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Padovan and Pell-Padovan Octonions

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Abstract. In this paper, we define the Padovan and Pell-Padovan octonions by using the Padovan and Pell-Padovan numbers. We give the generating functions, Binet's formulas, sums formulas and some properties for these octonions. We also present the matrix representations of the Padovan and Pell-Padovan octonions.

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1. Introduction

Octonion algebra is eight dimensional, non-commutative, non-associative and normed division algebra. Let O be the octonion algebra over the real number field $\mathbb{R}$. It is known, by the Cayley-Dickson process that any $p \in O$ can be written as

$$p = p\prime + p\prime\prime e$$

where $p\prime, p\prime\prime \in H = \{a_0 + a_1 i + a_2 j + a_3 k : i^2 = j^2 = k^2 = -1, ijk = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\}$, the real quaternion division algebra. The addition and multiplication of any two octonions, $p = p\prime + p\prime\prime e, q = q\prime + q\prime\prime e$, are defined by

$$p + q = (p\prime + q\prime) + (p\prime\prime + q\prime\prime)e$$

and

$$pq = (p\prime q\prime - \overline{q\prime\prime} p\prime\prime) + (q\prime\prime p\prime + p\prime\prime \overline{q\prime})e$$

where $\overline{q\prime}$, $\overline{q\prime\prime}$ denote the conjugates of the quaternions $q\prime$, $q\prime\prime$ respectively. Thus O is an eight-dimensional non-associative division algebra over the real numbers $\mathbb{R}$. A natural basis of this algebra as a space over $\mathbb{R}$ is formed by the elements

$e_0 = 1, e_1 = i, e_2 = j, e_3 = k, e_4 = e, e_5 = ie, e_6 = je, e_7 = ke.$

The multiplication table for the basis of O is

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.	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

Under this notation, all octonions take the form

$$p = \sum_{s=0}^{7} p_s e_s$$

where the coefficients p_s are real. Also, every $p \in O$ can be simply written as $p = Re(p) + Im(p)$, where $Re(p) = p_0$ and $Im(p) = \sum_{s=1}^{7} p_s e_s$ are called the real and imaginary parts, respectively. The conjugate of p is defined to be

$$\overline{p} = \overline{p\prime} - p\prime\prime e = Re(p) - Im(p).$$

This operation satisfies

$$\overline{\overline{p}} = p, \quad \overline{(p+q)} = \overline{p} + \overline{q}, \quad \overline{pq} = \overline{q}\,\overline{p}$$

for all $p, q \in O$. The norm of p is defined to be

$$N_p = p\overline{p} = \overline{p}p = \sum_{s=0}^{7} p_s^2.$$

The inverse of non-zero octonion $p \in O$ is

$$p^{-1} = \frac{\overline{p}}{N_p}.$$

For all $p, q \in O$

$$N_{pq} = N_p N_q,$$

$$(pq)^{-1} = q^{-1} p^{-1}.$$

O is non-commutative, non-associative but it is alternative

$$p(pq) = p^2 q, (qp)p = qp^2, (pq)p = p(qp) := pqp,$$

([2, 13, 14])

In the literature, many authors studied sequences of integer number defined by recurrence relations such as Fibonacci, Lucas, Pell, Jacobsthal, Tribonacci, Tribonacci-Lucas, Padovan, Pell-Padovan, Perrin sequences and their generalizations. For rich applications of these sequences in science and nature, one can see the citations in (see, for example, [11, 12]).

