Proceedings of International Conference on Mathematics
and Mathematics Education (ICMME 2019)
Turk. J. Math. Comput. Sci.
11(Special Issue)(2019) 40–45

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Astrohelicoidal Hypersurfaces in 4-space

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Received: 17-08-2019 • Accepted: 16-12-2019

Abstract. We consider an astrohelicoidal hypersurface which its profile curve has astroid curve in the four dimensional Euclidean space $\mathbb{E}^4$. We also calculate Gaussian curvature and the mean curvature, and Weingarten relation of the hypersurface. Moreover, projecting hypersurface into 3-spaces, we draw some figures.

2010 AMS Classification: Primary 53A05; Secondary 53C42.

Keywords: Astrohelicoidal hypersurface, Gauss map, Gaussian curvature, mean curvature.

1. Introduction

About (hyper)surfaces in the literature, we see some following papers: Arslan et al. [1], Ganchev and Milousheva [4], Güler et al. [6, 7], Güler and Turgay [8], and also some books: Eisenhart [2], Hacısalihoğlu [9], Nitsche [10].

In this paper, we consider the astrohelicoidal hypersurface in Euclidean 4-space $\mathbb{E}^4$. We give some fundamental notions of four dimensional Euclidean geometry in Section 2. In Section 3, we define helicoidal hypersurface. We obtain astrohelicoidal hypersurface, and calculate its curvatures in the last section.

2. Preliminaries

In the rest of this work, we shall identify a vector (a,b,c,d) with its transpose (a,b,c,d)^t. Let $\mathbf{M} = \mathbf{M}(u, v, w)$ be a hypersurface in $\mathbb{E}^4$.

The inner product of vectors $\overrightarrow{x} = (x_1, x_2, x_3, x_4)$ and $\overrightarrow{y} = (y_1, y_2, y_3, y_4)$ on $\mathbb{E}^4$ is defined by as follows:

$$\overrightarrow{x} \cdot \overrightarrow{y} = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4.$$

The vector product of vectors $\overrightarrow{x} = (x_1, x_2, x_3, x_4)$, $\overrightarrow{y} = (y_1, y_2, y_3, y_4)$ and $\overrightarrow{z} = (z_1, z_2, z_3, z_4)$ on $\mathbb{E}^4$ is defined by as follows:

$$\overrightarrow{x} \times \overrightarrow{y} \times \overrightarrow{z} = \begin{pmatrix} x_2y_3z_4 - x_2y_4z_3 - x_3y_2z_4 + x_3y_4z_2 + x_4y_2z_3 - x_4y_3z_2 \\ -x_1y_3z_4 + x_1y_4z_3 + x_3y_1z_4 - x_3z_1y_4 - y_1x_4z_3 + x_4y_3z_1 \\ x_1y_2z_4 - x_1y_4z_2 - x_2y_1z_4 + x_2z_1y_4 + y_1x_4z_2 - x_4y_2z_1 \\ -x_1y_2z_3 + x_1y_3z_2 + x_2y_1z_3 - x_2y_3z_1 - x_3y_1z_2 + x_3y_2z_1 \end{pmatrix}.$$

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Considering a hypersurface $\mathbf{M}$ in $\mathbb{E}^4$, we obtain

$$\det I = \det \begin{pmatrix} E & F & A \\ F & G & B \\ A & B & C \end{pmatrix} = (EG - F^2)C - A^2G + 2ABF - B^2E,$$

and

$$\det II = \det \begin{pmatrix} L & M & P \\ M & N & T \\ P & T & V \end{pmatrix} = \left(LN - M^2\right)V - P^2N + 2PTM - T^2L,$$

where

$$\begin{array}{l} E = \mathbf{M}_u \cdot \mathbf{M}_u,\ F = \mathbf{M}_u \cdot \mathbf{M}_v,\ G = \mathbf{M}_v \cdot \mathbf{M}_v, \\ L = \mathbf{M}_{uu} \cdot e, \quad M = \mathbf{M}_{uv} \cdot e, \quad N = \mathbf{M}_{vv} \cdot e, \\ A = \mathbf{M}_u \cdot \mathbf{M}_w,\ B = \mathbf{M}_v \cdot \mathbf{M}_w,\ C = \mathbf{M}_w \cdot \mathbf{M}_w, \\ P = \mathbf{M}_{uw} \cdot e, \quad T = \mathbf{M}_{vw} \cdot e, \quad V = \mathbf{M}_{ww} \cdot e, \end{array}$$

and e is the Gauss map

$$e(u, v, w) = \frac{\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w}{\|\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w\|}.$$

