On \(a^*-I\)-open Sets and a Decomposition of Continuity

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Abstract: In this paper, we introduce a new set namely \(a^*-I\)-open set in ideal topological spaces. Besides, we give some properties and characterizations of it. We obtain that it is stronger than pre\(a^*-I\)-open set with b-open set and weaker than \(\delta I\)-open set. Finally, we give a decomposition of continuity by using \(a^*-I\)-open set as stated the following: \(f : (X, \tau, I) \to (Y, \varphi)\) is continuous if and only if it is \(a^*-I\)-continuous and strongly \(A_I\)-continuous.

Keywords: \(a^*-I\)-open set, Decomposition of continuity, Ideal.

1 Introduction and preliminaries

Topic of ideals in topological spaces has been studied since beginning of 20th century. It has won repute in and importance in citevai. Throughout this paper, we will denote topological spaces by \((X, \tau)\) and \((Y, \varphi)\). For a subset \(A\) of a space \((X, \tau)\), the closure of \(A\) and the interior of \(A\) are denoted by \(Cl(A)\) and \(Int(A)\), respectively. It is well known that a subset \(A\) of a space \((X, \tau)\) is said to be regular open cideon (resp. semi-open, pre-open cideon) if \(A = \text{Int}(\text{Cl}(A))\). A subset \(A\) of a space \((X, \tau)\) is said to be \(\delta\)-open cideon if for each \(x \in A\) there exists a regular open set \(U\) such that \(x \in U \subseteq A\). \(A\) is \(\delta\)-closed cideon if \((X-A)\) is \(\delta\)-open. The set \(\{x \in X \mid x \in U \subseteq A\}\) for some regular open set \(U\) of \(X\) is called the \(\delta\)-interior of \(A\) and is denoted by \(\text{Int}_\delta(A)\) cideon. A point \(x \in X\) is called a \(\delta\)-cluster point of \(A\) if \(A \cap \text{Int}(\delta Cl(A)) \neq \emptyset\) for each open set \(V\) containing \(x\). The set of all \(\delta\)-cluster points of \(A\) is called the \(\delta\)-closure of \(A\) and denoted by \(\delta Cl(A)\) cideon. Of course, \(\delta\)-open sets form a topology \(\tau^\delta\) and then \(\tau^\delta \subset \tau\) holds cideon.

An ideal \(I\) on \(X\) is defined as a nonempty collection of subsets of \(X\) satisfying the following two conditions:

(1) If \(A \in I\) and \(B \subseteq A\), then \(B \in I\);
(2) If \(A \in I\) and \(B \in I\), then \(A \cup B \in I\).

Let \((X, \tau, I)\) be a topological space and \(I\) an ideal on \(X\). An ideal topological space is a topological space \((X, \tau, I)\) with an ideal \(I\) on \(X\) and is denoted by \((X, \tau, I)\). For a subset \(A \subseteq X\), \(A^+(I, \tau) = \{x \in X \mid U \cap A \not\subseteq I\text{ for each neighbourhood }U\text{ of }x\}\) is called the \(\tau\)-local function of \(A\) with respect to \(I\) and \(\tau\). Through this paper, we use \(A^+\) instead of \(A^+(I, \tau)\). Besides, in citejan, authors introduced a new Kuratowski closure operator \(\delta Cl^\ast\) defined by \(\delta Cl^\ast(A) = A \cup A^+(I, \tau)\) and obtained a new topology on \(X\) which is called an \(\ast\)-topology. This topology is denoted by \(\tau^\ast(I)\) which is finer than \(\tau\).

A point \(x\) in an ideal topological space is called \(\delta\)-\(\ast\)-cluster point of \(A\) if Int\((\delta Cl^\ast(A)) \cap A \neq \emptyset\) for each neighborhood \(U\) of \(x\). The set of all \(\delta\)-\(\ast\)-cluster points of \(A\) is called the \(\delta\)-\(\ast\)-closure of \(A\) and will be denoted by \(\delta Cl^\ast(I)(A)\) cideon. A is said to be \(\delta\)-\(\ast\)-continuous cideon if \(A = \delta Cl^\ast(I)(A)\). Of course, the complement of \(\delta I\)-open set is said \(\delta\)-\(I\)-closed cideon. The family of all \(\delta I\)-open sets in any ideal topological space \((X, \tau, I)\) form a topology \(\tau^\delta\) and then \(\tau^\delta \subset \tau\) holds cideon.

