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Null Cartan Helical Trajectories in Lorentzian 3-Space

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Abstract. In the present paper, we give an approach for null Cartan helices by using the null Cartan magnetic trajectories related to the Killing magnetic vector field. Additionally, we determine the Bishop curvatures and the explicit parametric equation of these curves by using Bishop curvatures. Finally, we give various examples and draw their images.

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1. Introduction

In Lorentzian 3-space, there exist three types of curves according to their causal characters. These curves are called a spacelike curve, timelike curve, and null(lightlike) curve. The null curves are different from the timelike and spacelike curves associated with the induced metric on these curves. Therefore, null curves are often more appropriate to explain physical phenomena (see [7–9, 11–13, 19, 20]).

On the other hand, a helix defined as a curve whose tangent vector makes a constant angle with a fixed direction. In nature, DNA, carbon nanotube, screws, springs, *etc.* have helical shapes. Moreover, the helix has various applications to natural scientists, mathematics, fractal geometry, computer-aided design, computer graphics, physics, *etc.*

In this work, we investigate the null Cartan helices with the help of the null Cartan magnetic trajectories in Lorentzian 3-space, $\mathbb{E}^3_1$. We investigate that when does the null Cartan magnetic curve be a null Cartan helix. Then, we determine the Bishop curvatures and parametric representations of all null Cartan helical trajectories. Furthermore, we give some related examples.

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2. Preliminaries

A non-degenerate metric tensor g on the Lorentzian manifold (F, g) is defined by

$$g(\alpha,\beta) = -\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3,$$

for all $\alpha = (\alpha_1, \alpha_2, \alpha_3), \beta = (\beta_1, \beta_2, \beta_3) \in \chi(F)$.

A Lorentzian (semi-Riemannian) curvature tensor Λ on the Lorentzian manifold (F, g) is given by

$$\Lambda(\alpha,\beta)\delta = -\nabla_\alpha\nabla_\beta\delta + \nabla_\beta\nabla_\alpha\delta + \nabla_{[\alpha,\beta]}\delta.$$

Then, the sectional curvature is defined by

$$K(u, v) = \frac{g(\Lambda(\alpha,\beta)\alpha,\beta)}{g(\alpha,\alpha)g(\beta,\beta) - g(\alpha,\beta)^2}.$$

If the sectional curvature on the Lorentzian manifold (F, g) is constant then the Lorentzian manifold is called Lorentzian space form and denoted by $(F(C), g)$, here C is the constant sectional curvature. In this situation, the Lorentzian curvature tensor reduces to the following equation

$$\Lambda(\alpha,\beta)\delta = C\{g(\delta,\alpha)\beta - g(\delta,\beta)\alpha\}.$$

Let γ be a curve in the Lorentzian space form $(F(C), g)$ then γ is called as a null (lightlike or isotropic) curve if the tangent vector to γ at any point is a null vector. A null curve γ is called a null Cartan curve, if it is parameterized by the pseudo-arc function s defined by

$$s(t) = \int_0^t g(\gamma''(t), \gamma''(t))dt.$$

The null Cartan curve γ has the unique Cartan frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ which satisfies

$$\begin{aligned} \mathbf{t}' &= \varkappa\mathbf{n}, \\ \mathbf{n}' &= -\varkappa\mathbf{t} + \tau\mathbf{b}, \\ \mathbf{b}' &= -\tau\mathbf{b}, \end{aligned}$$

where $\varkappa$, τ are the first curvature and second curvature of the curve γ, respectively. Also we have $\varkappa(s) = 1$ and $\tau(s)$ is an arbitrary function in pseudo-arc parameter s along the curve γ. If $\tau(s) = 0$, the null Cartan curve is called a *null Cartan cubic*. The Cartan frame vectors satisfy the following relations

$$\begin{aligned} g(\mathbf{t}, \mathbf{t}) &= g(\mathbf{b}, \mathbf{b}) = 0,\ g(\mathbf{n}, \mathbf{n}) = 1, \\ g(\mathbf{t}, \mathbf{n}) &= g(\mathbf{n}, \mathbf{b}) = 0,\ g(\mathbf{t}, \mathbf{b}) = -1 \end{aligned}$$

and the cross product satisfy

$$\mathbf{t}\times\mathbf{n} = -\mathbf{t},\ \mathbf{n}\times\mathbf{b} = -\mathbf{b},\ \mathbf{b}\times\mathbf{t} = \mathbf{n},$$

[6].