Calculating product matrices $(I)^{-1} \cdot (II)$, we get the matrix of the shape operator $\mathbf{S}$ as follows:

$$\mathbf{S} = \frac{1}{\det I} \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix},$$

$$\begin{aligned} s_{11} &= ABM - CFM - AGP + BFP + CGL - B^2L, \\ s_{12} &= ABN - CFN - AGT + BFT + CGM - B^2M, \\ s_{13} &= ABT - CFT - AGV + BFV + CGP - B^2P, \\ s_{21} &= ABL - CFL + AFP - BPE + CME - A^2M, \\ s_{22} &= ABM - CFM + AFT - BTE + CNE - A^2N, \\ s_{23} &= ABP - CFP + AFV - BVE + CTE - A^2T, \\ s_{31} &= -AGL + BFL + AFM - BME + GPE - F^2P, \\ s_{32} &= -AGM + BFM + AFN - BNE + GTE - F^2T, \\ s_{33} &= -AGP + BFP + AFT - BTE + GVE - F^2V. \end{aligned}$$

So, we get the following formulas of the Gaussian curvature:

$$K = \frac{\left(LN - M^2\right)V + 2MPT - P^2N - T^2L}{(EG - F^2)C + 2ABF - A^2G - B^2E},$$

and the mean curvature:

$$H = \frac{(EN + GL - 2FM)C + (EG - F^2)V - A^2N - B^2L - 2(APG + BTE - ABM - ATF - BPF)}{3\det I}.$$

3. Helicoidal Hypersurface

We consider a new kind helicoidal hypersurface which its profile curve has astroid curve in the four dimensional Euclidean space $\mathbb{E}^4$.

$\gamma : I \longrightarrow \Pi$ be a space curve for an open interval $I \subset \mathbb{R}$, and let ℓ be a line in Π. A rotational hypersurface is defined as a hypersurface rotating a curve γ profile curve around axis ℓ in $\mathbb{E}^4$.

When a profile curve γ rotates around the axis ℓ, it simultaneously displaces parallel lines which are orthogonal to the axis ℓ, so that the speed of displacement is proportional to the speed of rotation. Hence, obtaining hypersurface is named the helicoidal hypersurface has axis ℓ and pitches p and q for positive real numbers.

We may suppose that ℓ is the line spanned by the vector $(0,0,0,1)^t$. The orthogonal matrix which fixes the above vector is

$$Z(v,w) = \begin{pmatrix} \cos v \cos w & -\sin v & -\cos v \sin w & 0 \\ \sin v \cos w & \cos v & -\sin v \sin w & 0 \\ \sin w & 0 & \cos w & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $v, w \in \mathbb{R}$.

The matrix Z can be found by solving the following equations, simultaneously,

$$Z\ell = \ell, \quad Z^t Z = Z Z^t = I_4, \quad \det Z = 1.$$

When the axis of rotation is ℓ, there is an Euclidean transformation by which the axis is ℓ transformed to the x_4-axis of $\mathbb{E}^4$. Parametrization of the profile space curve is given by

$$\gamma(u) = (f(u), g(u), 0, \Psi(u)),$$

where $f(u), g(u), \Psi(u) : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ are differentiable functions for all $u \in I$.

Hence, the helicoidal hypersurface which is spanned by the vector $(0,0,0,1)^t$ is as follows

$$\mathbf{H}(u,v,w) = \underbrace{Z(v,w).\gamma(u)^t}_{rotation} + \underbrace{(pv + qw).\ell}_{translation},$$

where $0 \le u, v, w < 2\pi$.

