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Asymptotic Stability of Linear Delay Difference Equations Including Generalized Difference Operator

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Abstract. In this study, some necessary and sufficient conditions are given for the stability of linear delay difference equations involving generalized difference operator. For the root analysis Schur-Cohn criteria is used and some examples are given to verify the results.

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1. Introduction

Difference equations are the discrete analogues of differential equations and usually describe certain phenomena over the course of time. For a homogeneous linear difference equation, the stability of the or equilibrum point (or steady-state) zero is equivalent to the boundedness of all solutions for $n \geq 0$. For the asymptotic stability of the equilibrum point zero is equivalent to all solutions having zero limit as $n \to \infty$, which is true if and only if every root of the characteristic equation of homogeneous linear difference equation lies in the open disk $|\lambda| < 1$.

The basic theory of difference equations is based on the difference operator Δ defined as

$$\Delta y(n) = y(n+1) - y(n), \qquad n \in \mathbb{N}$$

where $\mathbb{N} = \{1, 2, ...\}$.

In [1, 7, 13] authors suggested the definition of Δ as

$$\Delta y(n) = y(n+l) - y(n), \qquad l \in \mathbb{N}.$$

In [14, 15] authors defined Δ_α as

$$\Delta_\alpha y(k) = y(k+1) - \alpha y(k)$$

where α is a fixed real constant and $k \in \{n_0, n_0+1, ...\}$ and n_0 is a given nonnegative integer.

Throughout this paper we define the operator $\Delta_{l,a}$ as

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$$\Delta_{l,a} y(n) = y(n+l) - ay(n), \quad n, l \in \mathbb{N}, \ a \in \mathbb{R}.$$

Stability of solutions of linear difference equations with constant coefficients requires analysis of root of characteristic equation. In [2, 4, 5, 8–12] authors found some stability results using root analysis. In our study firstly we will consider the asymptotic stability of the zero solution of the difference equation involving generalized difference of the form

$$\Delta^m_{l,a} y(n-l) + r\Delta_{l,a} y(n) + sy(n) = 0 \tag{1.1}$$

with the initial conditions

$$y(i) = \varphi_i \ , \ i = 0, 1, 2, \cdots, (m-1)l - 1 \tag{1.2}$$

where $a, r, s, \in \mathbb{R}$, $l, m, n \in \mathbb{N}$. By solution of equation (1.1) we mean a real sequence $y(n)$ which is defined for $n = 0, 1, 2, \cdots, (m-1)l - 1$ and reduce equation (1.1) to an identity over $\mathbb{N}$. Later we will consider the asymptotic stability of the zero solution of the delay difference equation involving generalized difference of the form

$$\Delta^m_{l,a} y(n-l) + r\Delta_{l,a} y(n) + sy(n-kl) = 0 \tag{1.3}$$

with the initial conditions

$$y(i) = \varphi_i \ , \ i = -kl, -kl+1, \cdots, (m-1)\,l - 1 \tag{1.4}$$

where $a, r, s, \in \mathbb{R}$, $k, l, m, n \in \mathbb{N}$ and $k > 1$. Similarly by solution of equation (1.3) we mean a real sequence $y(n)$ which is defined for $n = -kl, -kl+1, \cdots, (m-1)\,l - 1$ and reduce equation (1.3) to an identity over $\mathbb{N}$.

Paper is organized as follows: In Section 2 we give some definitions, properties of generalized difference operator and some basic lemmas and theorems. We will give stability results for equations (1.1) and (1.3) in section 3. Also we will give illustrative examples which verify the results obtained.

2. Some Definitions, Auxiliary Lemmas and Theorems

In this section we will give some definitions, auxiliary lemmas and theorems which we use throughout this study. For each positive integer m, we define the iterates $\Delta^m_{l,a}$ by

$$\Delta^m_{l,a} y(n) = \Delta_{l,a}\left(\Delta^{m-1}_{l,a} y(n)\right).$$

Basic property of the operator $\Delta_{l,a}$ is shown below.