Theorem 2.1. *Let γ be a null Cartan curve in Lorentzian manifold (F, g) parameterized by pseudo-arc s. Then the Bishop frame $\{\mathbf{t}_1, \mathbf{n}_1, \mathbf{n}_2\}$ and the Cartan frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ of the null Cartan curve γ have the relation:*

$$\begin{bmatrix} \mathbf{t}_1 \\ \mathbf{n}_1 \\ \mathbf{n}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -k_2 & 1 & 0 \\ \frac{k_2^2}{2} & -k_2 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}.$$

Then the null Cartan Bishop frame equations of the curve γ are given by

$$\begin{bmatrix} \mathbf{t}_1' \\ \mathbf{n}_1' \\ \mathbf{n}_2' \end{bmatrix} = \begin{bmatrix} k_2 & k_1 & 0 \\ 0 & 0 & k_1 \\ 0 & 0 & -k_2 \end{bmatrix} \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{n}_1 \\ \mathbf{n}_2 \end{bmatrix},$$

where k_1, k_2 are called first Bishop curvature and second Bishop curvature of the curve γ, respectively. The first Bishop curvature $k_1(s) = 1$ and the second Bishop curvature k_2 satisfies the following Riccati differential equation

$$k_2'(s) + \frac{1}{2}k_2^2(s) + \tau(s) = 0.$$

The Bishop frame $\{\mathbf{t}_1, \mathbf{n}_1, \mathbf{n}_2\}$ *satisfies*

$$\begin{aligned} g(\mathbf{t}_1, \mathbf{t}_1) &= g(\mathbf{n}_2, \mathbf{n}_2) = 0,\ g(\mathbf{n}_1, \mathbf{n}_1) = 1, \\ g(\mathbf{t}_1, \mathbf{n}_1) &= g(\mathbf{n}_1, \mathbf{n}_2) = 0,\ g(\mathbf{t}_1, \mathbf{n}_2) = -1, \end{aligned}$$

[10].

The cross products of the Bishop vectors $\{\mathbf{t}_1, \mathbf{n}_1, \mathbf{n}_2\}$ are obtained as

$$\mathbf{t}_1 \times \mathbf{n}_1 = -\mathbf{t}_1,\ \mathbf{n}_1 \times \mathbf{n}_2 = -\mathbf{n}_2,\ \mathbf{n}_2 \times \mathbf{t}_1 = \mathbf{n}_1,$$

[10].

Proposition 2.2. *Let* $\gamma : I \subset \mathbb{R} \to F$ *be a null Cartan curve in Lorentzian 3-manifold* (F, g) *and* V *be a Killing vector field along the null Cartan curve* γ*.Then, the variation of the speed function* $v(s, u)$*, and the Bishop curvature functions* $k_1(s, u)$ *and* $k_2(s, u)$ *at* $u = 0$ *satisfy the following conditions:*
i. $g(\nabla_{\mathbf{t}_1} V, \mathbf{t}_1) = 0,$
ii. $g(\nabla^2_{\mathbf{t}_1} V, \mathbf{n}_1) - k_2 g(\nabla_{\mathbf{t}_1} V, \mathbf{n}_1) = 0,$
iii. $g((k_1 - 1)\nabla^2_{\mathbf{t}_1} V - k_2 \nabla_{\mathbf{t}_1} V + k_1 k_2 \nabla_{T_1} V, N_2) + C(k_1 - 1) = 0.$

In the following section we investigate that when does the charged particle follow the null Cartan helical magnetic trajectory in the three dimensional Lorentzian space forms.

3. The Relation of The Magnetic Curves and Null Cartan Helices

The Lie derivative of a Killing vector field is zero on three-dimensional manifolds, that is, $\ell_V g = 0$. This gives the Killing vector field on (F, g) is divergence free. Thus, every Killing vector field on three-dimensional manifolds defines a magnetic field (for details, see [1]). If we denote that V is a Killing magnetic vector field then we have a magnetic force Ψ corresponding to the magnetic field V. The Ψ is a closed 2-form and has the notion $\Psi_V = i_v dv_g$, here i denotes the inner product and dv_g denotes a volume on F which satisfies $dv_g(\alpha, \beta, w) = g(\alpha \times \beta, w)$, for all $\alpha, \beta, w \in \chi(F)$. Besides, we have a force called as Lorentz force associated with the magnetic field $\Psi_V = i_v dv_g$. The Lorentz force ϕ associated with the magnetic field Ψ is a skew-symmetric operator on F and it is computed by

$$g(\phi(\alpha), \beta) = \Psi(\alpha, \beta) = (i_v dv_g)(\alpha, \beta) = dv_g(V, \alpha, \beta) = g(V \times \alpha, \beta),$$

for all $\alpha, \beta \in \chi(F)$.

This equation leads to the Lorentz force ϕ satisfies

$$\phi(\alpha) = V \times \alpha.$$

When a charged particle enters a magnetic field as defined above, it is affected by the field and as a result of this, the particle begins to follow a trajectory called the magnetic trajectory. This trajectory is calculated from the following equation

$$\phi(\mathbf{t}) = V \times \mathbf{t} = \nabla_{\mathbf{t}} \mathbf{t},$$

where ∇ is the Levi-Civita connection of the manifold M [1]. This trajectories are defined in the following articles [1, 2, 4, 5, 15, 16, 18].