In the end, we write following helicoidal hypersurface:

$$\mathbf{H}(u,v,w) = \begin{pmatrix} f(u) \cos v \cos w - g(u) \sin v \\ f(u) \sin v \cos w + g(u) \cos v \\ f(u) \sin w \\ \Psi(u) + pv + qw \end{pmatrix}.$$

4. Astrohelicoidal Hypersurface

Now, by using rotational matrix in $\mathbb{E}^4$, and profile curve γ with translation vector on axis x_4, we find helicoidal hypersurface which has astroid curve. Resulting hypersurface we called is astrohelicoidal hypersurface $\mathfrak{A}(u,v,w)$.

Considering function $\Psi(u)$ on the profile curve γ, we calculate the Gauss map of the hypersurface. Then we find the Gaussian curvature and the mean curvature of the astrohelicoidal hypersurface.

We also draw some figures of the astrohelicoidal hypersurface, and its Gauss map with projection from four dimensional Euclidean space to the three dimensional Euclidean space.

In $\mathbb{E}^4$, *astrohelicoidal hypersurface* $\mathfrak{A}(u,v,w)$ which is spanned by the vector $(0,0,0,1)^t$ for pitches $p, q \in \mathbb{R}^+$, and also $a \in \mathbb{R}$, is defined by as follows:

$$\mathfrak{A}(u,v,w) = \begin{pmatrix} -a \sin^3 u \sin v + a \cos^3 u \cos v \cos w \\ a \sin^3 u \cos v + a \cos^3 u \sin v \cos w \\ a \cos^3 u \sin w \\ \Psi(u) + pv + qw \end{pmatrix} = \begin{pmatrix} x_1(u,v,w) \\ x_2(u,v,w) \\ x_3(u,v,w) \\ x_4(u,v,w) \end{pmatrix}, \tag{4.1}$$

where the parametrization of the profile space curve is given by

$$\gamma(u) = \left(a \cos^3 u, a \sin^3 u, 0, \Psi(u)\right),$$

$\Psi : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable function for all $0 \le u, v, w < 2\pi$.

Taking $\Psi(u) = \cos^3 u$ in $\gamma(u)$, $w = \pi/4$ in (4.1), we get some projection surfaces into 3-space as in Figure 1, and in Figure 2.

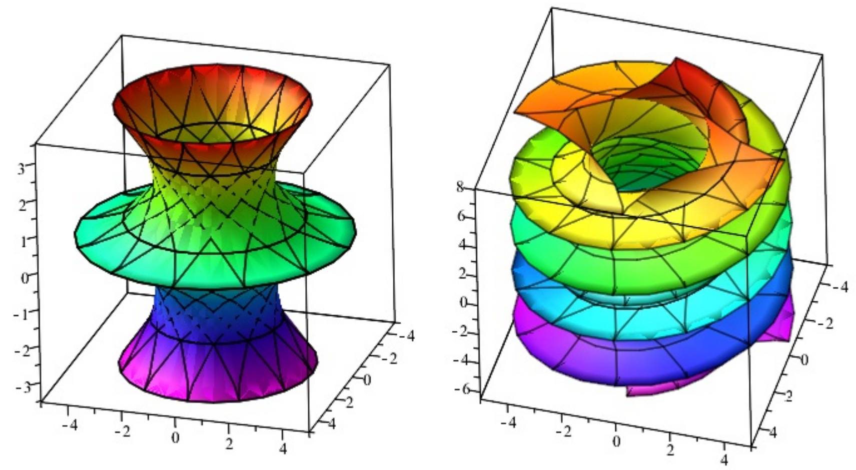

Figure 1. Projections of $\mathfrak{A}(u, v, w)$, **Left**: into $x_1x_2x_3$-space, **Right**: into $x_1x_2x_4$-space