Lemma 2.1. *For each positive integer m*

$$\Delta^m_{l,a} y(n) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} a^i y(n + (m-i)\,l). \tag{2.1}$$

Proof. The proof can be easily done by induction, so we omit it. □

Definition 2.2 ([3]). Let I be some intervals of real numbers and consider the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}) \tag{2.2}$$

where F is a function that maps some set I^{k+1} into I. Then a point $\bar{x}$ is called an equilibrium point of equation (2.2) if

$$x_n = \bar{x} \text{ for all } n \geq -k.$$

Definition 2.3 ([3]). Let $\bar{x}$ be an equilibrium point of equation (2.2).

(a) An equilibrium point $\bar{x}$ of equation (2.2) is called locally stable if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$is a solution of equation (2.1) with

$$\left|x_{-k} - \bar{x}\right| + \left|x_{-k+1} - \bar{x}\right| + ... + \left|x_0 - \bar{x}\right| < \delta,$$

then

$$\left|x_n - \bar{x}\right| < \varepsilon, \text{ for all } n \geq 0.$$

(b) An equilibrium point $\bar{x}$ of equation (2.2) is called locally asymptotically stable if $\bar{x}$ is locally stable, and if in addition there exists $\gamma > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$is a solution of equation (2.2) with

$$\left|x_{-k}-\bar{x}\right|+\left|x_{-k+1}-\bar{x}\right|+...+\left|x_{0}-\bar{x}\right|<\gamma,$$

then

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(c) An equilibrium point $\bar{x}$ of equation (2.2) is called global attractor if, for every solution $\{x_n\}_{n=-k}^{\infty}$ of equation (2.2) we have

$$\lim_{n\to\infty} x_n = \bar{x}$$

(d) An equilibrium point $\bar{x}$ of equation (2.2) is called globally asymptotically stable if $\bar{x}$ is locally stable, and $\bar{x}$ is also a global attractor of equation (2.2).

(e)An equilibrium point $\bar{x}$ of equation (2.2) is called unstable if it is not stable.

Consider the linear difference equation

$$x_{n+1} = a_0 x_n + a_1 x_{n-1} + \cdots + a_k x_{n-k} \tag{2.3}$$

where $a_i \in \mathbb{R}$, $i = 0, 1, ..., k$, $k \in \mathbb{N}$.

As is customary, a zero solution of (2.3) is said to be asymptotically stable iff all zeros of the corresponding characteristic equation are in the unit disk. Otherwise the zero solution is called unstable.

As it is well known, the asymptotic stability of the zero solution of the linear difference equation is determined by the location of the roots of the associated characteristic equation

$$\lambda^{k+1} - \sum_{i=0}^{k} a_i \lambda^{k-i} = 0.$$

Thus, for each particular choice of the coefficients $a_i, i = 0, ..., k$, one can use the so called Schur–Cohn criterion. However, with this method, it is very difficult to get explicit conditions for a general form of (2.3) depending on the coefficients. This kind of explicit conditions are of special importance in the applications, where the coefficients are meaningful parameters of the model [10].

Definition 2.4 (Inners of a matrix [6]). The inners of a matrix are the matrix itself and all the matrices obtained by omitting successively the first and the last rows and the first and the last columns.

The inners of the following matrix A are shown below.

$$A = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} \end{bmatrix}, \begin{bmatrix} b_{22} & b_{23} & b_{24} \\ b_{32} & b_{33} & b_{34} \\ b_{42} & b_{43} & b_{44} \end{bmatrix}, [b_{33}]$$

Definition 2.5 ([6]). A matrix is said to be innerwise if the determinants of all of its inners are positive.

Consider the linear homogeneous difference equation with constant coefficient

$$y(n+k) + p_1 y(n+k-1) + p_2 y(n+k-2) + \cdots + p_k y(n) = 0 \tag{2.4}$$

where $p_1, p_2, \cdots, p_k$ are real numbers. Then the zero solution of (2.4) is asymptotically stable iff $|\lambda| < 1$ for all characteristic roots λ of (2.4), that is for every zero λ of the characteristic polynomial

$$p(\lambda) = \lambda^k + p_1 \lambda^{k-1} + p_2 \lambda^{k-2} + \cdots + p_k. \tag{2.5}$$

Now the following theorem gives a necessary and sufficient conditions for the zeros of the polynomial (2.5) lie inside the unit disk $|\lambda| < 1$ [6, sec 5.1, page 246].