Definition 3.1. Let γ be a null Cartan curve with the Bishop frame $\{\mathbf{t}_1, \mathbf{n}_1, \mathbf{n}_2\}$ in Lorentzian 3-space. If there exist a Killing vector field along the curve γ such that $\phi(\mathbf{t}_1) = V \times \mathbf{t}_1 = \nabla_{\mathbf{t}} \mathbf{t}$, $\phi(\mathbf{n}_1) = V \times \mathbf{n}_1 = \nabla_{\mathbf{t}} \mathbf{n}_1$, and $\phi(\mathbf{n}_2) = V \times \mathbf{n}_2 = \nabla_{\mathbf{t}} \mathbf{n}_2$ then the curve γ is called as the first kind of null Cartan magnetic trajectory, second kind of null Cartan magnetic trajectory, and third kind of null Cartan magnetic trajectory, respectively.

Definition 3.2. Let γ be a null Cartan curve with the Bishop frame $\{\mathbf{t}_1, \mathbf{n}_1, \mathbf{n}_2\}$ in Lorentzian 3-space. If there exist a constant Killing vector field along the curve γ such that $g(V, \mathbf{t}_1) = const.$ then the curve γ is called as a null Cartan general helix.

Proposition 3.3. *Let $\{\mathbf{t}_1, \mathbf{n}_1, \mathbf{n}_2, k_1, k_2\}$ be the Bishop frame apparatus of the null Cartan curve γ. Then, the Lorentz force of the Bishop frame fields for the first kind of null Cartan magnetic trajectory γ is computed as*

$$\begin{bmatrix} \phi(\mathbf{t}_1) \\ \phi(\mathbf{n}_1) \\ \phi(\mathbf{n}_2) \end{bmatrix} = \begin{bmatrix} k_2 & k_1 & 0 \\ -\sigma & 0 & k_1 \\ 0 & -\sigma & -k_2 \end{bmatrix} \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{n}_1 \\ \mathbf{n}_2 \end{bmatrix},$$

[17].

Proposition 3.4. *Let γ be a curve with the Bishop frame apparatus $\{\mathbf{t}_1, \mathbf{n}_1, \mathbf{n}_2, k_1, k_2\}$ in the Lorentzian 3-manifold (F, g). Then γ is a first kind of null Cartan magnetic trajectory of a magnetic field V if and only if V can be written along γ as*

$$V(s) = \sigma\mathbf{t}_1 - k_2\mathbf{n}_1 + \mathbf{n}_2,$$

where the $\sigma(s)$ is an arbitrary function associated with each magnetic curve associated with the magnetic field V [17].

Remark 3.5. Under the pseudo-arc parameterization, generalized null cubics are represented as null helices of zero lightlike curvature. Moreover, such curves are unique up to Lorentz transformation. Null helices of zero lightlike curvature are cubic curves with respect to the pseudo-arc. Null helices of zero lightlike curvature called as null cubics [14].

Theorem 3.6. *Let γ be a null Cartan curve and V be a Killing vector field on a simply connected Lorentzian space form $(F(C), g)$. If the curve γ is one of the first kind of null Cartan magnetic trajectory of $(F(C), g; V)$, then γ is a null Cartan cubics with second Bishop curvature k_2 vanishes [17].*

Proposition 3.7. *Let γ be a null Cartan curve with the Bishop frame apparatus $\{\mathbf{t}_1, \mathbf{n}_1, \mathbf{n}_2, k_1, k_2\}$ in the Lorentzian 3-manifold (F, g). Then, the Lorentz force of the Bishop frame fields for the second kind of null Cartan magnetic curve γ is calculated as*

$$\begin{bmatrix} \phi(\mathbf{t}_1) \\ \phi(\mathbf{n}_1) \\ \phi(\mathbf{n}_2) \end{bmatrix} = \begin{bmatrix} -\varsigma & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \varsigma \end{bmatrix} \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{n}_1 \\ \mathbf{n}_2 \end{bmatrix},$$

where the function $\varsigma(s)$ is an arbitrary function associated with each magnetic trajectory of V.