Padovan sequence is defined by the initial values $P_0 = P_1 = P_2 = 1$ and the recurrence relation

$$P_n = P_{n-2} + P_{n-3} \tag{1.1}$$

for all $n \geq 3$. The first few values of the Padovan numbers are

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, \ldots.$$

Binet's formula of the nth Padovan number is given by

$$P_n = \alpha r_1^n + \beta r_2^n + \gamma r_3^n$$

where r_1, r_2 and r_3 are the roots of the equation $x^3 - x - 1 = 0$ and

$$\alpha = \frac{(r_2-1)(r_3-1)}{(r_1-r_2)(r_1-r_3)}, \beta = \frac{(r_1-1)(r_3-1)}{(r_2-r_1)(r_2-r_3)}, \gamma = \frac{(r_1-1)(r_2-1)}{(r_1-r_3)(r_2-r_3)}.$$

Pell-Padovan sequence is defined by the initial values $R_0 = R_1 = R_2 = 1$ and the recurrence relation

$$R_n = 2R_{n-2} + R_{n-3}, \tag{1.2}$$

for all $n \geq 3$. The first few values of the Pell-Padovan numbers are

$$1, 1, 1, 3, 3, 7, 9, 17, 25, 43, 67, 111, 177, 289, \ldots.$$

Binet's formula of the nth Pell-Padovan number is given by

$$R_n = 2\left(\frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2}\right) - 2\left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right) + r_3^{n+1}$$

where r_1, r_2 and r_3 are the roots of the equation $x^3 - 2x - 1 = 0$ [10].

On the other hand, several authors have defined new classes of quaternion and octonion numbers associated with these sequences of integer number. In [9] and [8], the authors defined the Fibonacci and Lucas quaternions, octonions with the classic Fibonacci and Lucas numbers and studied the properties of these quaternions and octonions. Several interesting and useful extensions of many of the familiar quaternions and octonions numbers such as the Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas, Tribonacci quaternions and octonions have been considered by many authors. In addition, generating functions, Binet's formulas and identities involving these octonions have been presented (see, for example, [1, 3–7]).

2. Main Results

In this paper, we aim at establishing new classes of octonion numbers associated with the Padovan and Pell-Padovan numbers and introduce the Padovan and Pell-Padovan octonions by using recurrence relations of the Padovan and Pell-Padovan sequence. It is introduced the Binet's formulas known as the general formulas and the generating functions, sums formulas and some properties for these octonions. We present the matrix representations of the Padovan and Pell-Padovan octonions and the terms of these octonions are derivated by the matrix.

2.1. **Padovan Octonions.**

Definition 2.1. For $n \geq 0$, the nth Padovan octonion is defined by

$$OP_n = \sum_{i=0}^{7} P_{n+i} e_i$$

where P_n is the nth Padovan number and $(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ is the standard octonion basis.

We now give the following theorem for the recurrence relation of the Padovan octonions.

Theorem 2.2. *Let OP_n be the nth Padovan octonion. The sequence $\{OP_n\}$ of the Padovan octonions satisfies following second order recurrence relation*

$$OP_n = OP_{n-2} + OP_{n-3}$$

with inital conditions $OP_0 = \sum_{i=0}^{7} P_i e_i, OP_1 = \sum_{i=0}^{7} P_{1+i} e_i, OP_2 = \sum_{i=0}^{7} P_{2+i} e_i$.

Proof. Using the (1.1) and Definiton 2.1 we have

$$\begin{aligned} OP_{n-2} + OP_{n-3} &= \sum_{i=0}^{7} P_{n-2+i}e_i + \sum_{i=0}^{7} P_{n-3+i}e_i \\ &= \sum_{i=0}^{7} (P_{n-2+i} + P_{n-3+i})e_i \\ &= \sum_{i=0}^{7} P_{n+i}e_i \\ &= OP_n. \end{aligned}$$

So theorem is completed. □

The next theorem gives the generating function for the Padovan octonions.

Theorem 2.3. *Let OP_n be the nth Padovan octonion. The generating function of the Padovan octonions is*

$$r(t) = \frac{OP_0 + OP_1 t + (OP_2 - OP_0 t^2)}{1 - t^2 - t^3}.$$

Proof. Let

$$r(t) = \sum_{n=0}^{\infty} OP_n t^n$$

be generating function of the Padovan octonions. On the other hand, multiplying both sides of this equation by t^2 and t^3, we obtain

$$\begin{aligned} r(t) &= OP_0 + OP_1 t + OP_2 t^2 + OP_3 t^3 \cdots + OP_n t^n + \cdots \\ t^2 r(t) &= OP_0 t^2 + OP_1 t^3 + OP_2 t^4 + OP_3 t^5 + \cdots + OP_{n-2} t^n + \cdots \\ t^3 r(t) &= OP_0 t^3 + OP_1 t^4 + OP_2 t^5 + OP_3 t^6 + \cdots + OP_{n-3} t^n + \cdots \end{aligned}$$

and we write

$$(1 - t^2 - t^3)r(t) = OP_0 + OP_1 t + (OP_2 - OP_0)t^2 + (OP_3 - OP_1 - OP_0)t^3 + ... + (OP_n - OP_{n-2} - OP_{n-3})t^n + ...$$