Theorem 2.6 (Schur-Cohn Criterion [6]). *The zeros of the characteristic polynomial (2.5) lie inside the unit disk if and only if the following hold:*

$$p(1) > 0\ ,\ (-1)^k p(-1) > 0$$

and $(k-1) \times (k-1)$ *matrices*

$$A_{k-1}^{\pm} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ p_1 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ p_{k-3} & p_{k-4} & \cdots & 1 & 0 \\ p_{k-2} & p_{k-3} & \cdots & p_1 & 1 \end{bmatrix} \pm \begin{bmatrix} 0 & 0 & \cdots & 0 & p_k \\ 0 & 0 & \cdots & p_k & p_{k-1} \\ \vdots & \vdots & & & \vdots \\ 0 & p_k & \cdots & & p_3 \\ p_k & p_{k-1} & \cdots & p_3 & p_2 \end{bmatrix}$$

are innervise.

In [6] using the Schur-Cohn Criterion (Theorem 2.1), necessary and sufficient conditions are given on the coefficients p_i such that the zero solution of (2.4) is asymptotically stable. Some compact necessary and sufficient conditions for the zero solutions of (2.4) to be asymptotically stable are available for lower order difference equations. Hence conditions for second and third order difference equations are given below.

For the second order difference equation

$$x(n+2) + p_1 x(n+1) + p_2 x(n) = 0$$

the zero solution is asymptotically stable if and only if

$$|p_1| < 1 + p_2 < 2. \tag{2.6}$$

For the third order difference equation

$$x(n+3) + p_1 x(n+2) + p_2 x(n+1) + p_3 x(n) = 0$$

a necessary and sufficient condition for the zero solution to be asymptotically stable is concluded as [6]

$$|p_1 + p_3| < 1 + p_2 \text{ and } |p_2 - p_1 p_3| < 1 - p_3^2 \tag{2.7}$$

3. Main Results

In this section we will give some stability results for the difference equation (1.1) with initial conditions (1.6), the difference equation (1.3) with initial conditions (1.4) and illustrative examples. For this we will use Schur-Cohn criterion .

Theorem 3.1. *Consider the difference equation* (1.1) *with initial conditions* (1.2) *Then the following statements are equivalent.*

(a) The zero solution of (1.1) *is asymptotically stable.*

(b) Followings hold;

$$\sum_{i=0}^{m-3} (-1)^i \binom{m}{i} a^i + (-1)^{m-2} \binom{m}{2} a^{m-2} + r + \quad (-1)^{m-1} m a^{m-1} - ar +$$
$$s + (-1)^m a^m > 0,$$

$$\sum_{i=0}^{m-3} \binom{m}{i} a^i + \binom{m}{2} a^{m-2} + (-1)^m r + m a^{m-1} +$$
$$(-1)^m (ar - s) + a^m > 0,$$

$$A_{m-1}^{\pm} = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ -\binom{m}{1} a & 1 & 0 & \ldots & 0 \\ \binom{m}{2} a^2 & -\binom{m}{1} a & \ddots & 0 \ldots & 0 \\ \vdots & & & 1 & \\ (-1)^{m-2} \binom{m}{m-2} a^{m-2} + r & (-1)^{m-3} \binom{m}{m-3} a^{m-3} & \cdots & & 1 \end{bmatrix} \pm$$

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & (-1)^m a^m \\ 0 & \dots & 0 & (-1)^m a^m & (-1)^{m-1} ma^{m-1} - ar + s \\ \vdots & & \iddots & (-1)^{m-1} ma^{m-1} - ar + s & \vdots \\ 0 & (-1)^m a^m & \iddots & & \vdots \\ (-1)^m a^m & (-1)^{m-1} ma^{m-1} - ar + s & \cdots & \dots & \binom{m}{2} a^2 \end{bmatrix}$$

matrices of dimension $(m-1) \times (m-1)$ *are innerwise. Here entries of* $A^{\pm}_{m-1}$ *is formed by the coefficients of* $p(t)$ *where*

$$p(t) = \sum_{i=0}^{m-3} (-1)^i \binom{m}{i} a^i t^{m-i} + \left((-1)^{m-2}\binom{m}{2}a^{m-2} + r\right)t^2 + \left((-1)^{m-1} ma^{m-1} - ar + s\right)t + (-1)^m a^m$$