Proof. Let γ be a second kind of null Cartan magnetic trajectory on the Lorentzian manifold (F, g) with the Bishop apparatus $\{\mathbf{t}_1, \mathbf{n}_1, \mathbf{n}_2, k_1, k_2\}$. From the definition of the second kind of magnetic trajectory we have

$$\phi(\mathbf{n}_1) = \nabla_{\mathbf{t}_1}\mathbf{n}_1 = \mathbf{n}_2.$$

Since $\phi(\mathbf{t}_1) \in span\{\mathbf{t}_1, \mathbf{n}_1, \mathbf{n}_2\}$ we can write

$$\phi(\mathbf{t}_1) = \lambda_1\mathbf{t}_1 + \mu_1\mathbf{n}_1 + \zeta_1\mathbf{n}_2.$$

This gives

$$\begin{aligned} \lambda_1 &= -g(\phi(\mathbf{t}_1), \mathbf{n}_2) = -\varsigma, \\ \mu_1 &= g(\phi(\mathbf{t}_1), \mathbf{n}_1) = -g(\phi(\mathbf{n}_1), \mathbf{t}_1) = 1, \\ \zeta_1 &= g(\phi(\mathbf{t}_1), \mathbf{t}_1) = 0. \end{aligned}$$

As a consequence the vector field $\phi(\mathbf{t}_1)$ calculated as follows

$$\phi(T_1) = -\varsigma\mathbf{t}_1 - k_2\mathbf{n}_1.$$

A similar computation leads to

$$\phi(\mathbf{n}_2) = \varsigma\mathbf{n}_2.$$

□

Proposition 3.8. *Let γ be a null Cartan curve with the Bishop frame apparatus $\{\mathbf{t}_1, \mathbf{n}_1, \mathbf{n}_2, k_1, k_2\}$ in the Lorentzian 3-manifold (F, g). Then γ is a second kind of magnetic trajectory of a magnetic field V if and only if V can be written along γ as*

$$V(s) = -\varsigma\mathbf{n}_1 + \mathbf{n}_2.$$

Proof. Let γ be a second kind of null Cartan magnetic trajectory of the Killing magnetic field V. Then, V can be written as

$$V(s) = \eta_1 \mathbf{t}_1 + \eta_2 \mathbf{n}_1 + \eta_3 \mathbf{n}_2, \tag{3.1}$$

where $\eta_i, i = 1, 2, 3$ are certain functions along a trajectory of V and assume V does not vanish on γ. From the Proposition 3.8 we calculate $\eta_1 = 0, \eta_2 = -\varsigma$ and $\eta_3 = 1$. Conversely, if we assume that eq.(3.1) holds then we obtain $V \times \mathbf{t}_1 = \phi(\mathbf{t}_1)$. This gives that γ is a second kind of magnetic trajectory of the Killing magnetic vector field V. □

Theorem 3.9. *Let γ be a second kind of null Cartan magnetic trajectory of the Killing magnetic field V on a simply connected space form $(F(C), g)$. If the curve γ is one of the second kind of null Cartan magnetic trajectory of $(F(C), g; V)$, then γ has the following curvatures*

$$k_2(s) = -A \text{ or } k_2(s) = -\sqrt{2c_1} \tanh \frac{\sqrt{2c_1}}{2}(s + c_2) ,$$

where c_1, c_2, A are some constants.

Proof. Let γ be a second kind of null Cartan magnetic trajectory of the Killing magnetic field V on a simply connected space form $(F(C), g)$. Then V may be written as

$$V(s) = -\varsigma \mathbf{n}_1 + \mathbf{n}_2. \tag{3.2}$$

By differentiating eq.(3.2) with respect to s, we have

$$\nabla_{\mathbf{t}_1} V = -\varsigma' \mathbf{n}_1 + (-\varsigma - k_2)\mathbf{n}_2. \tag{3.3}$$

From the first equation of the Proposition 2.2 we calculate

$$k_2(s) = -\varsigma. \tag{3.4}$$

Then, using the derivative of the eq.(3.3) we get

$$\nabla^2_{\mathbf{t}_1} V = -\varsigma'' \mathbf{n}_1 - \varsigma' \mathbf{n}_2.$$

After then, we calculate the Lorentzian curvature tensor satisfies

$$\Lambda(V, \mathbf{t}_1)\mathbf{t}_1 = -C\mathbf{t}_1. \tag{3.5}$$

Considering the eq.(3.3), eq.(3.4), and eq.(3.5) with second equation in the Proposition 2.2, we obtain the following second order non-linear differential equation

$$\varsigma'' + \varsigma\varsigma' = 0.$$

The solutions of the differential equation are computed as

$$\varsigma(s) = A \text{ or } \varsigma(s) = \sqrt{2c_1} \tanh \frac{\sqrt{2c_1}}{2}(s + c_2).$$

These imply

$$k_2(s) = -A \text{ or } k_2(s) = -\sqrt{2c_1} \tanh \frac{\sqrt{2c_1}}{2}(s + c_2) .$$

In these cases, the last equation in the Proposition 2.2 is provided automatically. □

Corollary 3.10. *Let γ be a second kind of null Cartan magnetic trajectory of the Killing magnetic field V on a simply connected space form $(F(C), g)$. Then the ratio of the curvature and torsion of the curve γ is constant, namely, $\frac{\varkappa}{\tau} = const$.*