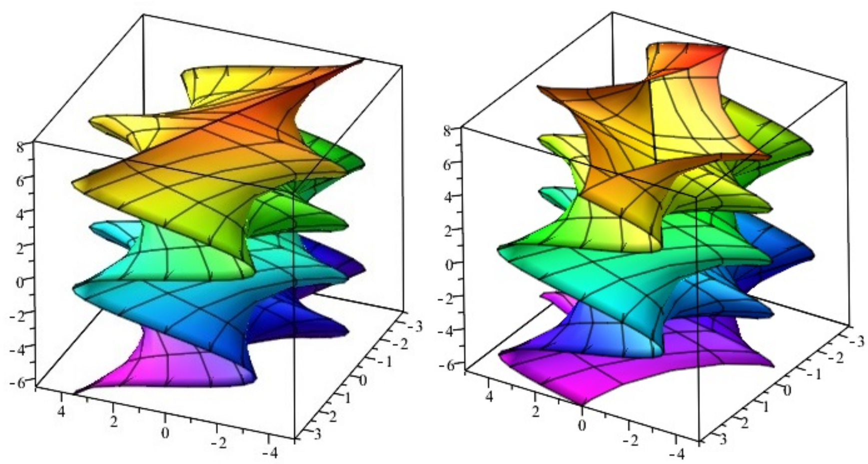

Figure 2. Projections of $\mathfrak{A}(u, v, w)$, **Left**: into $x_1x_3x_4$-space, **Right**: into $x_2x_3x_4$-space

Using the first differentials of (4.1) with respect to u, v, w, we get the first quantities as follows

$$\begin{aligned} E &= 9a^2 \sin^2 u \cos^2 u \cos w + \Psi'^2, \\ F &= 3 \sin^2 u \cos^2 u \cos w + p\Psi', \\ G &= -a^2 \cos^2 u \left(\cos^4 u \sin^2 w + 3 \sin^2 u\right), \\ A &= q\Psi', \\ B &= a^2 \sin^3 u \cos^3 u \sin w + pq, \\ C &= a^2 \cos^6 u + q^2, \end{aligned}$$

and have

$$\begin{aligned} \det I = \; & \{3 \cos^4 u \cos^2 w \left(1 - 3 \sin^2 u \cos^2 u\right) \Psi'^2 - 6 \sin^2 u \cos^3 u (p \cos^3 u - q \sin^3 u \sin w)\Psi' \\ & + \sin^2 u[-36a^2 \cos^8 u \sin^2 u \cos^2 w + 9((a^2 + q^2) \cos^2 w + \left(p^2 - q^2\right)) \cos^6 u \\ & -18pq \sin^3 u \cos^3 u \sin w - 9q^2 \sin^2 u \cos^2 u \left(\cos^3 w + 3\right)]\}a^4 \cos^2 u. \end{aligned}$$

The Gauss map $e_{\mathfrak{A}}(u, v, w)$ of the astrohelicoidal hypersurface $\mathfrak{A}(u, v, w)$ is as follows

$$e_{\mathfrak{A}}(u,v,w) = \frac{1}{W}\begin{pmatrix} e_1(u,v,w) \\ e_2(u,v,w) \\ e_3(u,v,w) \\ e_4(u,v,w) \end{pmatrix}, \tag{4.2}$$

where

$$\begin{aligned}
e_1 &= \Psi'(\cos^3 u\cos v\cos w - \sin^3 u\sin v)\cos^2 u\cos w \\
&\quad +3p(-\sin^2 u\cos v\cos w + \sin u\cos u\sin v)\cos^3 u \\
&\quad +3q((-\cos u\sin v + \sin u\cos v\cos w)\cos^4 u + \left(2\cos^2 u - 1\right)\cos u\sin v)\sin w, \\
e_2 &= \Psi'(\cos^3 u\sin v\cos w + \sin^3 u\cos v)\cos^2 u\cos w \\
&\quad +3p(-\sin^2 u\sin v\cos w - \sin u\cos u\cos v)\cos^3 u \\
&\quad +3q((\cos u\cos v + \sin u\sin v\cos w)\cos^4 u - \left(2\cos^2 u - 1\right)\cos u\cos v)\sin w, \\
e_3 &= \Psi'\cos^5 u\sin w\cos w - 3p\sin^2 u\cos^3 u\sin w + 3q(\cos^4 u\sin^2 w - \left(2\cos^2 u - 1\right))\sin u, \\
e_4 &= 3a(2\cos^2 u - 1)\sin u\cos^3 u\cos w,
\end{aligned}$$

and

$$\begin{aligned}
W &= \{3\cos^4 u\cos^2 w\left(1 - 3\sin^2 u\cos^2 u\right)\Psi'^2 - 6\sin^2 u\cos^3 u(p\cos^3 u - q\sin^3 u\sin w)\Psi' \\
&\quad + \sin^2 u[-36a^2\cos^8 u\sin^2 u\cos^2 w + 9((a^2+q^2)\cos^2 w + \left(p^2 - q^2\right))\cos^6 u \\
&\quad -18pq\sin^3 u\cos^3 u\sin w - 9q^2\sin^2 u\cos^2 u\left(\cos^3 w + 3\right)]\}^{1/2}.
\end{aligned}$$