Proof. $(a) \implies (b)$. Suppose that zero solution of (1.1) is asymptotically stable. Since (1.1) is a linear difference equation with constant coefficients then the roots of the corresponding characteristic equation must be in the unit disk. Using Lemma 1.1 we reduce (1.1) to

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} a^i y\left(n + (m-1-i)\,l\right) + ry(n+l) - ary(n) + sy(n) = 0. \tag{3.1}$$

Rearranging (3.1) we get

$$\sum_{i=0}^{m-3} (-1)^i \binom{m}{i} a^i y\left(n + (m-1-i)\,l\right) + \left((-1)^{m-2}\binom{m}{2}a^{m-2} + r\right) y\left(n+l\right) +$$
$$\left((-1)^{m-1} ma^{m-1} - ar + s\right) y(n) + (-1)^m a^m y(n-l) = 0. \tag{3.2}$$

The characteristic equation of (3.2) is

$$\sum_{i=0}^{m-3} (-1)^i \binom{m}{i} a^i \lambda^{(m-i)l} + \left((-1)^{m-2}\binom{m}{2}a^{m-2} + r\right)\lambda^{2l} +$$
$$\left((-1)^{m-1} ma^{m-1} - ar + s\right)\lambda^l + (-1)^m a^m = 0. \tag{3.3}$$

Getting $\lambda^l = t$ in (3.3) we obtain

$$\sum_{i=0}^{m-3} (-1)^i \binom{m}{i} a^i t^{m-i} + \left((-1)^{m-2}\binom{m}{2}a^{m-2} + r\right)t^2 +$$
$$\left((-1)^{m-1} ma^{m-1} - ar + s\right)t + (-1)^m a^m = 0. \tag{3.4}$$

In (3.4) taking

$$p(t) = \sum_{i=0}^{m-3} (-1)^i \binom{m}{i} a^i t^{m-i} + \left((-1)^{m-2}\binom{m}{2}a^{m-2} + r\right)t^2 +$$
$$\left((-1)^{m-1} ma^{m-1} - ar + s\right)t + (-1)^m a^m, \tag{3.5}$$

$p_i = (-1)^i \binom{m}{i} a^i$ for $1 \le i \le m-3$, $p_{m-2} = (-1)^{m-2}\binom{m}{2}a^{m-2} + r$, $p_{m-1} = (-1)^{m-1} ma^{m-1} - ar + s$ and $p_m = (-1)^m a^m$. In view of Theorem 2.1 following conditions are necessary and sufficient conditions for the roots of polynomial in (3.5) to be inside the unit disk $|t| < 1$.

$$p(1) = \sum_{i=0}^{m-3} (-1)^i \binom{m}{i} a^i + (-1)^{m-2}\binom{m}{2}a^{m-2} + r + (-1)^{m-1} ma^{m-1} - ar + s + (-1)^m a^m > 0,$$

$$\begin{aligned} lll\,(-1)^m\,p(-1) &= (-1)^m \sum_{i=0}^{m-3}(-1)^i\binom{m}{i}a^i(-1)^{m-i} + (-1)^m\left((-1)^{m-2}\binom{m}{2}a^{m-2}+r\right) \\ &+ (-1)^m\left((-1)^{m-1}\,ma^{m-1}-ar+s\right)(-1)+(-1)^m\,(-1)^m\,a^m \\ &= \sum_{i=0}^{m-3}\binom{m}{i}a^i+\binom{m}{2}a^{m-2}+(-1)^m\,r+ma^{m-1}+(-1)^m\,(ar-s)+a^m>0, \end{aligned} \tag{3.6}$$

and the matrices $A^{\pm}_{m-1}$ whose entries are formed the coefficients of $p(t)$ must be innerwise where

$$A^{\pm}_{m-1} = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ -\binom{m}{1}a & 1 & 0 & \dots & 0 \\ \binom{m}{2}a^2 & -\binom{m}{1}a & \ddots & 0\dots & 0 \\ \vdots & & & 1 & \\ (-1)^{m-2}\binom{m}{m-2}a^{m-2}+r & (-1)^{m-3}\binom{m}{m-3}a^{m-3} & \dots & & 1 \end{bmatrix} \pm$$

$$\begin{bmatrix} 0 & 0 & \dots & 0 & (-1)^m\,a^m \\ 0 & \dots & 0 & (-1)^m\,a^m & (-1)^{m-1}\,ma^{m-1}-ar+s \\ \vdots & & \iddots & (-1)^{m-1}\,ma^{m-1}-ar+s & \vdots \\ 0 & (-1)^m\,a^m & \iddots & & \vdots \\ (-1)^m\,a^m & (-1)^{m-1}\,ma^{m-1}-ar+s & \dots & & \binom{m}{2}a^2 \end{bmatrix}.$$