Corollary 3.11. *Let γ be a second kind of null Cartan magnetic trajectory of the Killing magnetic field V on a simply connected space form $(F(C), g)$. Then γ follows the helical trajectory if and only if γ has the following curvatures and parametric representation*

$$k_1(s) = 1 \text{ and } k_2(s) = -A, \text{ or } k_2 = 0,$$

$$\gamma(s) = b_1 + b_2 s + \frac{b_3}{A^2} e^{\sqrt{A^2}s} + \frac{b_4}{A^2} e^{-\sqrt{A^2}s} \text{ or } \gamma \text{ is a null cubic},$$

where A is a non-zero constant and $b_i \in \mathbb{E}^3_1$, $i = 1, 2, 3, 4$.

Proof. Let γ be a second kind of null Cartan magnetic trajectory of the Killing magnetic field V on a simply connected space form $(F(C), g)$. Then the curve γ has the following curvatures

$$k_2(s) = -A \text{ or } k_2(s) = -\sqrt{2c_1}\tanh\frac{\sqrt{2c_1}}{2}(s + c_2).$$

Since we want to a helical trajectory with the axis V the Killing vector field V must satisfies $g(V, V) = const$. This condition is satisfied when the second kind null Cartan magnetic curve equal to $k_2(s) = -A$ or $k_2 = 0$ $(c_1 = 0)$. In this situation we obtain that the null Cartan helices have the constant torsion and the position vector of the helices satisfy the following higher-order linear ordinary differential equation:

$$\gamma^{(4)} + 2\tau\gamma'' = 0,\ \tau = -k_2' - \frac{1}{2}k_2^2.$$

The solution of the differential equation give that the helical trajectory is null cubic or has following parametric representation

$$\gamma(s) = b_1 + b_2 s + \frac{b_3}{A^2}e^{\sqrt{A^2}s} + \frac{b_4}{A^2}e^{-\sqrt{A^2}s}, \quad \tau = -\frac{A^2}{2} < 0,$$

where $b_i \in \mathbb{E}_1^3$, $i = 1, 2, 3, 4$. □

Remark 3.12. A second kind of magnetic trajectory that has the second Bishop curvature $k_2(s) = -\sqrt{2c_1}\tanh\frac{\sqrt{2c_1}}{2}(s + c_2)$ is a helix but the Killing vector field V does not the axis of this helix.

Proposition 3.13. *Let* $\{\mathbf{t}_1, \mathbf{n}_1, \mathbf{n}_2, k_1, k_2\}$ *be the Bishop frame apparatus of the null Cartan curve* γ*. Then, the Lorentz force of the Bishop frame fields for the third kind of null Cartan magnetic trajectory* γ *is calculated as*

$$\begin{bmatrix} \phi(\mathbf{t}_1) \\ \phi(\mathbf{n}_1) \\ \phi(\mathbf{n}_2) \end{bmatrix} = \begin{bmatrix} k_2 & \varrho & 0 \\ 0 & 0 & \varrho \\ 0 & 0 & -k_2 \end{bmatrix} \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{n}_1 \\ \mathbf{n}_2 \end{bmatrix}.$$

Proof. Let γ be a second kind of null Cartan magnetic trajectory on the Lorentzian manifold (F, g) with the Bishop apparatus $\{\mathbf{t}_1, \mathbf{n}_1, \mathbf{n}_2, k_1, k_2\}$. From the definition of the third kind of magnetic trajectory we have

$$\phi(\mathbf{n}_2) = \nabla_{\mathbf{t}_1}\mathbf{n}_2 = -k_2\mathbf{n}_2. \tag{3.6}$$

Since $\phi(\mathbf{n}_1) \in span\{\mathbf{t}_1, \mathbf{n}_1, \mathbf{n}_2\}$ we may write

$$\phi(\mathbf{n}_1) = \lambda_2\mathbf{t}_1 + \mu_2\mathbf{n}_1 + \zeta_2\mathbf{n}_2. \tag{3.7}$$

Then, we obtain $\lambda_2 = 0, \mu_2 = 0, \zeta_2 = \varrho$ these gives that the vector field $\phi(\mathbf{n}_1)$ is obtained as

$$\phi(\mathbf{n}_1) = \varrho\mathbf{n}_2. \tag{3.8}$$

Finally, the similar computations give

$$\phi(\mathbf{t}_1) = k_2\mathbf{t}_1 + \varrho\mathbf{n}_2. \tag{3.9}$$

□

Proposition 3.14. *Let* γ *be a null Cartan curve with the Bishop frame apparatus* $\{\mathbf{t}_1, \mathbf{n}_1, \mathbf{n}_2, k_1, k_2\}$ *in the Lorentzian 3-manifold* (F, g)*. Then* γ *is a third kind of magnetic trajectory of a magnetic field V if and only if V can be written along* γ *as*