Using the sequence $\{OP_n\}$ of the Padovan octonions satisfies following second order recurrence relation

$$OP_n = OP_{n-2} + OP_{n-3}$$

with inital conditions $OP_0 = \sum_{i=0}^{7} P_i e_i, OP_1 = \sum_{i=0}^{7} P_{1+i} e_i, OP_2 = \sum_{i=0}^{7} P_{2+i} e_i$. Then we obtain

$$r(t) = \frac{OP_0 + OP_1 t + (OP_2 - OP_0 t^2)}{1 - t^2 - t^3}.$$

So theorem is completed. □

The next theorem gives the Binet's formula for the Padovan octonions.

Theorem 2.4. *For $n \geq 0$, the Binet's formula for the Padovan octonions is*

$$OP_n = \alpha^* \alpha r_1^n + \beta^* \beta r_2^n + \gamma^* \gamma r_3^n$$

where r_1, r_2 and r_3 are the roots of the equation $x^3 - x - 1 = 0$ and

$$\alpha^* = \sum_{i=0}^{7} r_1^i e_i, \beta^* = \sum_{i=0}^{7} r_2^i e_i, \gamma^* = \sum_{i=0}^{7} r_3^i e_i,$$

$$\alpha = \frac{(r_2 - 1)(r_3 - 1)}{(r_1 - r_2)(r_1 - r_3)}, \ \beta = \frac{(r_1 - 1)(r_3 - 1)}{(r_2 - r_1)(r_2 - r_3)}, \ \gamma = \frac{(r_1 - 1)(r_2 - 1)}{(r_1 - r_3)(r_2 - r_3)}.$$

Proof. Consider the Binet's formula of the Padovan sequence is

$$P_n = \alpha r_1^n + \beta r_2^n + \gamma r_3^n$$

where r_1, r_2 and r_3 are the roots of the equation $x^3 - x - 1 = 0$ and

$$\alpha = \frac{(r_2-1)(r_3-1)}{(r_1-r_2)(r_1-r_3)}, \beta = \frac{(r_1-1)(r_3-1)}{(r_2-r_1)(r_2-r_3)}, \gamma = \frac{(r_1-1)(r_2-1)}{(r_1-r_3)(r_2-r_3)}.$$

On the other hand, from Definition 2.1 we have

$$OP_n = \sum_{i=0}^{7} P_{n+i}e_i = P_n + P_{n+1}e_1 + P_{n+2}e_2 + P_{n+3}e_3 + P_{n+4}e_4 + P_{n+5}e_5 + P_{n+6}e_6 + P_{n+7}e_7.$$

Then we obtain

$$\begin{aligned} OP_n &= \sum_{i=0}^{7} P_{n+i}e_i \\ &= \sum_{i=0}^{7} [\alpha r_1^{n+i} + \beta r_2^{n+i} + \gamma r_3^{n+i}]e_i \\ &= \alpha^* \alpha r_1^n + \beta^* \beta r_2^n + \gamma^* \gamma r_3^n \end{aligned}$$

where $\alpha^* = \sum_{i=0}^{7} r_1^i e_i, \beta^* = \sum_{i=0}^{7} r_2^i e_i, \gamma^* = \sum_{i=0}^{7} r_3^i e_i$. So theorem is completed. □

Theorem 2.5. *Let OP_n be the nth Padovan octonion. Then we get the following sums formulas*

i. $\sum_{m=0}^{n} OP_m = OP_{n+3} + OP_{n+2} - OP_4$,

ii. $\sum_{m=0}^{n} OP_{2m} = OP_{2n+3} - OP_1$,

iii. $\sum_{m=0}^{n} OP_{2m+1} = OP_{2n+4} - OP_2$.