Definition 1. some label A subset \(A\) of an ideal topological space \((X, \tau, I)\) is said to be \(\alpha\)-open cideon (resp. semi-open, pre-open cideon) if \(A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))\) (resp. \(A \subseteq \text{Int}(\text{Cl}(A)), A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))\)).

Definition 2. some label A subset \(A\) of an ideal topological space \((X, \tau, I)\) is said to be pre-\(I\)-open cideon (resp. semi-\(I\)-open, \(\alpha\)-open cideon) if \(A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))\) (resp. \(A \subseteq \text{Cl}(\text{Int}(A)), A \subseteq \text{Cl}(\text{Int}(\text{Cl}(A)))\)).

Definition 3. A subset \(A\) of an ideal topological space \((X, \tau, I)\) is said to be \(\delta\)-\(\alpha\)-I-open cideon if \(A \subseteq \text{Int}(\text{Cl}(\delta \text{Int}(A)))\) (resp. \(A \subseteq \text{Cl}(\delta \text{Int}(A)), A \subseteq \text{Cl}(\text{Int}(\delta \text{Int}(A)))\)).

Related to above definitions, one can find the following diagram in citehat4. None of these implications are reversible in generally as shown in the related papers.
Lemma 1. For a subset $A$ of an ideal topological space $(X, \tau, I)$, the following properties are hold:

1. If $U$ is an open set, then $U \cap Cl^*(A) \subseteq Cl^*(U \cap A)$.
2. If $U$ is an open set, then $\delta Cl_I(U) = Cl(U)$.

2. $a^*\text{-}I$-open sets

In this section, to give a decomposition of open set we introduce a new set which name is $a^*\text{-}I$-open set and obtain some properties and characterizations of it.

Definition 4. A subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be an $a^*\text{-}I$-open if

$$A \subset Int(\delta Cl_I(A)) \cup Cl(Int(A)).$$

The complement of an $a^*\text{-}I$-open set said to be an $a^*\text{-}I$-closed. It is obvious that $A$ is an $a^*\text{-}I$-closed if and only if $Cl(\delta Int_I(A)) \cap Int(Cl(A)) \subseteq A$.

Corollary 1. It is obtained from Definition 4, $\emptyset$ and $X$ are both $a^*\text{-}I$-open sets and $a^*\text{-}I$-closed sets.

Proposition 1. Let $(X, \tau, I)$ be an ideal topological space. Then, the following properties are hold:

1. If $A$ is $pre^*\text{-}I$-open, then it is $a^*\text{-}I$-open,
2. If $A$ is $b$-open, then it is $a^*\text{-}I$-open,
3. If $A$ is $a^*\text{-}I$-open, then it is $\delta \beta_1$-open.

Proof: The proof of (1) is clear from Definitions 1, 3 and 4. The others are obtained by using related set definitions. The following diagram is obtained by using Proposition 3 and several sets defined above.

![Diagram II](image)

Remark 1. The converses of each statements in Proposition 3 are not true in generally as shown in the next examples.

Example 1. Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ and $I = \{\emptyset\}$. (1) Set $A = \{a, d\}$. Then, $A$ is an $a^*\text{-}I$-open but it is not $pre^*\text{-}I$-open. (2) Set $A = \{a, b\}$. Then, $A$ is an $a^*\text{-}I$-open but it is not $b$-open.

Example 2. Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$ and $I = \{\emptyset\}$. For $A = \{b, d\}$ is $\delta \beta_1$-open, but it isn’t $a^*\text{-}I$-open.

We have the following diagram.
Since sufficiency is stated in Proposition 3(2), we prove only necessity. Let $a$ be an open operator, we have

$$\delta \text{Cl}_1(A) = \text{Cl}(A).$$

Let $\alpha \text{Cl}_1(A) = \text{Cl}(A)$, if $\alpha$ is an $a^* - I$-open set and hence every $a^* - I$-open set is $b$-open.