$$V(s) = k_2\mathbf{n}_1 + \varrho\mathbf{n}_2.$$

Proof. Let γ be a second kind of null Cartan magnetic trajectory of the Killing magnetic field V. Then, V can be written as

$$V(s) = \varsigma_1\mathbf{t}_1 + \varsigma_2\mathbf{n}_1 + \varsigma_3\mathbf{n}_2, \tag{3.10}$$

where ς_i, $i = 1, 2, 3$ are certain functions along a trajectory of V and assume V does not vanish on γ. From the eqs. (3.6)-(3.9) we calculate $\varsigma_1 = 0, \varsigma_2 = k_2$, and $\varsigma_3 = \varrho$. Conversely, we assume that eq.(3.10) holds then we get $V \times \mathbf{t}_1 = \phi(\mathbf{t}_1)$. Therefore, the curve γ is a null Cartan magnetic trajectory of the magnetic vector field V. □

Theorem 3.15. *Let γ be a second kind of null Cartan magnetic trajectory of the Killing magnetic field V on a simply connected space form $(F(C), g)$. If the curve γ is one of the second kind of null Cartan magnetic trajectory of $(F(C), g; V)$, then γ has the following curvatures*

$$k_2 = B, \; k_2(s) = \sqrt{2c_3}\tan\frac{\sqrt{2c_3}}{2}(s + c_4),$$

where c_3, c_4, B are constants.

Proof. Let γ be a second kind of null Cartan magnetic trajectory of the Killing magnetic field V on a simply connected space form $(F(C), g)$. Then V have the form in eq.(3.10). If we differentiate the eq.(3.10) we have

$$\nabla_{\mathbf{t}_1} V = k_2'\mathbf{n}_1 + (k_2 + \varrho' - k_2\varrho)\mathbf{n}_2. \tag{3.11}$$

Using the first equation in the Proposition 2.2 we compute

$$k_2 + \varrho' - k_2\varrho = 0.$$

Then, the differentiation of eq.(3.11) gives

$$\nabla^2_{\mathbf{t}_1} V = k_2''\mathbf{n}_1 + k_2'\mathbf{n}_2. \tag{3.12}$$

Since γ is a third kind of null Cartan magnetic trajectory we compute the Lorentzian curvature tensor Λ as follows

$$\Lambda(V, \mathbf{t}_1)\mathbf{t}_1 = -C\varrho\mathbf{t}_1. \tag{3.13}$$

Considering the eq.(3.11), eq.(3.12) and eq.(3.13) with second equation in Proposition 2.2, we have the following second order non-linear differential equation

$$k_2'' - k_2k_2' = 0.$$

The solutions of the differential equation calculated as

$$k_2 = B, \; k_2(s) = \sqrt{2c_3}\tan\frac{\sqrt{2c_3}}{2}(s + c_4).$$

In these cases, the last equation in Proposition 2.2 is provided automatically. □

Corollary 3.16. *Let γ be a third kind of null Cartan magnetic trajectory of the Killing magnetic field V on a simply connected space form $(F(C), g)$. Then the ratio of the curvature and torsion of the curve γ is constant, namely, $\frac{\varkappa}{\tau} = const$.*

Corollary 3.17. *Let γ be a second kind of null Cartan magnetic trajectory of the Killing magnetic field V on a simply connected space form $(F(C), g)$. Then γ follows the helical trajectory if and only if γ has the following curvatures and parametric representations*

$$k_1(s) = 1 \text{ and } k_2 = 0 \text{ or } k_2(s) = B,$$
$$\gamma(s) = b_1 + b_2 s + \frac{b_3}{B^2}e^{\sqrt{B^2}s} + \frac{b_4}{B^2}e^{-\sqrt{B^2}s} \text{ or } \gamma \text{ is a null cubic},$$

where B is a non-zero constant and $b_i \in \mathbb{E}^3_1$, $i = 1, 2, 3, 4$.

Proof. Let γ be a third kind of null Cartan magnetic trajectory of the Killing magnetic field V on a simply connected space form $(F(C), g)$. Then the curve γ has the following curvatures

$$k_2(s) = B \text{ or } k_2(s) = \sqrt{2c_3}\tanh\frac{\sqrt{2c_3}}{2}(s + c_4).$$

Since we want to a helical trajectory with the axis V the Killing vector field V must satisfies $g(V, V) = const$. This condition is satisfied when the second kind null Cartan magnetic curve equal to $k_2(s) = B$. In this situation we obtain that the null Cartan helices have the constant torsion and the position vector of the helices satisfy the following higher-order linear ordinary differential equation:

$$\gamma^{(4)} - B^2\gamma'' = 0.$$

The solution of the differential equation give that the helical trajectory is a null cubic or has the following parametric representation

$$\gamma(s) = d_1 + d_2 s + \frac{d_3}{B^2}e^{\sqrt{B^2}s} + \frac{d_4}{B^2}e^{-\sqrt{B^2}s}, \quad \tau = -\frac{B^2}{2} < 0,$$

where $d_i \in \mathbb{E}^3_1$, $i = 1, 2, 3, 4$. □

Remark 3.18. A third kind of magnetic trajectory that has the second Bishop curvature $k_2(s) = -\sqrt{2c_1}\tanh\frac{\sqrt{2c_1}}{2}(s + c_2)$ is a helix but the Killing vector field V does not the axis of this helix.