Since $|t| = \left|\lambda^l\right| < 1$ and $l > 0$ we can see that $|\lambda| < 1$. Hence (b) is satisfied.

$(b) \implies (a)$. If (3.1), (3.1) and (3.1) hold then for characteristic polynomial of (1.1) conditions of Schur-Cohn criteria are satisfied. So roots of characteristic polynomial be inside the unit disk. Hence the zero solution of (1.1) is asymptotically stable. □

Corollary 3.2. *Consider the difference equation involving generalized difference*

$$\Delta^2_{l,a}y(n-l) + r\Delta_{l,a}y(n) + sy(n) = 0$$

with the initial conditions

$$y\,(i) = \varphi_i\ ,\ i = -l, -l+1, \cdots, l-1$$

where $a, r, s, \in \mathbb{R}, r \neq -1,\ l, n \in \mathbb{N}$. Then the following statements are equivalent.

(a) The zero solution of (3.2) *is asymptotically stable.*

(b) $\left|\frac{s-ar-2a}{1+r}\right| < \frac{a^2}{1+r} + 1 < 2$ holds.

Proof. $(a) \implies (b)$. Suppose that the zero solution of (3.2) is asymptotically stable. For $m = 2$ equation (1.1) reduces to equation (3.2). Using definition of $\Delta_{l,a}$ (3.2) reduces to

$$(1+r)y(n+l) + (s-ar-2a)\,y(n) + a^2y(n-l) = 0,$$

which is equivalent to

$$y(n+l) + \frac{s-ar-2a}{1+r}y(n) + \frac{a^2}{1+r}y(n-l) = 0. \tag{3.7}$$

In view of Theorem 3.1 and (2.6) , the roots of the characteristic polynomial of (3.7) be inside the unit disk $|\lambda| < 1$ if and only if

$$\left|\frac{s-ar-2a}{1+r}\right| < \frac{a^2}{1+r} + 1 < 2$$

holds. Hence (b) is satisfied.

$(b) \implies (a)$. If $\left|\frac{s-ar-2a}{1+r}\right| < \frac{a^2}{1+r} + 1 < 2$ holds then for characteristic polynomial of (3.2), conditions of Schur-Cohn criteria are satisfied. So roots of characteristic polynomial be inside the unit disk. Hence the zero solution of (3.2) is asymptotically stable. □

Example 3.3. Consider the generalized difference equation of the form

$$\Delta^2_{5,1/2}y(n-5) + 2\Delta_{5,1/2}y(n) + \frac{1}{4}y(n) = 0 \tag{3.8}$$

where $l = 5$, $a = 1/2$, $r = 2$, $s = 1/4$. For $m = 2$ all the conditions of Corollary 3.2 are satisfied. Hence the zero solution of equation (3.8) is asymptotically stable.

Corollary 3.4. *Consider the difference equation involving generalized difference*

$$\Delta^3_{l,a}y(n-l) + r\Delta_{l,a}y(n) + sy(n) = 0 \tag{3.9}$$

with the initial conditions

$$y(i) = \varphi_i \; for \; i = -l, -l+1, \cdots, 2l-1$$

where $a, r, s, \in \mathbb{R}$, $n, l \in \mathbb{N}$. Then the following statements are equivalent.
(a) The zero solution of (3.9) *is asymptotically stable.*
(b) $\left|r - 3a - a^3\right| < 1 + 3a^2 - ar + s$ and $\left|3a^2 - ar + s + ra^3 - 3a^4\right| < 1 - a^6$ hold.

