalues of this problem are real. Furthermore, by applying the definition of the conformable Wronskian function discovered

BSEU Journal of Science
DOI: 10.35193/bseufbd.668272

e-ISSN: 2458-7575 (http://dergipark.gov.tr/bseufbd)

the representation of the solution to the Sturm-Liouville eigenvalue problem having modified Coulomb potential. We have known that the outcomes of the solution of conformable Sturm-Liouville eigenvalue problem having modified Coulomb potential in the homogeneous case are different with distinct order of derivative, see Fig. 1. Also, the series solutions are increasing on different positive scalar multiplication with order $\alpha = 0.5$ and they are different, can see Fig. 2. The solution of a non- homogeneous case approach to negative infinite number for negative eigenvalues λ, see Fig. 4. and Fig. 6.

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