Example 3.19. We consider a null helices which is a null Cartan magnetic curve in Lorentzian 3-space defined by

$$\gamma(s) = (s^3 + \frac{s}{12}, \frac{s^2}{2}, s^3 - \frac{s}{12}).$$

The curve has the following Bishop curvatures:

$$k_1(s) = 1,\ k_2(s) = 0.$$

Then we can easily calculate that $\sigma = 0$, and so γ is a null Cartan helical magnetic trajectory with the null Killing axis

$$V(s) = (-6, 0, 6).$$

The image of this trajectory is rendered in Figure 1.

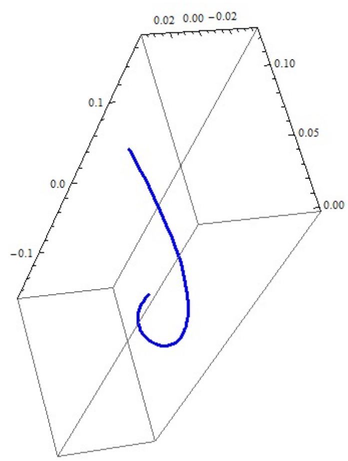

Figure 1. The helical trajectory of a charged particle in the null killing magnetic field.

Example 3.20. Using the appropriate coefficient selections, the following second (third) types of helical magnetic trajectory is determined

$$\gamma(s) = (\frac{e^s}{2} - \frac{e^{-s}}{2}, s, \frac{e^s}{2} + \frac{e^{-s}}{2}).$$

The curve has the following curvatures and axis, respectively,

$$k_1 = 1 \text{ and } k_2 = \pm 1,$$

$$V(s) = (-\frac{e^s}{4} - \frac{e^{-s}}{4}, -\frac{3}{2}, -\frac{e^s}{4} + \frac{e^{-s}}{4})$$

The image of the trajectory is plotted in Figure 2.

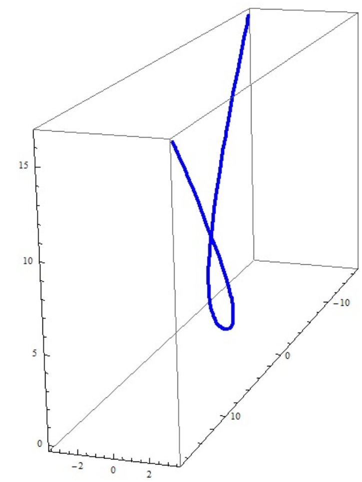

FIGURE 2. The helical trajectory of a charged particle in the spacelike Killing magnetic field.

4. CONCLUSIONS

From the Theorem 3.6, Theorem 3.9, and Theorem 3.15 we obtain that when a charged particle enter in a Killing magnetic field it follows a helical trajectory that has the Killing axis V if and only if γ has the form illustrated in Table 1.

Helix type	Parametric representation	Curvature	Axis
First kind	$\gamma(s) = a_1 + a_2 s + a_3 s^2 + a_4 s^3$	$k_2 = 0$	$V = \sigma \mathbf{t}_1 + \mathbf{n}_2$
Second kind	$\gamma(s) = b_1 + b_2 s + \frac{b_3}{A^2} e^{\sqrt{A^2} s} + \frac{b_4}{A^2} e^{-\sqrt{A^2} s}$	$k_2 = -A$	$V = -A\mathbf{n}_1 + \mathbf{n}_2$.
Third kind	$\gamma(s) = d_1 + d_2 s + \frac{d_3}{B^2} e^{\sqrt{B^2} s} + \frac{d_4}{B^2} e^{-\sqrt{B^2} s}$	$k_2 = B$	$V = -B\mathbf{n}_1 + \mathbf{n}_2$.

TABLE 1. Trajectories according to different types of helices.

We can see the helical trajectory with the null Killing axis in the structure of black holes. The rays coming out of the black hole shown in Figure 3 are the lines that consist of null points. The trajectory in Figure 3 is a null Cartan helix that accepts these lines as axes. If we consider null curves as the path of light in space, such helices are null. Therefore, null curves with the Killing axis can match this structure. That is, if a charged particle enters a magnetic field such that the magnetic vector field and velocity vector field makes a constant angle, it follows a trajectory in the form of a helical curve. In the photo below, the simulation of the image of a star resulting from the gravitational force of a black hole is given (see [21]). We show the null Cartan helical trajectory in Figure 3.