Proof. i. We can complete the proof by induction method on n. For $n = 0$ and $n = 1$, we obtain

$$\begin{aligned} \sum_{m=0}^{0} OP_m &= OP_0 = (OP_1 + OP_0) + OP_2 - (OP_2 + OP_1) = OP_3 + OP_2 - OP_4, \\ \sum_{m=0}^{1} OP_m &= OP_0 + OP_1 = OP_3 + OP_4 - OP_4. \end{aligned}$$

We assume that it is true for $n \in \mathbb{Z}^+$, namely

$$\sum_{m=0}^{n} OP_m = OP_{n+3} + OP_{n+2} - OP_4.$$

Now we shall show it is true for $n+1$. Indeed we have

$$\sum_{m=0}^{n+1} OP_m = \sum_{m=0}^{n} OP_m + OP_{n+1}.$$

Using our assumption for $n+1$ we have

$$\sum_{m=0}^{n+1} OP_m = OP_{n+3} + OP_{n+2} - OP_4 + OP_{n+1}.$$

By Theorem 2.2, since $OP_{n+4} = OP_{n+2} + OP_{n+1}$ we obtain

$$\sum_{m=0}^{n+1} OP_m = OP_{n+4} + OP_{n+3} - OP_4.$$

The proofs of ii. and iii. are obtained by induction method on n. □

Theorem 2.6. *Let for $n \geq 1$ be integer. We have*

i. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} OP_2 \\ OP_1 \\ OP_0 \end{bmatrix} = \begin{bmatrix} OP_{n+2} \\ OP_{n+1} \\ OP_n \end{bmatrix}$,

ii. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} OP_2 & OP_1 & OP_0 \\ OP_1 & OP_0 & OP_{-1} \\ OP_0 & OP_{-1} & OP_{-2} \end{bmatrix} = \begin{bmatrix} OP_{n+2} & OP_{n+1} & OP_n \\ OP_{n+1} & OP_n & OP_{n-1} \\ OP_n & OP_{n-1} & OP_{n-2} \end{bmatrix}$

where OP_n is the nth Padovan octonion.

Proof. i. We can complete the proof by induction method on n. If $n = 0$ and $n = 1$ then the result is obviosly true. We assume that it is true for $n \in Z^+$, namely

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} OP_2 \\ OP_1 \\ OP_0 \end{bmatrix} = \begin{bmatrix} OP_{n+2} \\ OP_{n+1} \\ OP_n \end{bmatrix}.$$

Now we shall show that it is true for $n + 1$. For $n + 1$ by using our assumption, we obtain

$$\begin{aligned} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{n+1} \begin{bmatrix} OP_2 \\ OP_1 \\ OP_0 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} OP_2 \\ OP_1 \\ OP_0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} OP_{n+2} \\ OP_{n+1} \\ OP_n \end{bmatrix} \\ &= \begin{bmatrix} OP_{n+3} \\ OP_{n+2} \\ OP_{n+1} \end{bmatrix} \end{aligned}$$

where $OP_{n+3} = OP_{n+1} + OP_n$ from Theorem 2.2. □

The proof of ii. is obtained by induction on n.

2.2. **Pell-Padovan Octonions.**

Definition 2.7. 2. For $n \geq 0$, the nth Pell-Padovan octonion is defined by

$$OR_n = \sum_{i=0}^{7} R_{n+i} e_i$$

where R_n is the nth Pell-Padovan number and $(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ is the standard octonion basis.

We now give the following theorem for the recurrence relation of the Pell-Padovan octonions.

Theorem 2.8. *Let OR_n be the nth Pell-Padovan octonion. The sequence $\{OR_n\}$ of the Pell-Padovan octonions satisfies following second order recurrence relation*

$$OR_n = 2OR_{n-2} + OR_{n-3}$$

with inital conditions $OR_0 = \sum_{i=0}^{7} R_i e_i, OR_1 = \sum_{i=0}^{7} R_{1+i} e_i, OR_2 = \sum_{i=0}^{7} R_{2+i} e_i$.