Proof. $(a) \Longrightarrow (b)$. Suppose that the zero solution of (3.2) is asymptotically stable. For $m = 3$ equation (1.1) reduces to equation (3.9) which is equivalent to

$$y(n+2l) + (r-3a)\,y(n+l) + \left(3a^2 - ar + s\right)y(n) - a^3y(n-l) = 0. \tag{3.10}$$

In view of Theorem 3.1 and (2.7), the roots of the characteristic polynomial of (3.10) be inside the unit disk $|\lambda| < 1$ if and only if

$$\left|r - 3a - a^3\right| < 1 + 3a^2 - ar + s$$

and

$$\left|3a^2 - ar + s + ra^3 - 3a^4\right| < 1 - a^6$$

hold. Hence (b) is satisfied.
$(b) \Longrightarrow (a)$. Proof is same as in proof of Corollary 3.2. □

Example 3.5. Consider the generalized difference equation of the form

$$\Delta^3_{4,\frac{1}{4}}y(n-4) - \frac{1}{8}\Delta_{4,\frac{1}{4}}y(n) = 0, \tag{3.11}$$

where $l = 4$, $a = 1/4$, $r = -1/8$, $s = 0$. For $m = 3$ all the conditions of Corollary 3.4 are satisfied. Hence the zero solution of equation (3.11) is asymptotically stable.

Theorem 3.6. *Consider the delay difference equation* (1.3) *with initial conditions* (1.4)*. Then the following statements are equivalent.*
(a) The zero solution of (1.3) *is asymptotically stable.*
(b) Followings hold ;

$$p(1) = \sum_{i=0}^{m-3}(-1)^i\binom{m}{i}a^i + (-1)^{m-2}\binom{m}{2}a^{m-2} + r + (-1)^{m-1}\,ma^{m-1}$$

$$-ar + (-1)^m\,a^m + s > 0,$$

$$(-1)^{m+k-1}\,p(-1) = \sum_{i=0}^{m-3}\binom{m}{i}a^i + (-1)^m\left[(-1)^{m-2}\binom{m}{2}a^{m-2} + r\right]$$

$$+ (-1)^{m-1}\left[(-1)^{m-1}\,ma^{m-1} - ar\right] + (-1)^{m+k-1}\,a^m + s > 0,$$

$$A^{\pm}_{m+k-2}=\begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ -\binom{m}{1}a & 1 & 0 & & \cdots & 0 & 0 \\ \binom{m}{2}a^2 & -\binom{m}{1}a & 1 & 0 & & \vdots & \vdots \\ \vdots & \binom{m}{2}a^2 & \ddots & \ddots & \ddots & & \vdots \\ p_{m+k-5} & & \ddots & \ddots & 1 & 0 & 0 \\ p_{m+k-4} & p_{m+k-5} & \cdots & \binom{m}{2}a^2 & -\binom{m}{1}a & 1 & 0 \\ p_{m+k-3} & p_{m+k-4} & p_{m+k-5} & \cdots & \binom{m}{2}a^2 & -\binom{m}{1}a & 1 \end{bmatrix} \pm$$

$$\begin{bmatrix} 0 & 0 & \cdots & & & 0 & s \\ 0 & & & & & s & p_{m+k-2} \\ \vdots & & & & s & p_{m+k-2} & p_{m+k-3} \\ & & \iddots & \iddots & \iddots & p_{m+k-3} & \\ & 0 & s & \iddots & \iddots & & \\ 0 & s & p_{m+k-2} & p_{m+k-3} & & & -\binom{m}{3}a^3 \\ s & p_{m+k-2} & p_{m+k-3} & & & -\binom{m}{3}a^3 & \binom{m}{2}a^2 \end{bmatrix}$$

matrices of dimension $(m+k-2)\times(m+k-2)$ are innerwise. Here entries of $A^{\pm}_{m-1}$ is formed by the coefficients of $p(t)$ where

$$p(t)=\sum_{i=0}^{m-3}(-1)^i\binom{m}{i}a^it^{m-i}+\left((-1)^{m-2}\binom{m}{2}a^{m-2}+r\right)t^2+$$
$$\left((-1)^m ma^{m-1}-ar\right)t+(-1)^m a^m+s.$$

Proof. The proof is similar to the proof of Theorem 3.1, so we omit it. □

Conflicts of Interest

The author declare that there are no conflicts of interest regarding the publication of this article.

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