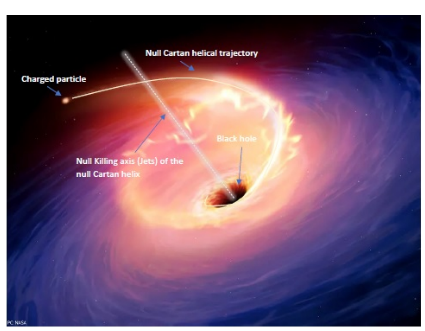

FIGURE 3. The relation Black hole and Null Cartan helical trajectory.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

References

[1] Barros, M., Cabrerizo, J.L., Fernández, M., Romero, A., *Magnetic vortex filament flows.* J. Math. Phys. **48**(2007), 1–27. 3

[2] Barros, M., Ferrández, A., Lucas, P., Merono, M.A., *General helices in the 3-dimensional Lorentzian space forms.* Rocky. Mt. J. Math. **31**(2001), 373–388. 3

[3] Bishop, L.R., *There is more than one way to frame a curve.* Amer. Math. Monthly. **82 (3)**(1975), 246–251.

[4] Bozkurt, Z., Gök, İ., Yaylı, Y., Ekmekci, F.N., *A new approach for magnetic curves in 3D Riemannian manifolds.* J. Math. Physics. **55**(2014), 053501. 3

[5] Druta-Romaniuc, S.L., Munteanu, M.I., *Killing magnetic curves in a Minkowski 3-space.* Nonlinear Anal. Real World Appl. **14**(2013), 383–396. 3

[6] Duggal, K.L., Jin, D.H., Null Curves and Hypersurfaces of Semi-Riemannian Manifolds. World Scientific, Singapore, 2007. 2

[7] Ferrández, A., Giménez, A. & Lucas, P., *Geometrical particles models on 3D null curves.* Physics Letters B, **543(3–4)**(2002), 311–317. 1

[8] Ferrández, A., Giménez, A., Lucas, P., *Relativistic particles and the geometry of 4D null curves.* J. Geom. Phys. **57(10)**(2007), 2124–2135 . 1

[9] Giménez, A., *Relativistic particles along null curves in 3D Lorentzian space forms.* Int. J. Bifurcation and Chaos in Appl. Sci. and Engineering, **20(9)**(2010), 2851–2859 . 1

[10] Grbović, M., Nešović, E., *On the Bishop frames of pseudo null and null Cartan curves in Minkowski 3-space.* J. Math. Anal. Appl. **461**(2018), 219–233. 2.1, 2

[11] Hughston, L.P., Shaw, W.T., *Real classical strings.* Proc. Roy. Soc. London Ser. A. **414**(1987), 415–422. 1

[12] Hughston, L.P., Shaw, W.T., *Classical strings in ten dimensions.* Proc. Roy. Soc. London Ser. A. **414**(1987), 423–431. 1

[13] Hughston, L.P., Shaw, W.T., *Constraint-free analysis of relativistic strings.* Classical Quantum Gravity. **5**(1988), 69–72. 1

[14] Inoguchi, J. & Lee, S., *Null curves in Minkowski 3-space.* Int. Electronic J. Geom. **1(2)**(2008), 40–83. 3.5

[15] Kazan, A., Karadag, H.B., *Magnetic pseudo null and magnetic null curves in Minkowski 3-space.* Int. Math. Forum. **12(3)**(2017), 119–132. 3

[16] Kazan, A., Karadag, H.B., *Magnetic Curves According to Bishop Frame and Type-2 Bishop Frame in Euclidean 3-Space.* British J. Math. Comp. Sci. **22(4)**(2017), 1–18. 3

[17] Özdemir, Z., *Null Cartan Curve Variations in 3D semi-Riemannian Manifold.* Submitted to the journal, 2019. 3.3, 3.4, 3.6

[18] Özdemir, Z., Gök, İ., Yaylı Y., Ekmekci, F.N.: *Notes on Magnetic Curves in 3D semi-Riemannian Manifolds*. Turk. J. Math. **39**(2015), 412–426. 3

[19] Shaw, W. T., Twistors and strings (Santa Cruz, CA, 1986), 337–363. Amer. Math. Soc., RI, 1988. 1

[20] Urbantke, H., On Pinl's representation of null curves in n-dimensions. In Relativity Today, (Budapest, 1987), 34–36. World Sci. Publ., Teaneck, New York, 1988. 1

[21] https://www.brecorder.com/2019/07/08/508109/in-a-first-x-ray-helps-nasa-capture-spinning-black-holes-billions-of-light-years-away/ 4