Proof. By the equation (1.2) and Definiton 2.7, we have

$$\begin{aligned} 2OR_{n-2} + OR_{n-3} &= \sum_{i=0}^{7} 2R_{n-2+i}e_i + \sum_{i=0}^{7} R_{n-3+i}e_i \\ &= \sum_{i=0}^{7}(2R_{n-2+i} + R_{n-3+i})e_i \\ &= \sum_{i=0}^{7} R_{n+i}e_i \\ &= OR_n. \end{aligned}$$

So theorem is completed. □

The next theorem gives the generating function for the Pell-Padovan octonions.

Theorem 2.9. *Let OR_n be the nth Pell-Padovan octonion. The generating function for the Pell-Padovan octonions is*

$$s(t) = \frac{OR_0 + OR_1 t + (OR_2 - 2OR_0 t^2)}{1 - 2t^2 - t^3}.$$

Proof. Let

$$s(t) = \sum_{n=0}^{\infty} OR_n t^n$$

be generating function of the Pell-Padovan octonions. On the other hand, multiplying both sides of this equation by $2t^2$ and t^3, we obtain

$$s(t) = OR_0 + OR_1 t + OR_2 t^2 + OR_3 t^3 + \cdots + OR_n t^n + \cdots$$

$$2t^2 s(t) = 2OR_0 t^2 + 2OR_1 t^3 + 2OR_2 t^4 + 2OR_3 t^5 + \cdots + 2OR_{n-2} t^n + \cdots$$

$$t^3 s(t) = OR_0 t^3 + OR_1 t^4 + OR_2 t^5 + OR_3 t^6 \cdots + OR_{n-3} t^n + \cdots$$

Then we write

$$(1 - 2t^2 - t^3)s(t) = OR_0 + OR_1 t + (OR_2 - 2OR_0)t^2 + (OR_3 - 2OR_1 - OR_0)t^3 + ... + (OR_n - 2OR_{n-2} - OR_{n-3})t^n + ...$$

where the sequence $\{OR_n\}$ of the Pell-Padovan octonions satisfies following second order recurrence relation,

$$OR_n = 2OR_{n-2} + OR_{n-3}$$

with inital conditions $OR_0 = \sum_{i=0}^{7} R_i e_i$, $OR_1 = \sum_{i=0}^{7} R_{1+i} e_i$, $OR_2 = \sum_{i=0}^{7} R_{2+i} e_i$. Then we obtain

$$s(t) = \frac{OR_0 + OR_1 t + (OR_2 - 2OR_0 t^2)}{1 - 2t^2 - t^3}.$$

So theorem is completed. □

The next theorem gives the Binet's formulas for the Pell-Padovan octonions.

Theorem 2.10. *For $n \geq 0$, the Binet's formula for the Pell-Padovan octonions is*

$$OR_n = 2\left(\frac{\alpha^* r_1^{n+1} - \beta^* r_2^{n+1}}{r_1 - r_2}\right) - 2\left(\frac{\alpha^* r_1^{n} - \beta^* r_2^{n}}{r_1 - r_2}\right) + r_3^{n+1}\gamma^*$$

where r_1, r_2 and r_3 are the roots of the equation $x^3 - 2x - 1 = 0$ and

$$\alpha^* = \sum_{i=0}^{7} r_1^i e_i, \beta^* = \sum_{i=0}^{7} r_2^i e_i, \gamma^* = \sum_{i=0}^{7} r_3^i e_i.$$

Proof. Consider the Binet's formula of the Pell-Padovan sequence is

$$R_n = 2\left(\frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2}\right) - 2\left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right) + r_3^{n+1}$$

where r_1, r_2 and r_3 are the roots of the equation $x^3 - 2x - 1 = 0$. On the other hand, from Definition 2.7 we have

$$OR_n = \sum_{i=0}^{7} R_{n+i}e_i = R_n + R_{n+1}e_1 + R_{n+2}e_2 + R_{n+3}e_3 + R_{n+4}e_4 + R_{n+5}e_5 + R_{n+6}e_6 + R_{n+7}e_7.$$