**Remark 2.** The notions of $a^* - I$-open set and $b$-open set are independent each other. Indeed in Example 2, set $A = \{a, b\}$ is $b$-open, but it isn’t $a^* - I$-open. Besides in Example 1(2), set $A = \{a, b\}$ is an $a^* - I$-open but it is not $b$-open.

**Proposition 3.** Let $(X, \tau, I)$ be an ideal topological space with an arbitrary index set $\Delta$. If $\{A_\alpha : \alpha \in \Delta\} \subset a^* I O(X, \tau)$, then $\bigcup \{A_\alpha : \alpha \in \Delta\} \in a^* I O(X, \tau)$.

**Proposition 4.** Let $(X, \tau, I)$ be an ideal topological space and $A, U$ are subsets of $X$. If $A$ is an $a^* - I$-open set and $U$ is $\delta - I$-open set. Then $(A \cap U)$ is an $a^* - I$-open set.

**Definition 5.** A subset $A$ of an ideal topological space $(X, \tau, I)$ is called

1) strongly $I$-set if $\text{Int}(\delta \text{Cl}_1(A)) = \text{Int}(A)$,
2) strongly $A_1$-set if $A = U \cap V$, where $U \in \tau$ and $V$ is strongly $I$-set and $\text{Int}(\delta \text{Cl}_1(V)) = \text{Cl}(\text{Int}(V))$.

**Theorem 1.** The following properties hold for a subset $A$ of an ideal topological space $(X, \tau, I)$:

1) If $A$ is strongly $I$-set and $\text{Int}(\delta \text{Cl}_1(A)) = \text{Cl}(\text{Int}(A))$, then it is strongly $A_1$-set,
2) If $A$ is open set, then it is strongly $A_1$-set.
Proof: (1) Since A is strongly t-I-set with $\text{Int}(\delta \text{Cl}_I(A)) = \text{Cl}(\text{Int}(A))$ and $A \in \tau$, the proof of (1) is obvious.

Proof: (2) Since X is strongly t-I-set with $\text{Int}(\delta \text{Cl}_I(X)) = \text{Cl}(\text{Int}(X))$ and $A \in \tau$, the proof of (2) is obtained.

Theorem 2. For a subset A of $(X, \tau, I)$, the following properties are equivalent:

1. A is open.
2. A is an $a^* - I$-open and strongly $A_I$-set.

Proof: (1) $\implies$ (2): By Diagram II, every open set is $a^* - I$-open. Besides, we have every open set is strongly $A_I$-set according to Theorem 7(2).

(2) $\implies$ (1): Let A is an $a^* - I$-open and strongly $A_I$-set. Then, we have $A \subseteq \text{Int}(\delta \text{Cl}_I(A)) \cup \text{Cl}(\text{Int}(A))$ and strongly $A_I$-set if $A = U \cap V$, where $U \in \tau$ and V is strongly t-I-set and $\text{Int}(\delta \text{Cl}_I(V)) = \text{Cl}(\text{Int}(V))$, respectively. Therefore, we have $A \subseteq \text{Int}(\delta \text{Cl}_I(U \cap V)) \cup \text{Cl}(\text{Int}(U \cap V)) \subseteq \text{Int}(\delta \text{Cl}_I(U)) \cap \text{Int}(\delta \text{Cl}_I(V)) \cup \text{Cl}(\text{Int}(U)) \cap \text{Cl}(\text{Int}(V)) = \text{Int}(\delta \text{Cl}_I(U)) \cap \text{Cl}(\text{Int}(V)) \cup \text{Cl}(\text{Int}(U)) \cap \text{Cl}(\text{Int}(V))$. According to Lemma 1(2), since $U \in \tau$, it is obvious that $\delta \text{Cl}_I(U) = \text{Cl}(U)$ and $\text{Int}(\delta \text{Cl}_I(U)) = \text{Int}(\text{Cl}(U))$. So, we have