Finally, the Gaussian curvature of the astrohelicoidal hypersurface is as follows:

$$K = \frac{\Theta}{(\det I)^{5/2}},$$

where

$\Theta = -144\sqrt{2}C^7(C^4-C^2+1/6)(C^4-C^2+1/2)\Psi'^3 - 1/2C^2(((C^{12}-3/2\,C^{10}+5/6\,C^8-1/6\,C^6)\Psi'' - 41C^4 - 20C^{12} - 1 + 72C^{10} - 223/2C^8 + 91C^6 + 10C^2)\sqrt{2}S - 30C^{13} + 87C^{11} - 99C^9 + 58C^7 - 18C^5 + 2C^3)\Psi'^2 - 4S^2C((\sqrt{2}C^2(C^2-1/2)(C^8-19/8\,C^6+5/2\,C^4-5/4\,C^2+1/4)\Psi'' - 3C^{10} - 399/16\,C^6 + 105/8\,C^8 - 63/8\,C^2 + 9/8 + 339/16\,C^4 + 3/4C^{12}) - 3/2C^5S((C^6-3/2\,C^4+5/6\,C^2-1/6)\Psi'' - 8C^6 + 29/2\,C^4 - 37/4\,C^2 + 11/4))\Psi' - 3/2(C-1)((C^2(C^2-1/2)(C^8+12C^6-19C^4+10C^2-2)\Psi'' - 12 + 48C^{12} + 12C^{10} - 567/2\,C^8 + 867/2\,C^6 - 291C^4 + 93C^2)\sqrt{2}S + (-16C^{13} + 46C^{11} - 49C^9 + 23C^7 - 4C^5)\Psi'' - 30C^{13} + 273C^{11} - 600C^9 + 543C^7 - 222C^5 + 36C^3)(C+1))C^3.$

The mean curvature of it is as follows:

$$H = \frac{\Omega}{(\det I)^{3/2}},$$

where

$\Omega = 1/12\,((-6C^{11}+6C^9-2C^7)\Psi'^3 - 36C^2S(C^8-9/4\,C^6+2C^4-5/6\,C^2+1/6)\Psi'^2 + (36a^2C^{15} - 72a^2C^{13} + (30a^2-6)C^{11} + (6a^2+186)C^9 + (-3a^2-453)C^7 + 420C^5 - 186C^3 + 30C)\Psi' + 18(C^2(C^2-1/2)(2/3 + a^2C^{10} - a^2C^8 + 1/3(1+a^2)C^6 + 2C^4 - 2C^2)\Psi'' + 1 + 12a^2C^{14} - 36a^2C^{12} + (97/2\,a^2+10)C^{10} + (-77/2\,a^2 - 7/2)C^8 + (17a^2 - 43/2)C^6 + (-3a^2+22)C^4 - 8C^2)S)\sqrt{2} + 27(C-1)((1/9\,C^4 - 1/27\,C^2 - 5/27\,C^6)\Psi'^2 - 2/27\,CS(C^4+2C^2+1/2)\Psi' + (-2/27C^6 + 1/9\,C^4 - 1/27\,C^2)\Psi'' + 5/9 + a^2C^{10} - 13/6\,a^2C^8 + (14/9\,a^2 - 2/9)C^6 + (11/6 - 1/3\,a^2)C^4 - 2C^2)C^3(C+1).$

Here, $C := \cos u$ and $S := \sin u$.

Corollary 1. *Let $\mathfrak{A} : M^3 \longrightarrow \mathbb{E}^4$ be an immersion given by (4.1). Then M^3 has following Weingarten relation*

$$3\Theta\,H + W^{1/2}\Omega\,K = 0,$$

where Θ and Ω are the numerator functions of K and H, respectively.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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