Then we obtain

$$\begin{aligned} OR_n &= \sum_{i=0}^{7} R_{n+i}e_i \\ &= \sum_{i=0}^{7}\left(2\left(\frac{r_1^{n+1+i} - r_2^{n+1+i}}{r_1 - r_2}\right) - 2\left(\frac{r_1^{n+i} - r_2^{n+i}}{r_1 - r_2}\right) + r_3^{n+1+i}\right)e_i \\ &= 2\left(\frac{r_1^{n+1}\alpha^* - r_2^{n+1}\beta^*}{r_1 - r_2}\right) - 2\left(\frac{r_1^n\alpha^* - r_2^n\beta^*}{r_1 - r_2}\right) + r_3^{n+1}\gamma^* \end{aligned}$$

where $\alpha^* = \sum_{i=0}^{7} r_1^i e_i, \beta^* = \sum_{i=0}^{7} r_2^i e_i, \gamma^* = \sum_{i=0}^{7} r_3^i e_i$. So theorem is completed. □

Theorem 2.11. *Let OR_n be the nth Pell-Padovan octonion. Then we get*

$$\sum_{m=0}^{n} OR_m = \frac{1}{2}(OR_{n+2} + OR_{n+1} + OR_n - OR_2 - OR_1 + OR_0).$$

Proof. We can complete the proof by induction method on n. For $n = 0$ and $n = 1$, we obtain

$$\begin{aligned} \sum_{m=0}^{0} OR_m &= OR_0 = \frac{1}{2}(OR_2 + OR_1 + OR_0 - OR_2 - OR_1 + OR_0) \\ \sum_{m=0}^{1} OR_m &= OR_0 + OR_1 \\ &= \frac{1}{2}(2OR_1 + OR_0 + OR_0) \\ &= \frac{1}{2}(OR_3 + OR_2 + OR_1 - OR_2 - OR_1 + OR_0) \end{aligned}$$

We assume that it is true for $n \in \mathbb{Z}^+$, namely

$$\sum_{m=0}^{n} OR_m = \frac{1}{2}(OR_{n+2} + OR_{n+1} + OR_n - OR_2 - OR_1 + OR_0).$$

Now we shall show that it is true for $n + 1$. Indeed we have

$$\sum_{m=0}^{n+1} OR_m = \sum_{m=0}^{n} OR_m + OR_{n+1}.$$

Using our assumption for $n + 1$ we have

$$\sum_{m=0}^{n+1} OR_m = \frac{1}{2}(OR_{n+2} + OR_{n+1} + OR_n - OR_2 - OR_1 + OR_0 + 2OR_{n+1}).$$

By Theorem 2.8, since $OR_{n+3} = 2OR_{n+1} + OR_n$ we obtain

$$\sum_{m=0}^{n+1} OR_m = \frac{1}{2}(OR_{n+3} + OR_{n+2} + OR_{n+1} - OR_2 - OR_1 + OR_0).$$

So theorem is completed. □

Theorem 2.12. *Let for $n \geq 1$ be integer. We have*

i. $$\begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} OR_2 \\ OR_1 \\ OR_0 \end{bmatrix} = \begin{bmatrix} OR_{n+2} \\ OR_{n+1} \\ OR_n \end{bmatrix},$$

ii. $$\begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} OR_2 & OR_1 & OR_0 \\ OR_1 & OR_0 & OR_{-1} \\ OR_0 & OR_{-1} & OR_{-2} \end{bmatrix} = \begin{bmatrix} OR_{n+2} & OR_{n+1} & OR_n \\ OR_{n+1} & OR_n & OR_{n-1} \\ OR_n & OR_{n-1} & OR_{n-2} \end{bmatrix}$$

where OR_n is the nth Pell-Padovan octonion.

Proof. The theorem is proved by induction method on n. □

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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