$A \subseteq \text{Int}(\text{Cl}(U)) \cup \text{Cl}(\text{Int}(V)) \cup \text{Cl}(\text{Int}(U)) \cap \text{Cl}(\text{Int}(V)) = \text{Int}(\text{Cl}(U)) \cup \text{Cl}(\text{Int}(U)) \cap \text{Cl}(\text{Int}(V))$. Consequently, since $A \subseteq U$, we obtain $A \subseteq U \cap (\text{Int}(\text{Cl}(U)) \cup \text{Cl}(\text{Int}(U)) \cap \text{Cl}(\text{Int}(V))) = \{U \cap \text{Int}(\text{Cl}(U)) \cup \text{Cl}(\text{Int}(U)) \cap \text{Cl}(\text{Int}(V)) = \{U \cap \text{Int}(\text{Cl}(U)) \cup (U \cap \text{Cl}(\text{Int}(U))) \cap \text{Cl}(\text{Int}(V)) = U \cap \text{Int}(V) = \text{Int}(U \cap V) = \text{Int}(A)$. Hence A is an open.

3 Decomposition of continuity

In this section, we introduce the notions of $a^* - I$-continuity, strongly $A_I$-continuity and obtain a decomposition of continuity.

Definition 6. A function $f: (X, \tau) \rightarrow (Y, \varphi)$ is said to be b-continuous citeel-a if $f^{-1}(V)$ is a b-open set in $(X, \tau)$ for every open set $V$ in $(Y, \varphi)$.

Definition 7. A function $f: (X, \tau, I) \rightarrow (Y, \varphi)$ is said to be pre-$a^* - I$-continuous citeelk (resp. $\delta \beta_I$-continuous citeel4, $a^* - I$-continuous strongly $A_I$-continuous ) if $f^{-1}(V)$ is a pre-$a^* - I$-open (resp. $\delta \beta_I$-open, $a^* - I$-open set, strongly $A_I$-set ).

Proposition 5. For a function $f: (X, \tau, I) \rightarrow (Y, \varphi)$, the following properties are hold: (1) If f is pre-$a^* - I$-continuous, then f is $a^* - I$-continuous. (2) If f is b-continuous, then f is $a^* - I$-continuous. (3) If f is $a^* - I$-continuous, then f is $\delta \beta_I$.

Proof: The proofs are omitted from Proposition 3 as consequences by using Definitions 6 and 7.

Remark 3. The converses of each statements in Proposition 9 are not true in generally as shown in the next examples.

Example 5. Let $(X, \tau, I)$ be an ideal topological space as same as in Example 1 and $Y = \{a, b\}$, $\varphi = \{Y, \varnothing, \{a\}\}$. (1) Let $f: (X, \tau, I) \rightarrow (Y, \varphi)$ be a function defined as $f(a) = f(d) = a$, $f(b) = f(c) = b$. Then f is $a^* - I$-continuous, but it isn’t pre-$a^* - I$-continuous.

(2) Let $f: (X, \tau, I) \rightarrow (Y, \varphi)$ be a function defined as $f(b) = f(d) = a$, $f(a) = f(c) = b$. Then f is $a^* - I$-continuous, but it isn’t b-continuous.

Example 6. Let $(X, \tau, I)$ be an ideal topological space as same as in Example 2 and $Y = \{a, b\}$, $\varphi = \{Y, \varnothing, \{a\}\}$. Let $f: (X, \tau, I) \rightarrow (Y, \varphi)$ be a function defined as $f(a) = f(d) = a$, $f(b) = f(c) = b$. Then f is $\delta \beta_I$-continuous, but it isn’t $a^* - I$-continuous.

It is known that a function $f: (X, \tau) \rightarrow (Y, \varphi)$ is continuous if $f^{-1}(V)$ is an open set in $(X, \tau)$ for every open set $V$ in $(Y, \varphi)$.

Theorem 3. For a function $f: (X, \tau, I) \rightarrow (Y, \varphi)$, the following statements are equivalent: (1) f is continuous, (2) f is $a^* - I$-continuous and strongly $A_I$-continuous.

Proof: This follows from Theorem 8.

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4 References


